# Irregular conformal blocks and Painleve equations

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Bogolyubov Institute for Theoretical Physics

6 March 2025

KMPB-Ukraine Workshop

# Literature

This talk is partially based on a paper:

- Hasmik Poghosyan, Rubik Poghossian, "A note on rank 5/2 Liouville irregular block, Painlevé I and the $\mathcal{H}_0$ Argyres-Douglas theory", arXiv:2308.09623

The Painlevé I equation is defined by

$$\frac{d^2q}{dt^2} = 6q^2 + t.$$

It's a Hamiltonian system

$$H(p,q,t) = \frac{p^2}{2} - 2q^3 - tq.$$

The isomonodromic **tau function** $\tau(t)$ defined by

$$\partial_t \log \tau(t) = H(t).$$

https://en.wikipedia.org/wiki/Painlevé_transcendents

# Linear problem for PI

Consider a system of matrix differential equations

$$\partial_z \Phi = A(z,t)\Phi, \qquad \partial_t \Phi = B(z,t)\Phi,$$

with $2 \times 2$ matrices $A(z,t)$ and $B(z,t)$

$$A(z,t) = A_2(t)z^2 + A_1(t)z + A_0(t),$$

$$B(z,t) = B_1(t)z + B_0(t),$$

Zero curvature condition gives PI equation

$$\partial_t A - \partial_z B + [A,B] = 0.$$

O. Lisovyy, J. Roussillon, "On the connection problem for Painlevé I", arXiv:1612.08382

# Types of singularities

Connection matrix $A(z,t)$ can have singularities in $z$-plane

$$A(z,t) \sim \frac{A^{\mathrm{res}}}{(z-a)^{1+r}} \quad \Rightarrow \quad \Phi(z) \sim \exp\left(-\frac{A^{\mathrm{res}}}{r}(z-a)^{-r}\right)$$

- $r = 0$ – regular singularity
- $r > 0$ – irregular singularity

In PI case the only singular point is $z = \infty$ of rank $r = 3$. However, the residue $A^{\mathrm{res}}$ is not diagonalizable $\Rightarrow$ rank of asymptotics is $r = 5/2$

# Motivation: short

Why do we study the Painlevé equations?

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Why do we study the Painlevé equations?

Because of a huge interplay between different branches of theoretical physics and mathematics.

# Motivation: less short

In early 2010's it was found remarkable formulas for **tau functions** of Painlevé equations near **regular singularities**.

- Gamayun O, Iorgov N and Lisovyy O, Conformal field theory of Painlevé VI, 2012
- Gamayun O, Iorgov N and Lisovyy O, How instanton combinatorics solves Painlevé VI, V and IIIs, 2013

Symbolically, tau functions have the form

$$\tau(t) = \sum_{n \in \mathbb{Z}} e^{in\rho} Z(\nu + n; \boldsymbol{\theta}; t)$$

where $\rho, \nu$ – monodromy parameters (initial data), $\boldsymbol{\theta}$ – parameters of equation, $Z(\nu; \boldsymbol{\theta}; t)$ – Nekrasov's partition function.

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$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\rho} Z_{AD}(\nu + n; \boldsymbol{\theta}; t),$$

but $Z_{AD}$ is partition function of Argyres–Douglas theories.

- Giulio Bonelli, Oleg Lisovyy, Kazunobu Maruyoshi, Antonio Sciarappa and Alessandro Tanzini, On Painlevé/gauge theory correspondence, 2016

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Remark: parameter $t$ lies on some critical rays and goes to infinity.

# Asymptotics of PI tau function

- Asymptotics for $t \to -\infty$

$$\tau(t) = \sum_{n \in \mathbb{Z}} e^{in\rho} Z_{AD}(\nu + n; t),$$

- Fourier coefficients $Z_{AD}$

$$Z_{AD}(\nu, t) = C(\nu) Z_{\mathrm{tree}}(\nu, t) Z_{\mathrm{inst}}(\nu, t),$$

- Nice time variable

$$s = 24^{\frac{1}{4}} (-t)^{\frac{5}{4}},$$

- Nonperturbative part

$$Z_{\mathrm{tree}}(\nu, t) = \exp\left(\frac{s^2}{45} + \frac{4i\nu s}{5}\right) s^{-\frac{1}{60} - \frac{\nu^2}{2}},$$

- Perturbative part

$$Z_{\mathrm{inst}}(\nu, t) = 1 + \sum_{k=1}^{\infty} \frac{B_k(\nu)}{s^k}.$$

# Argyres–Douglas partition function from CFT

There is a conjecture that $Z_{AD}$ can be found using irregular representations of Virasoro algebra

1. integer rank
   - D. Gaiotto and J. Teschner, Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories, 2012
   - H. Nagoya, Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations, 2015
2. half integer rank
   - H. Poghosyan, R. Poghossian, "A note on rank 5/2 Liouville irregular block, Painlevé I and the $\mathcal{H}_0$ Argyres-Douglas theory", 2023
   - T. Nishinaka et al, "Liouville Irregular States of Half-Integer Ranks", 2024

# Virasoro algebra

- Generators of Virasoro algebra $\{L_n, c\}$ comute as

$$[L_n, L_m] = (n-m)L_{n+m} + c\delta_{n+m,0}\frac{n^3-n}{12}, \qquad [L_n, c] = 0.$$

with $n \in \mathbb{Z}$.

Wikipedia: Virasoro algebra

- We fix central charge $c$ to be

$$c = 1 + 6Q^2$$

# Representation theory

- Highest weight module $\mathcal{V}_\Delta$ is built using vector $|\Delta\rangle$

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_{n>0}|\Delta\rangle = 0.$$

- Generators with negative $n$ generate module from $|\Delta\rangle$

$$\mathcal{V}_\Delta = \mathrm{Span}\left\{ L_{-Y}|\Delta\rangle \,|\, Y \in \text{Young diagrams} \right\}$$

where

$$L_{-Y} = L_{-k}^{n_k} \dots L_{-2}^{n_2} L_{-1}^{n_1}, \qquad |Y| = n_1 + 2n_2 + \cdots kn_k.$$

# Generalized representations

- Rank-$n$ representations is built using vector $|I^{(n)}\rangle$

$$L_k|I^{(n)}\rangle = \mathcal{L}_k|I^{(n)}\rangle,$$

  where $\mathcal{L}_k = 0$ for $k > 2n$ and nonzero number for $k \in \{n, n+1, \ldots, 2n\}$
- Generators with $k \in \{0, 1, \ldots n-1\}$ can be built as differential operators.

Conjectured in: D. Gaiotto, ArXiv:0908.0307

Existence theorem: V. Mazorchuk, K. Zhao, ArXiv: 1205.5937

# Example: integer rank

Let us present rank-2 representation

$$L_k|I^{(2)}\rangle = \mathcal{L}_k|I^{(2)}\rangle, \qquad k \geq 0,$$

with $\mathcal{L}_{k>4} = 0$ and

$$\mathcal{L}_0 = c_0(Q - c_0) + c_1\frac{\partial}{\partial c_1} + 2c_2\frac{\partial}{\partial c_2},$$
$$\mathcal{L}_1 = 2c_1(Q - c_0) + c_2\frac{\partial}{\partial c_1},$$
$$\mathcal{L}_2 = c_2(3Q - 2c_0) - c_1^2,$$
$$\mathcal{L}_3 = -2c_1c_2, \qquad \mathcal{L}_4 = -c_2^2.$$

D. Gaiotto and J. Teschner, ArXiv:1203.1052

# Example: half-integer rank

Let us present rank-5/2 representation

$$L_n|I^{(5/2)}\rangle = \mathcal{L}_n|I^{(5/2)}\rangle, \qquad n \geq 0,$$

with $\mathcal{L}_{n>5} = 0$ and

$$\mathcal{L}_0 = c_1\frac{\partial}{\partial c_1} + 2c_2\frac{\partial}{\partial c_2} + 5\Lambda_5\frac{\partial}{\partial \Lambda_5},$$
$$\mathcal{L}_1 = \frac{2c_1^2c_2^2}{\Lambda_5} + \frac{2c_2^3 - 3c_1\Lambda_5}{2c_2^2}\frac{\partial}{\partial c_1} + \frac{3\Lambda_5}{2c_2}\frac{\partial}{\partial c_2},$$
$$\mathcal{L}_2 = \frac{\Lambda_5}{2c_2}\frac{\partial}{\partial c_1},$$
$$\mathcal{L}_3 = -2c_1c_2, \qquad \mathcal{L}_4 = -c_2^2, \qquad \mathcal{L}_5 = -\Lambda_5.$$

H. Poghosyan, R. Poghossian, arXiv:2308.09623

# Irregular conformal blocks

- We can built correlation function

$$\mathcal{Z} = \langle 0 | I^{(5/2)} \rangle \sim Z_{AD},$$

where $\langle 0|$ is the vaccum state

$$\langle 0 | L_n = 0, \qquad n \leq 1.$$

- Rank $5/2$ state can be embeded (conjecturally) into rank 2 module

$$|I^{(5/2)}\rangle = f|\psi\rangle,$$

where $f$ is a function of $c_1, c_2, \Lambda_5$ defined as

$$f^{-1}\mathcal{L}_k^{(5/2)} f = \mathcal{L}_k^{(2)} + O(\Lambda_5).$$

and $|\psi\rangle$ is a vector in rank-2 module

$$|\psi\rangle = |I^{(2)}\rangle + \sum_{k=1}^{\infty} \Lambda_5^k |I_k^{(2)}\rangle, \qquad \langle 0|\psi\rangle \sim Z_{\text{inst}}$$

# Our results

- Vector $|\psi\rangle$ can be defined as a solution of

$$L_5|\psi\rangle = \varepsilon|\psi\rangle, \qquad L_4|\psi\rangle = \frac{1}{4}|\psi\rangle, \qquad L_3|\psi\rangle = 0,$$

$$\left(L_2 - 2\varepsilon L_1 + 6\varepsilon^2 L_0\right)|\psi\rangle = \left(30\varepsilon^3 \frac{\partial}{\partial\varepsilon} - \nu\right)|\psi\rangle,$$

where $\varepsilon \sim (-t)^{-5/8}$.

- Vector $|\psi\rangle$ can be found uniquely in the form

$$|\psi\rangle = G|I^{(2)}\rangle, \qquad G = 1 + \sum_{k=1}^{\infty} \mu^k G_k,$$

$$G_k = \sum_{\substack{l=0 \\ l=k \mod 2}}^{k} \sum_{2m_0+|Y|+m_1=l} C_{Y;m_0,m_1} L_{-Y} L_0^{m_0} L_1^{m_1}.$$

# Outline

1. To find recursion relations for other irregular conformal blocks
2. Relation with topological recursion construction of K.Iwaki, ArXiv: 1902.06439

$$\psi(z) = \text{"Baker–Akhiezer function"} = \frac{\langle 0 | V_{\mathrm{deg}}(z) | I^{(5/2)} \rangle}{\langle 0 | I^{(5/2)} \rangle}.$$

# End

Thanks to organisers of "KMPB-Ukraine Workshop" for their work!