# Research in Symplectic Topology

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Humboldt-Universität zu Berlin

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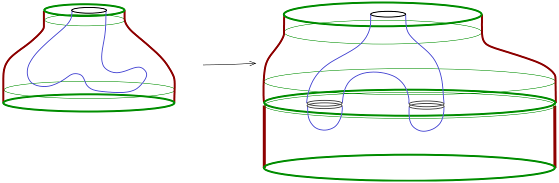

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$$\Rightarrow \quad \textbf{WOW!}$$

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## Examples

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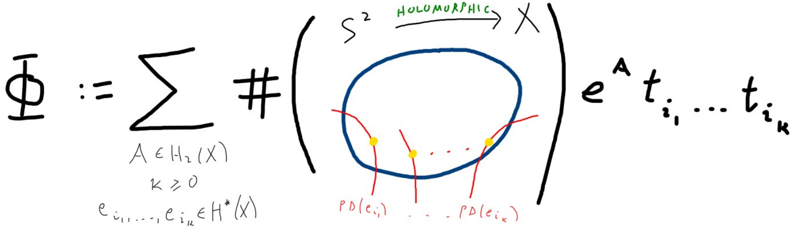
Each closed symplectic manifold $X$ has a **Gromov-Witten potential**:



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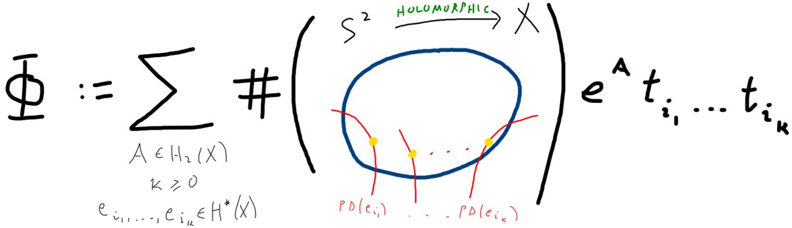


This is a **symplectic** invariant, but is studied more in **algebraic** geometry.

# Why symplectic *topology*?

Darboux's theorem: For every symplectic manifold $(X, \omega)$,

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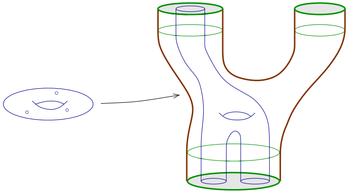
**Message**: *All interesting symplectic invariants are* **global**, *not local.*

# Symplectic field theory (Eliashberg-Givental-Hofer 2000)

SFT generalizes GW-theory to study **punctured** holomorphic curves in **symplectic cobordisms** with **contact boundary**: $\partial X = Y_+ \sqcup (-Y_-)$.



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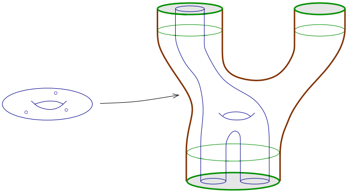

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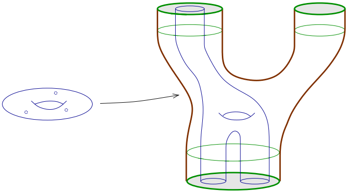

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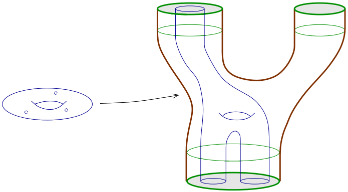

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2. Given two contact manifolds $Y_\pm$, do there exist **symplectic cobordisms** from $Y_-$ to $Y_+$?
3. How are the answers to those questions related to the **classification of contact structures** on a given manifold?

# Rigidity and flexibility

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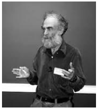

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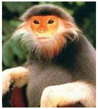

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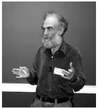

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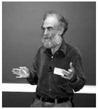

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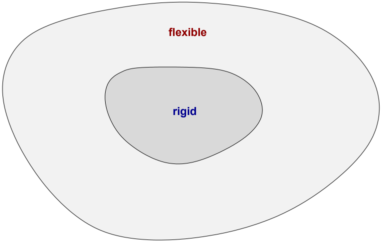


*Basic insight:* The most interesting stuff is on the **borderline**.

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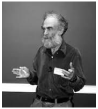

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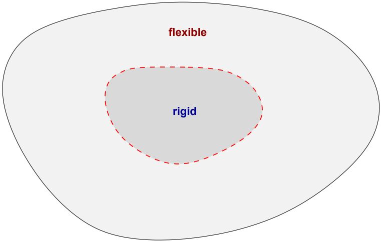


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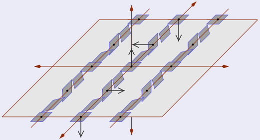
  *(Gromov '85 + Eliashberg '89 + Niederkrüger '06)*
- The $3$-torus admits an infinite sequence of contact structures that are **homotopic** as $2$-plane fields but **not isotopic**. *(Giroux '94)*



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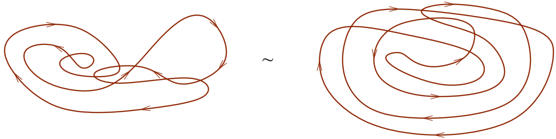

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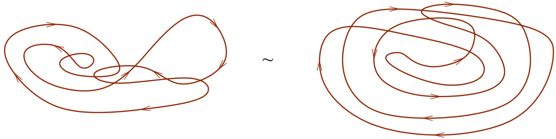

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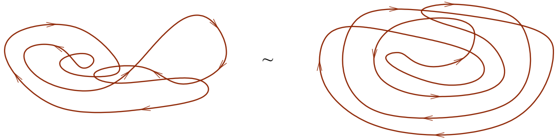

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- **Open** manifolds admit **symplectic** structures if and only if they admit **almost complex** structures. *(Gromov '69)*
- Two **overtwisted** contact structures are **isotopic** if and only if they are **homotopic**. *(Eliashberg '89 + Borman-Eliashberg-Murphy '15)*

# Algebra of SFT

Contact manifold $Y \rightsquigarrow$ supersymmetric operator algebra:
closed Hamiltonian **orbits** $\gamma$ in $\mathbb{R} \times Y \rightsquigarrow$ **operators** $p_\gamma, q_\gamma$ satisfying

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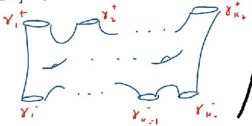
Counting holomorphic curves in $\mathbb{R} \times Y \rightsquigarrow$ **SFT generating function**:

$$H := \sum_{\substack{g, k_+, k_- \geq 0 \\ \gamma_1^+, \dots, \gamma_{k_+}^+ \\ \gamma_1^-, \dots, \gamma_{k_-}^- \\ A \in H_2(M, \gamma^+ - \gamma^-)}} \# \left( \Sigma_g \setminus \{k_+ + k_- \text{ pts}\} \xrightarrow{\text{HOL.}} \mathbb{R} \times Y \right) \hbar^{g-1} e^A q_{\gamma_1^-} \cdots q_{\gamma_{k_-}^-} p_{\gamma_1^+} \cdots p_{\gamma_{k_+}^+}$$



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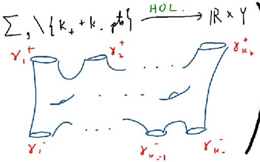


This is *not* an invariant, but. . .

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**Theorem/definition**

$\mathbf{H}^2 = \frac{1}{2}[\mathbf{H}, \mathbf{H}] = 0$, and in certain super-Lie-algebra representations (e.g. setting $p_\gamma := \hbar \frac{\partial}{\partial q_\gamma}$), $\mathbf{H}$ defines the differential in a homological **contact invariant** $H_*^{\mathrm{SFT}}(Y)$, functorial under **symplectic cobordisms**.

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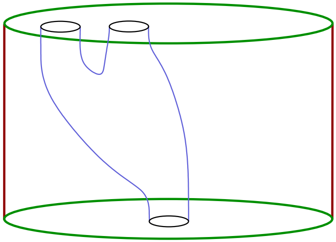

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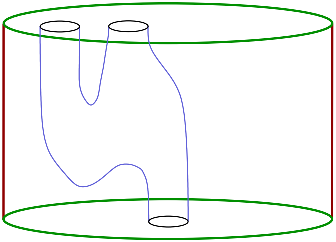

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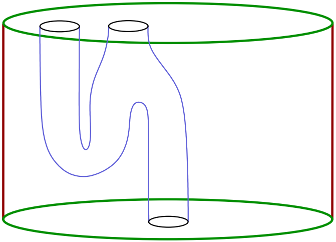

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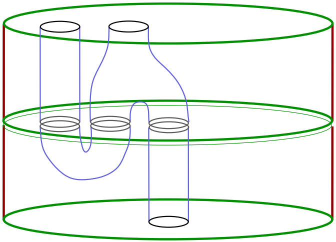

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## Simplistic example

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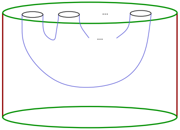

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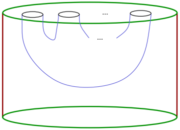

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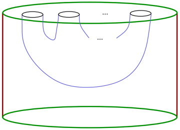

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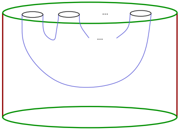

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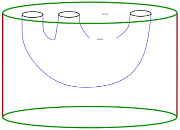
# Algebraic torsion in contact manifolds

## Simplistic example

Suppose $\mathbb{R} \times Y$ has **exactly one** rigid holomorphic curve, with **genus** $0$, **no negative ends**, and positive ends at orbits $\gamma_1, \ldots, \gamma_k$.



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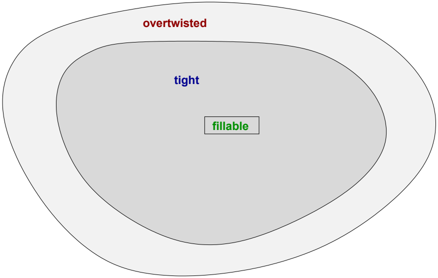
## Definition

$\mathrm{AT}(Y) \in \mathbb{N} \cup \{0, \infty\}$ is the smallest $k$ such that $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(Y)$, or $\infty$ if no such $k < \infty$ exists.

# Algebraic torsion in contact manifolds



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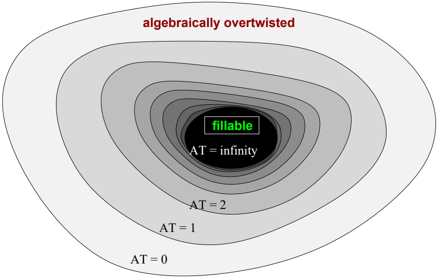

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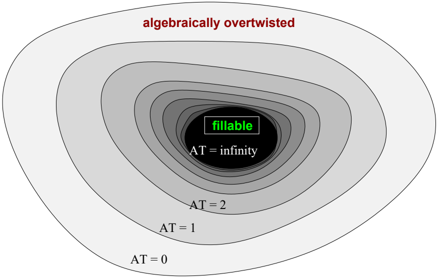


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1. $\mathrm{AT}(Y) = 0$ if and only if $Y$ is **algebraically overtwisted**.

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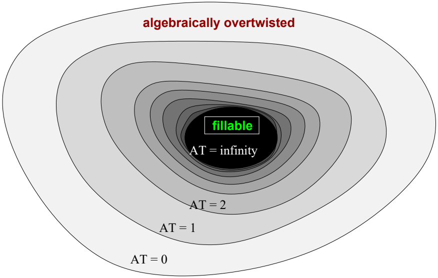


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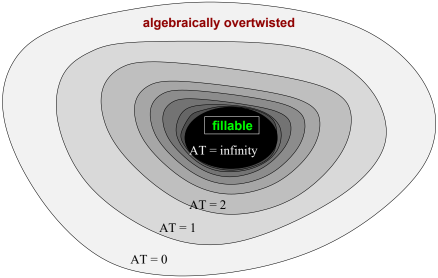


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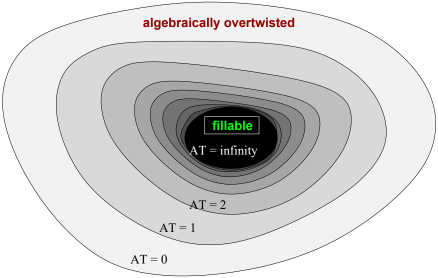


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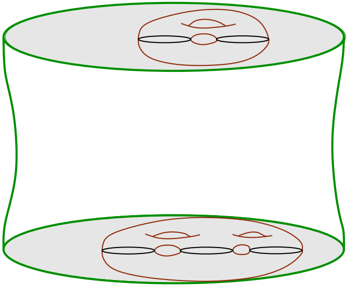
3. $\exists$ **exact symplectic cobordism** $Y_- \rightsquigarrow Y_+ \Longrightarrow \mathrm{AT}(Y_-) \leq \mathrm{AT}(Y_+)$.
4. There exist contact manifolds $Y$ taking **all possible values** of $\mathrm{AT}(Y)$.

# Cobordism obstructions from holomorphic curves

**Example**: What goes wrong if $\mathrm{AT}(Y_-) = 2$ and $\mathrm{AT}(Y_+) = 1$?



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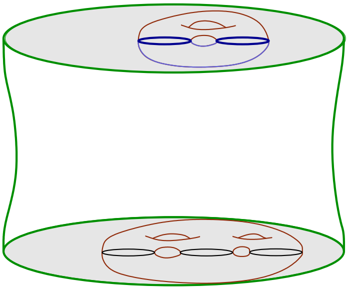

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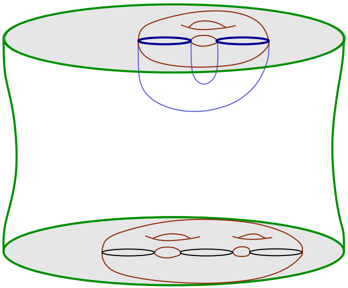

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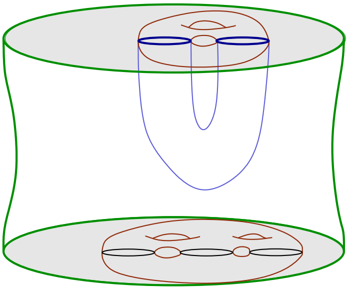

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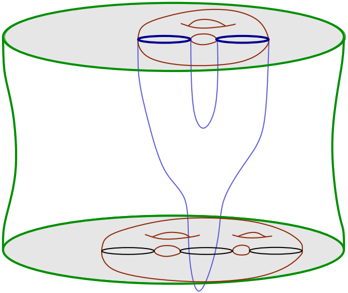

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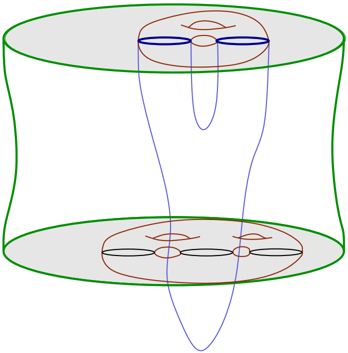

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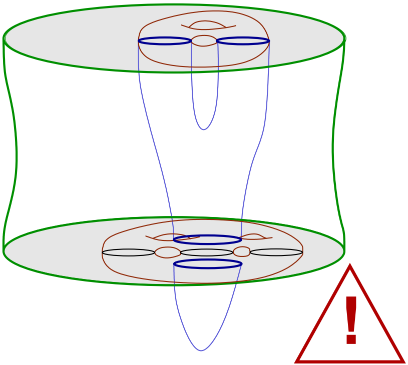

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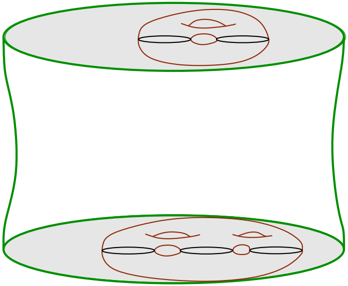

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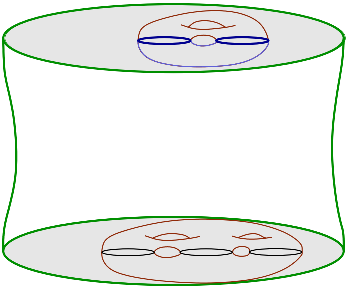

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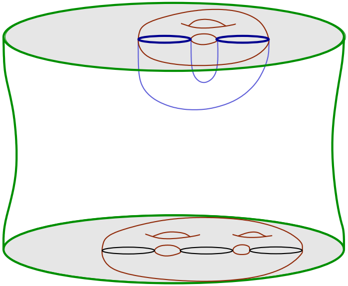

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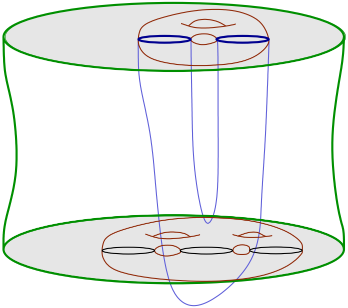

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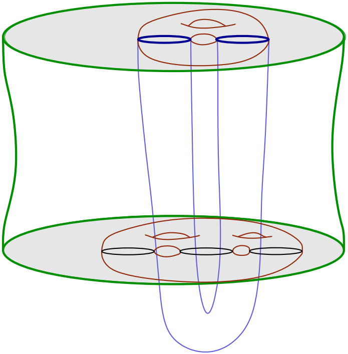

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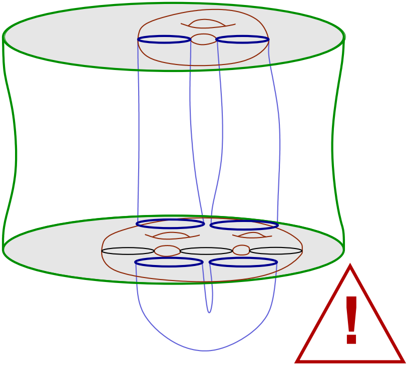

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*Work in progress:*
Understand **bifurcations** in $\mathcal{M}(J)$ under generic $1$-parameter deformations $\{J_s\}_{s\in[0,1]}$. This should lead to a mathematical definition of the **BPS invariants** $n_{A,h} \in \mathbb{Z}$ appearing in the **Gopakumar-Vafa formula**:

$$\sum_{A\neq 0,\ g\geq 0} \mathrm{GW}_{g,A}\, t^{2g-2} q^A = \sum_{A\neq 0,\ h\geq 0} n_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left(2\sin\frac{kt}{2}\right)^{2h-2} q^{kA}$$