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Simple layer potential expansion for solving and optimizing contact interaction by doubly connected domains

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ЗАПОРІЗЬКА ПОЛІТЕХНІКА
НАЦІОНАЛЬНИЙ УНІВЕРСИТЕТ

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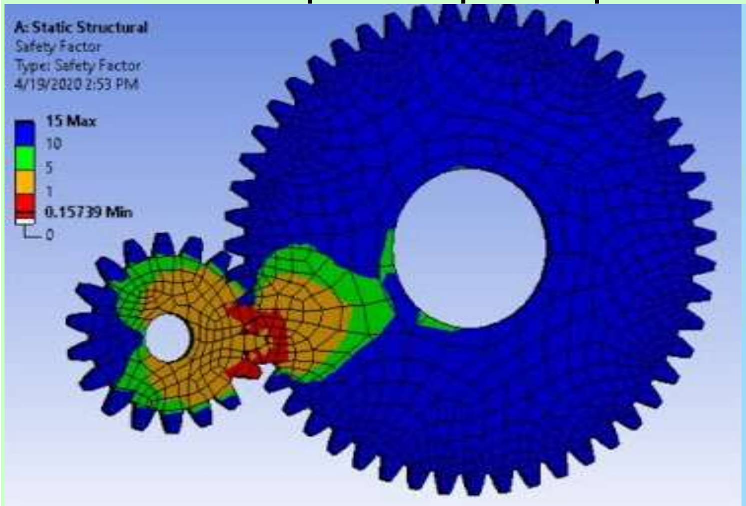
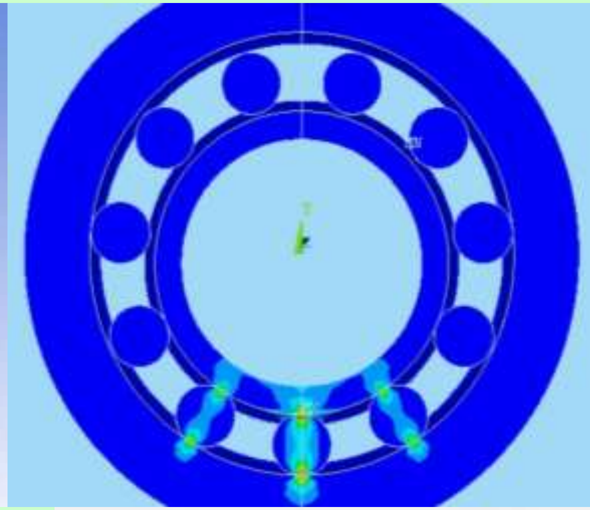
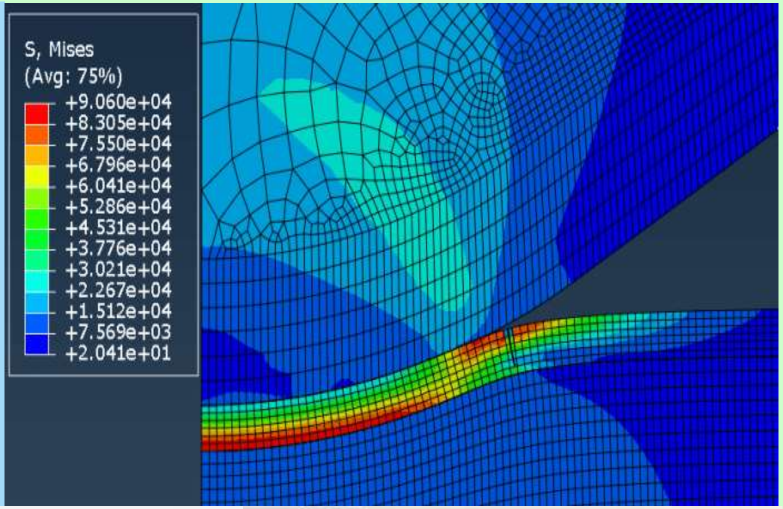
Mainly the contact interaction problems are the problems with mixed boundary conditions that lead to integral equations, which require special solution methods.

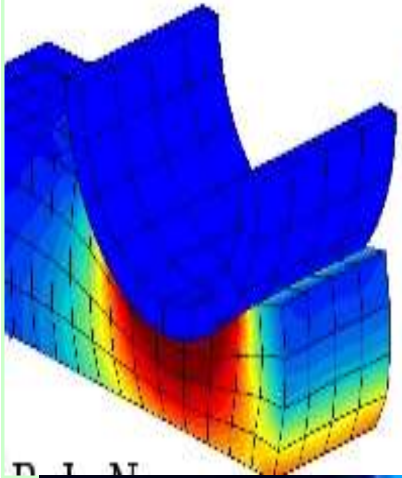
Numerical solution of three-dimensional problems have difficulties through worsening convergence.

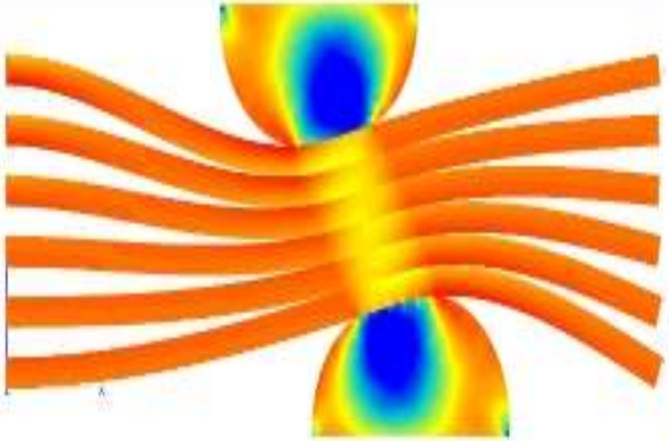
Aims: ♦ to work out solving methods for the problem about indentation of doubly-connected punch into elastic half-space to define normal pressure distribution, indentation value and optimal punch shape for optimal pressure distribution.

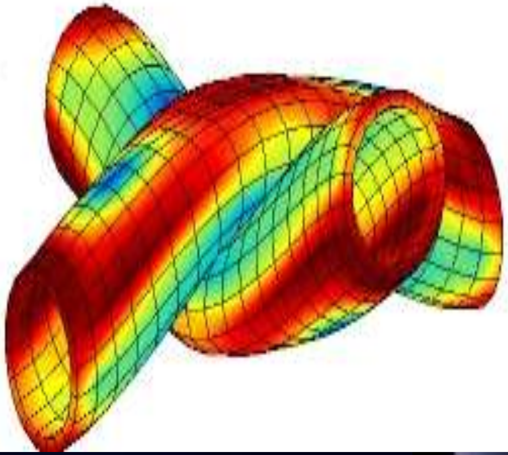


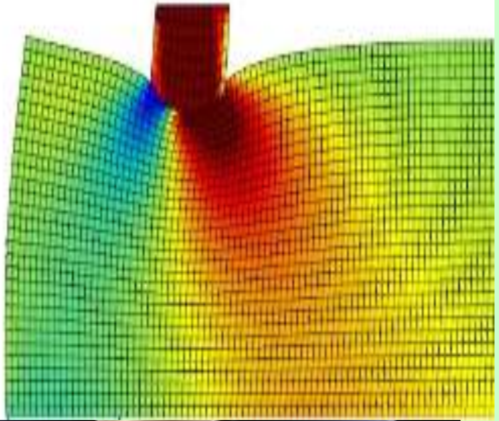






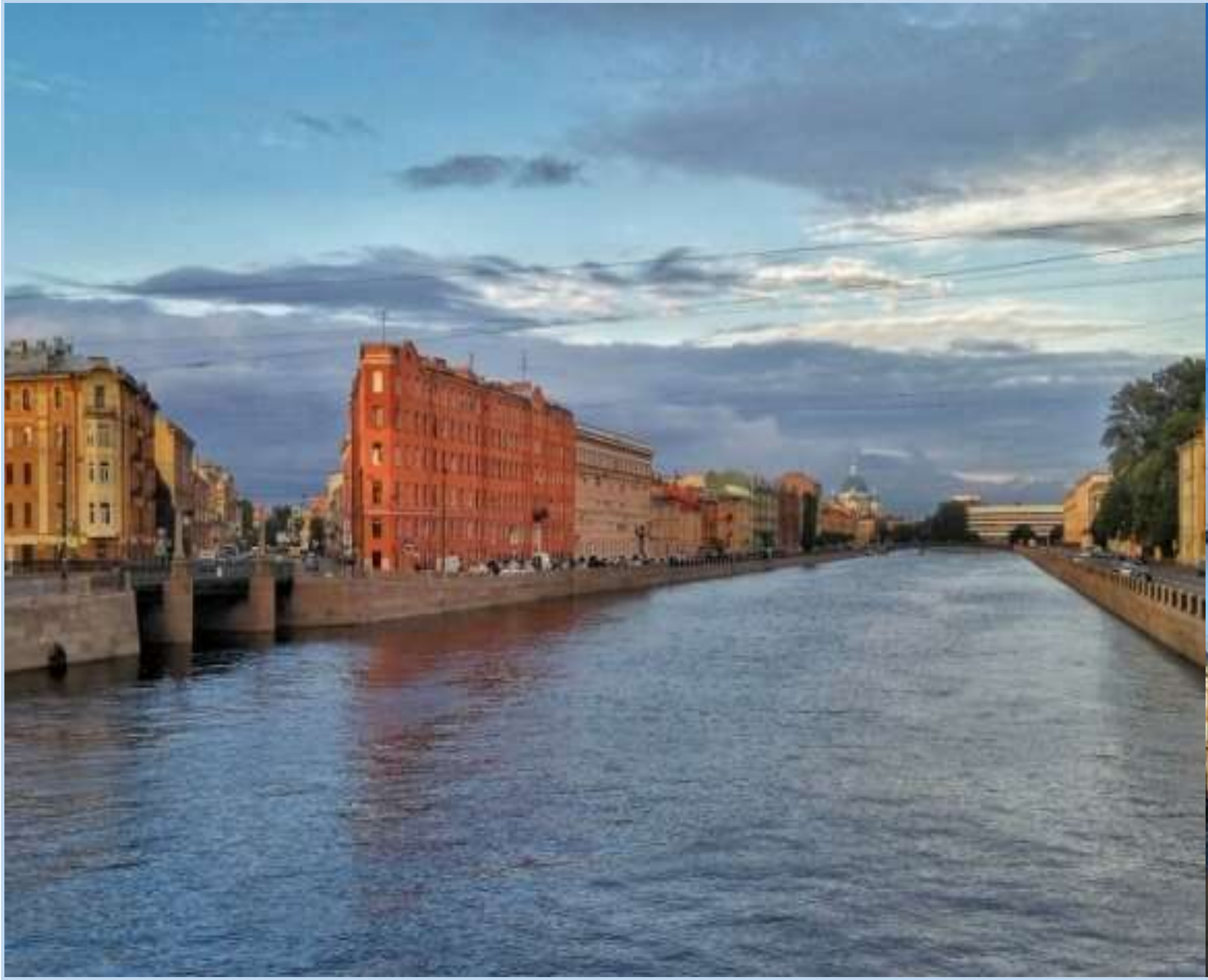


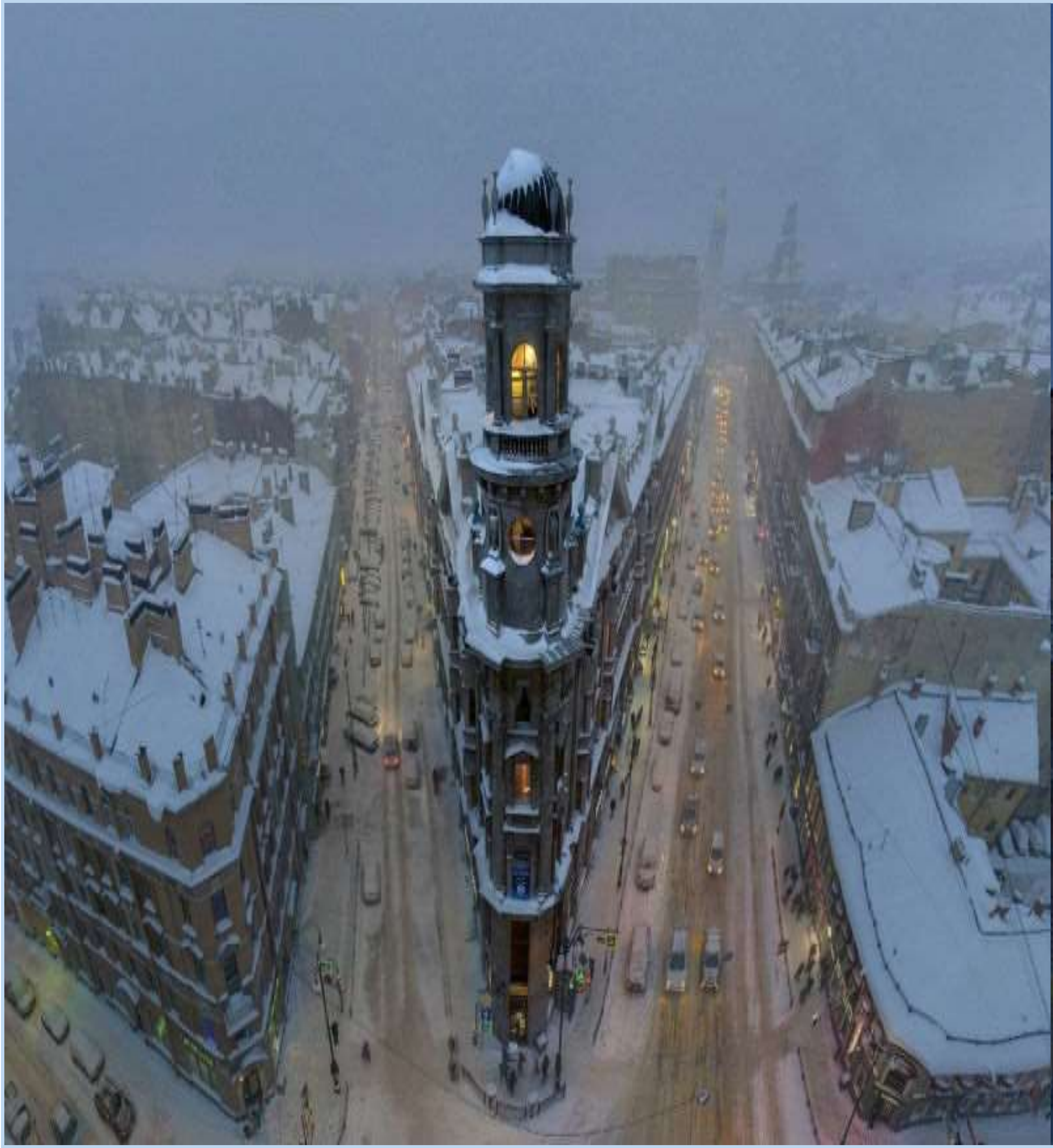




From a mechanical point of view, the calculation of slab structures on an elastic foundation is a contact problem of interacting bodies.


БОГЕМИУС






АСТРОНОМ

FORMULATION OF THE PROBLEM

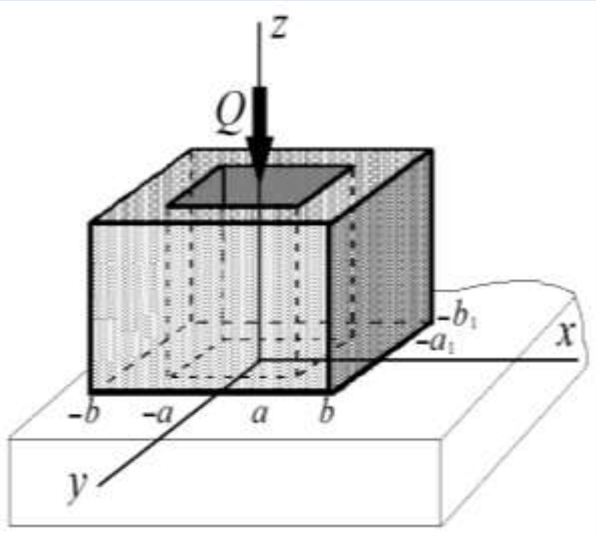


Fig.1 Scheme of a plane doubly connected punch contact

$$\delta - \beta_2 x + \beta_1 y - z(x,y) = \frac{1-\nu}{2\pi G} \iint_\Omega \frac{p(x',y')\,d\Omega'}{\sqrt{(x-x')^2 + (y-y')^2}}, \qquad (1)$$

$$\sigma_{23} = \sigma_{31} = 0, \qquad z = 0,$$

$$\sigma_{33}(x,y,0) = \begin{cases} -p(x,y,0), \varsigma(x,y,0) \in \Omega, \\ 0, \qquad \varsigma(x,y,0) \notin \Omega, \end{cases} \qquad (2)$$

$$Q = \iint_\Omega p(x,y)\,d\Omega, \quad M_1 = \iint_\Omega y \cdot p(x,y)\,d\Omega, M_2 = \iint_\Omega x \cdot p(x,y)\,d\Omega. \qquad (3)$$

$$\tilde{\varphi}(p(\rho_0,\theta_0)) + \frac{1-\nu^2}{\pi E} \iint_\Omega \frac{p(\rho,\theta)}{r} d\Omega = \\ = \delta - \beta_2 \rho_0 \cos\theta_0 + \beta_1 \rho_0 \sin\theta_0 - z(\rho_0,\theta_0), \qquad (4)$$

Therefore, the contact problem reduced to the solution of the system of equilibrium equations and equation containing integrals with weak singularity. Existence and uniqueness of the problem is proved by Ya. Novitskiy

The analytic solution method based on small parameter expansion of simple layer potential type integrals distributed on doubly-connected (and simply-connected) domain is developed. Unknown contact domain is found out.

SHAPE OPTIMIZATION IN 3D CONTACT PROBLEMS

The solution method was proposed by Banichuk and Ivanova.

The first problem boundary conditions

$$\sigma_{13} = \sigma_{23} = 0,\ \omega = \omega(x),\ \text{if } x \in \Omega,\ \text{ and } \sigma_{33} = \sigma_{31} = \sigma_{23} = 0,\ \text{if } x \in \Omega_0, \tag{5}$$

$p(x) = \sigma_{33}(x),\ x \in \Omega$, can be written as $p(\omega) = L_a \omega$, where L_a is a linear operator.

The second problem boundary conditions

$$\sigma_{13} = \sigma_{23} = 0,\ \sigma_{33} = p(x),\ \text{if } x \in \Omega,\ \text{and } \sigma_{33} = \sigma_{31} = \sigma_{23} = 0,\ x \in \Omega_0, \tag{6}$$

the vertical displacement

$$x_3 = 0, \quad (x_1, x_2) \in \Omega: \ \omega(x_1, x_2) = \delta - \beta_2 x_1 + \beta_1 x_2 - f(x_1, x_2). \tag{7}$$

The boundary condition for the vertical displacement of the domain points is reduced to a two-dimensional integral equation

$$\omega(x_1, x_2) = (1-\nu)\cdot(2\pi G)^{-1/2} \iint_\Omega p(x_1', x_2') \cdot \left((x_1 - x_1')^2 + (x_2 - x_2')^2\right)^{-1/2} \mathrm{d}\, x_1' \,\mathrm{d}\, x_2', \tag{8}$$

the equilibrium conditions of the punch (3).

$p_g = p_g(x) \geq 0$ describes some given pressure distribution in the contact domain Ω.

As a functional to be optimized, let us consider the integral

$$J_f = J(p(f)) = \iint_\Omega (p_g - p(g))^2 d\Omega. \tag{9}$$

The following optimization problem is formulated, which describes the punch shape and delivers a minimum to the mismatch functional

$$J_f = J(p(f)) \to \min_f,$$

satisfying the equilibrium conditions

$$P(p(f)) = P^*, \;\; M_{x1}(p(f)) = M^*_{x1}, \;\; M_{x2}(p(f)) = M^*_{x2}, \qquad \text{and } p(x) \geq 0.$$

The feasible set of shapes and the optimization problem

$$\Lambda_f = \left\{ \begin{matrix} f : (L_a f))_\Omega \;\; \geq 0, \;\; \iint_\Omega L_a f \, d\Omega = P^*, \\ \iint_\Omega x_2 L_a f \, d\Omega = M^*_{x1}, \;\; \iint_\Omega x_1 L_a f \, d\Omega = M^*_{x2} \end{matrix} \right\},$$

$$J^*_f = J(p(f_*)) \to \min_{f \in \Lambda_f} J(p(f)).$$

the feasible set and the following auxiliary problem are

$$\Lambda_p = \left\{ \begin{array}{l} p:(p)_\Omega \ \geq 0, \quad \iint_\Omega p d\Omega = P^*, \\ \iint_\Omega x_2 p d\Omega = M_{x1}^*, \quad \iint_\Omega x_1 p d\Omega = M_{x2}^* \end{array} \right\}, \quad (10)$$

$$J_p^* = J(p_*) \to \min_{p \in \Lambda_p} J(p) = \min_{p \in \Lambda_p} \iint_\Omega \left[p_g - p\right]^2 d\Omega. \ (11)$$

$$\min_{f \in \Lambda_f} J(p(f)) = \min_{p \in \Lambda_p} J(p), \qquad J_f^* = J_p^*. \qquad (12)$$

When searching for the optimal pressure distribution $p_*(x)$ based on the solution of problem (10), (11), an auxiliary variable $\psi(x)$ is introduced, satisfying the relation

$$p(x) - \psi^2(x) = 0.$$

The solution method that we use here for contact pressure optimization is developed Banichuk and Ivanova.

An extended Lagrange functional is given by

$$J^L = \iint_\Omega \left(\left(p - p_g\right)^2 - \lambda_1 p - \lambda_2 x_1 p - \lambda_3 x_2 p - \chi(p - \psi^2)\right) d\Omega,$$

The extremum conditions $\quad p = p_g + 0.5 \cdot (\lambda_1 + \lambda_2 x_1 + \lambda_3 x_2 + \chi), \ \ \psi\chi = 0, \ x \in \Omega$

Simple Layer Potential Expansion for Annular Ring with Nonsymmetrical Density Distribution

$$\iint_D \frac{\sigma_2(\rho)\cos 2m\theta}{r} ds = 2\pi \cos 2m\theta_0 \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} \right]^2 C_{m,n} U_{2n}(\rho), \quad (5)$$

$$C_{m,n} = \frac{2n(2n-2)...(2n-2m+2)(2n+1)(2n+3)...(2n+2m-1)}{(2n+2)(2n+4)...(2n+2m)(2n-1)(2n-3)...(2n-2m+1)},$$

$$U_{2n}(\rho) = \int_a^{\rho_0} \sigma_2(\rho)\left(\rho/\rho_0\right)^{2n+1} d\rho + \int_{\rho_0}^{b} \sigma_2(\rho)\left(\rho_0/\rho\right)^{2n} d\rho.$$

The Problem Reduction to the Sequence of Problems for Annular Ring

Equations of bounder lines of contact domain: $\Gamma_1:\ \rho = a(1+ f_1(\varepsilon,\theta));$
$\Gamma_2:\ \rho = b(1+ f_2(\varepsilon,\theta)),$ (7)

$$f_1(\varepsilon,\theta) = \sum_{i=1}^{\infty}\varepsilon^i f_{i0}(\theta);\ f_2(\varepsilon,\theta) = \sum_{k=1}^{\infty}\varepsilon^k f_{0k}(\theta),\ \ p(\rho,\theta) = \sum_{i=0}^{\infty} p_i(\rho,\theta)\varepsilon^i \quad \beta_j = \sum_{i=0}^{\infty}\beta_{ji}\varepsilon^i;\ j = 1,2$$

$$\delta = \sum_{i=0}^{\infty}\delta_i\varepsilon^i;\ U = \iint_{\Omega}\frac{p(\rho,\theta)\,\rho\, d\rho\, d\theta}{r} = \sum_{i=0}^{\infty}U_i\varepsilon^i.$$

normal pressure function $p(\rho(R,\varphi,\varepsilon),\varphi) = \sum_{i=0}^{\infty} P_i(R,\varphi)\varepsilon^i$ (8)

Simple layer potential expansion using new variables

$$\iint_{\Omega}\frac{p(\rho(R,\varphi,\varepsilon),\varphi)}{r(R,R_0)}d\Omega = \sum_{k=0}^{\infty}\varepsilon^k\left[\iint_{D}\frac{P_k(R,\varphi)}{r(R,R_0)}ds + \Phi_k(P_0,P_1,\ldots,P_{k-1})\right], \quad (9)$$

$$\Phi_0(P_0) = 0,$$

$$\Phi_1(P_0) = \left(1 - R_0\frac{\partial}{\partial R_0}\right)\iint_{D}\frac{P_0(R,\varphi)}{r(R,R_0)}f_1^{(1)}(R,\varphi)ds + \iint_{D}\frac{P_0(R,\varphi)}{r(R,R_0)}f_1^{(2)}(R,\varphi)ds$$

Considering the doubly connected triangular punch, the equations for the boundary lines (7), using Fourier series contain the functions for $\varepsilon = 0.1406$:

$$f_1(\theta) = -2.6295 + 0.039\cos 2\theta,$$

$$f_2(\theta) = 0.5584\cos\theta + 8.9052\cos 3\theta + 0.4949\cos 4\theta;$$

the functions have the form for ones close to a square:

$$f_1(\theta) = 0.8178 - \cos 4\theta, \;\; f_2(\theta) = 1.2677\cos 8\theta;$$

to a regular hexagon:

$$f_1(\theta) = -0.6504 + 0.3758\cos 6\theta, \;\; f_2(\theta) = 0.7410\cos 12\theta;$$

to a regular octagon:

$$f_1(\theta) = -0.5626 + 0.1556\cos 4\theta, \quad f_2(\theta) = 1.5092\cos 8\theta.$$

to a rectangular: $\varepsilon = 0.3583$,

$$f_1(\theta) = 0{,}5\cos 2\theta + 0{,}06170;\; f_2(\theta) = -0{,}4958\cos 2\theta - \cos 4\theta.$$

The contact under the triangular ring punch

$$p(\rho,\theta) = P_0 + \varepsilon \cdot P_1 + \varepsilon^2 P_2, \qquad (8)$$

where $P_0 = \tilde{Q}\,\sigma_0$, $P_1 = \tilde{Q}\left[5.2589\sigma_0 + (0.0018\sigma_0 - 0.0652\sigma_1)\cos 2\theta_*\right]$,

$$P_2 = \tilde{Q}\left[20.6182\sigma_0 + 0.0060\sigma_1 + (0.0014\sigma_0 - 0.5015\sigma_1)\cos 2\theta_* \right.$$

$$+ (0.2403\sigma_0 + 1.6227\sigma_1 + 0.0851\sigma_2 - 3.1701\sigma_3)\cdot \cos 4\theta_*$$

$$\cdot + (1.6296\sigma_0 + 0.4693\sigma_1)\,\rho_{\Gamma_1}/\rho \cos\theta_* + (5.1564\sigma_0$$

$$-4.9808\sigma_1 + 2.3093\sigma_2)\;\rho_{\Gamma_1}/\rho\cos 3\theta_* \qquad (9)$$

$$+\frac{0.5584}{a+b}\left[\sigma_0'\left(\frac{\rho(a+b)}{\rho_{\Gamma_2}} - \frac{b\rho^2}{\rho_{\Gamma_2}^2} - a\right) - \frac{3b\rho}{\rho_{\Gamma_2}}\sigma_0\right]\cos\theta_*$$

$$\left. + \frac{8.9052\sigma_0'}{a+b}\left(\frac{\rho(a+b)}{\rho_{\Gamma_2}} - \frac{b\rho^2}{\rho_{\Gamma_2}^2} - a\right)\cos 3\theta_*\right].$$

$$\sigma_i = \frac{\pi\gamma}{2}\sum_{k,p=0}^{\infty}\left[\left(\frac{\rho}{\rho_{\Gamma_2}}\right)^{2k}\alpha_{pk}^{(i)} + \left(\frac{\rho_{\Gamma_1}}{\rho}\right)^{2k+3}\beta_{pk}^{(i)}\right]\left(\frac{a}{b}\right)^{p}, \qquad (10)$$

where $\gamma = \sum_{k,p=0}^{\infty}\frac{\pi}{2}\left(\frac{a}{b}\right)^{p}\left(\frac{\alpha_{pk}^{(0)}}{2k+2} - \frac{\beta_{pk}^{(0)}}{2k+1}\left(\frac{a}{b}\right)^{2k+3}\right).$

the sum of discarded terms of the series does not exceed the following value:

$$\left|R_{mn}\right| \le \frac{b}{b-a}\cdot\left\{\frac{\rho_{\Gamma_2}^2}{\rho_{\Gamma_2}^2-\rho^2}\cdot\left[\left(\frac{a}{b}\right)^{m}+\left(\frac{\rho}{\rho_{\Gamma_2}}\right)^{2n}\right]+\frac{\rho_{\Gamma_1}^2}{(\rho^2-\rho_{\Gamma_1}^2)}\cdot\left[\left(\frac{a}{b}\right)^{m}+\left(\frac{\rho_{\Gamma_1}}{\rho}\right)^{2n}\right]\right\}.$$

The load-displacement dependence has the form:

$$\delta = (1-\nu)\cdot Q\cdot(1+\varepsilon+0.9818\cdot\varepsilon^2)/(4b\gamma\cdot G). \qquad (11)$$

ADVANTAGES OF THE METHOD:

- exact formulae in each approximation
- the formulae are acceptable for qualitative analysis
- the formulae are convenient for engineering practice

The numerical analytic solution

B is the coefficient characterizing the deformation properties of the surface roughness of the elastic half-space

$$(1-\alpha)\cdot P(\rho,\theta)+\iint_{\Omega}\frac{P}{2\pi br}\cdot \mathrm{d}S=\frac{\delta-\beta_2\rho_0\cos\theta_0+\beta_1\rho_0\sin\theta_0}{2\pi b}, \qquad (12)$$

Where $P(\rho,\theta)=\left(1-\nu^2\right)/(\pi E)\cdot p(\rho,\theta)$, $B_1/(2\pi)=1-\alpha$, $0<\alpha<1$,

$$B_1=B\pi E/\left(\left(1-\nu^2\right)\cdot b\right).$$

the closer the value of the coefficient α is to one, the smoother the surface we consider. When $\alpha<1$, this equation can be solved by the method of successive approximations.

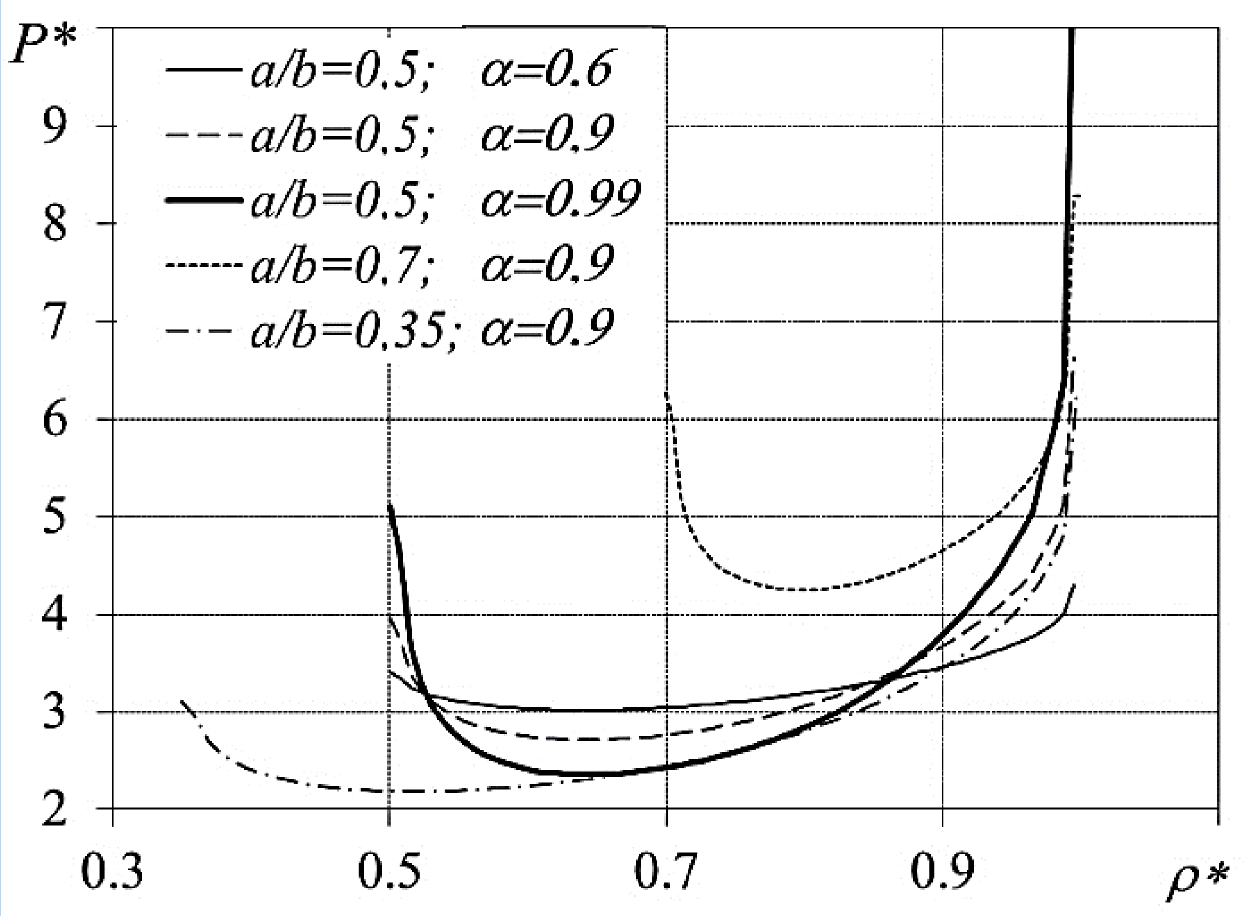
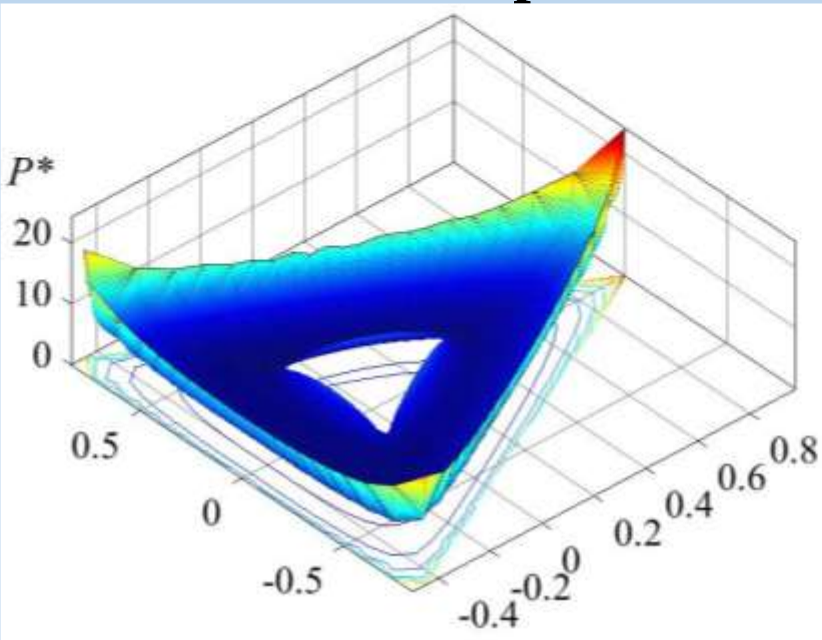
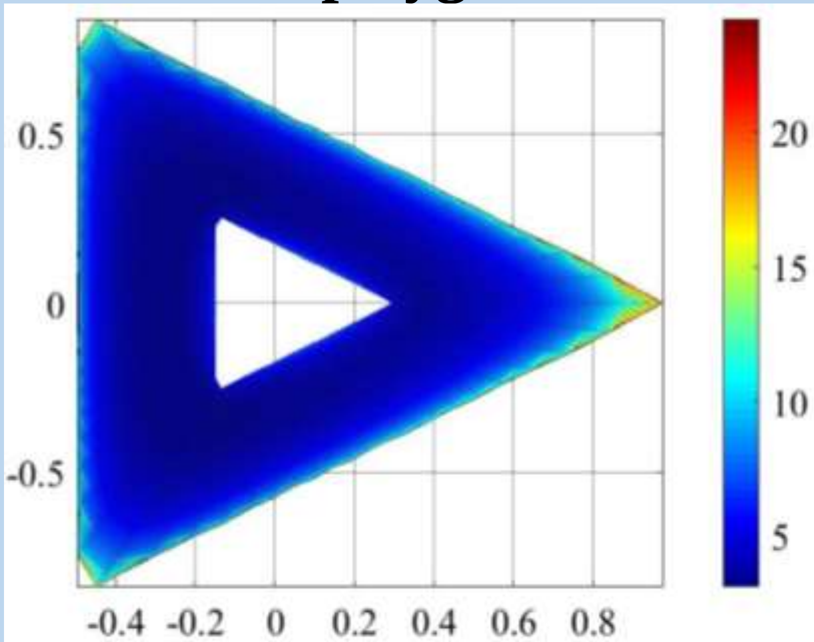
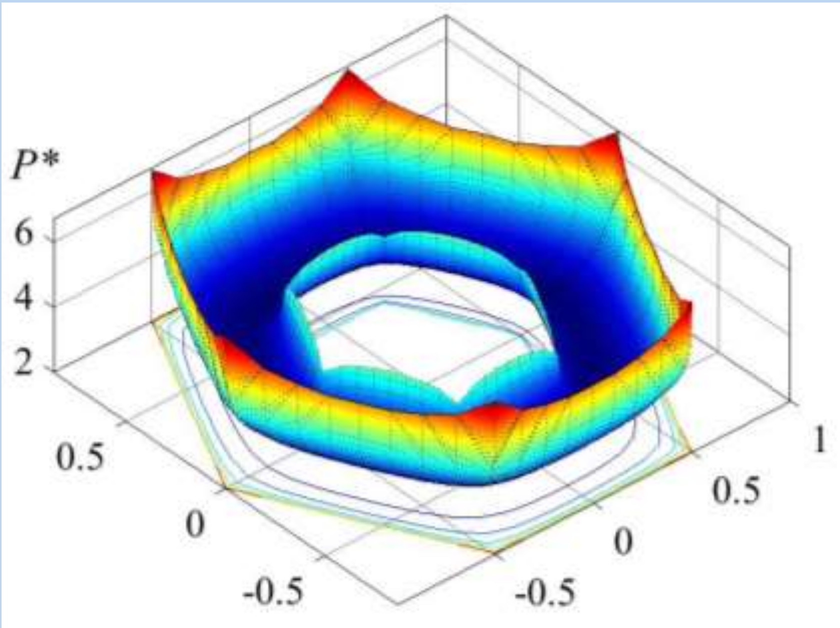


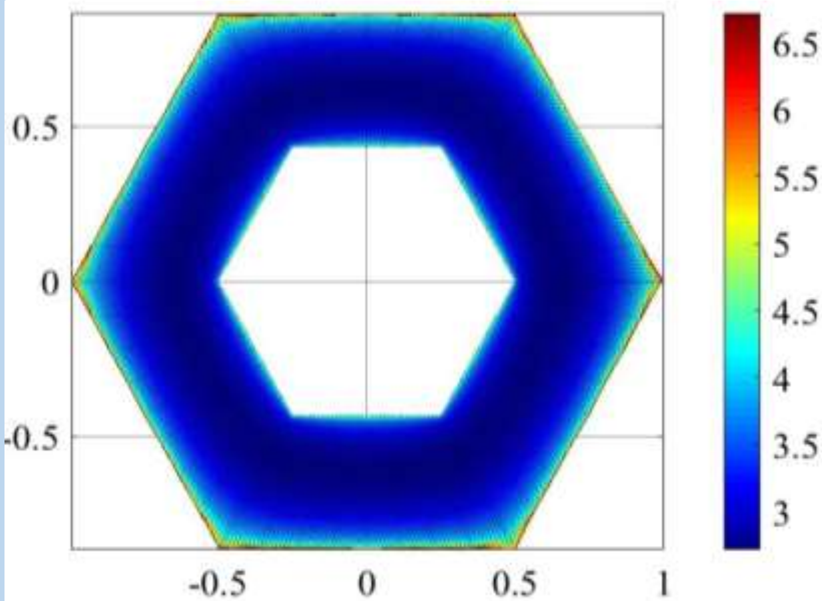
Figure 2. Graphs of pressure distribution under doubly connected hexagonal punches in section $\theta = 0$ with different width a/b and roughness α

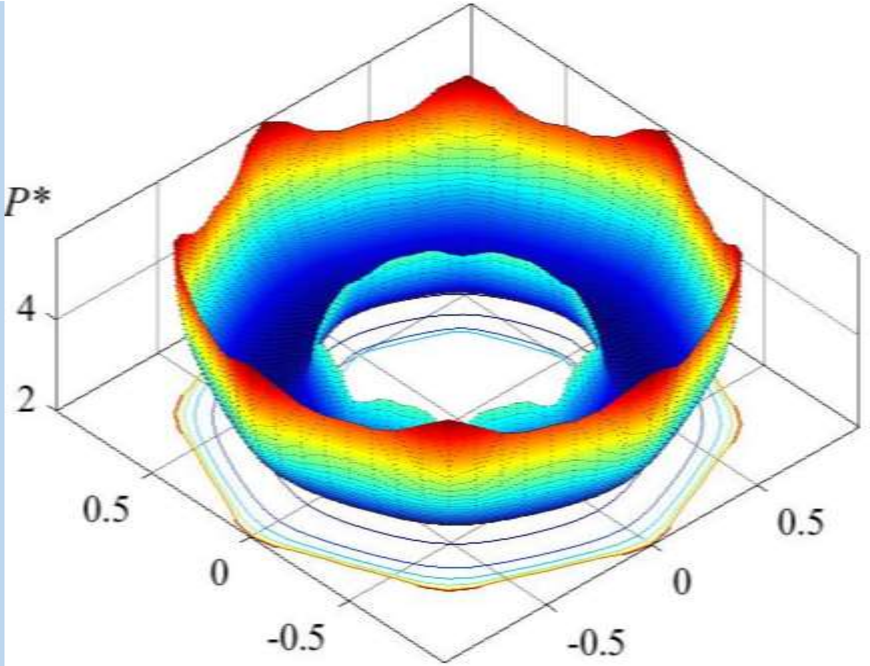
Distribution of contact pressure under the punches with polygonal bases







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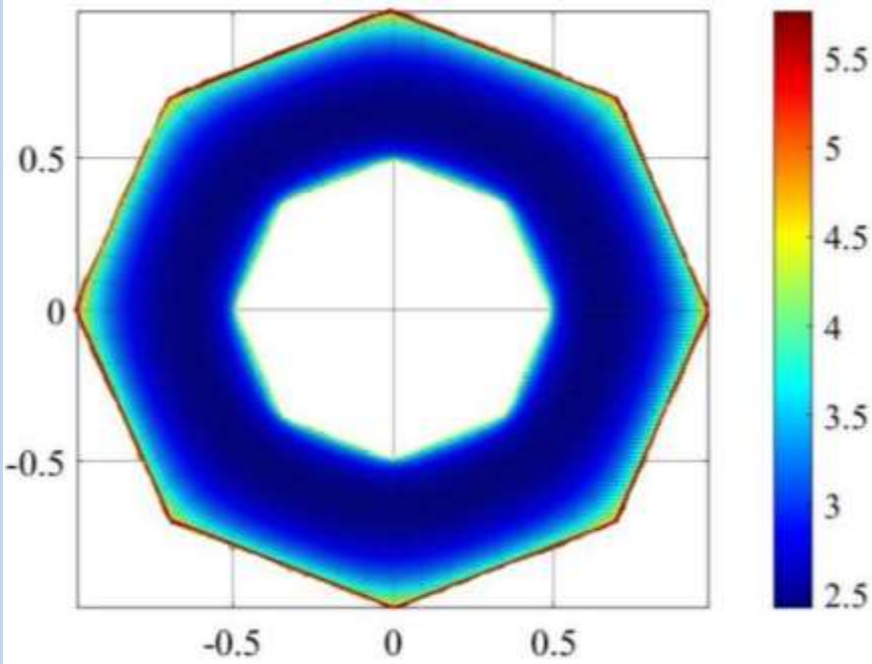
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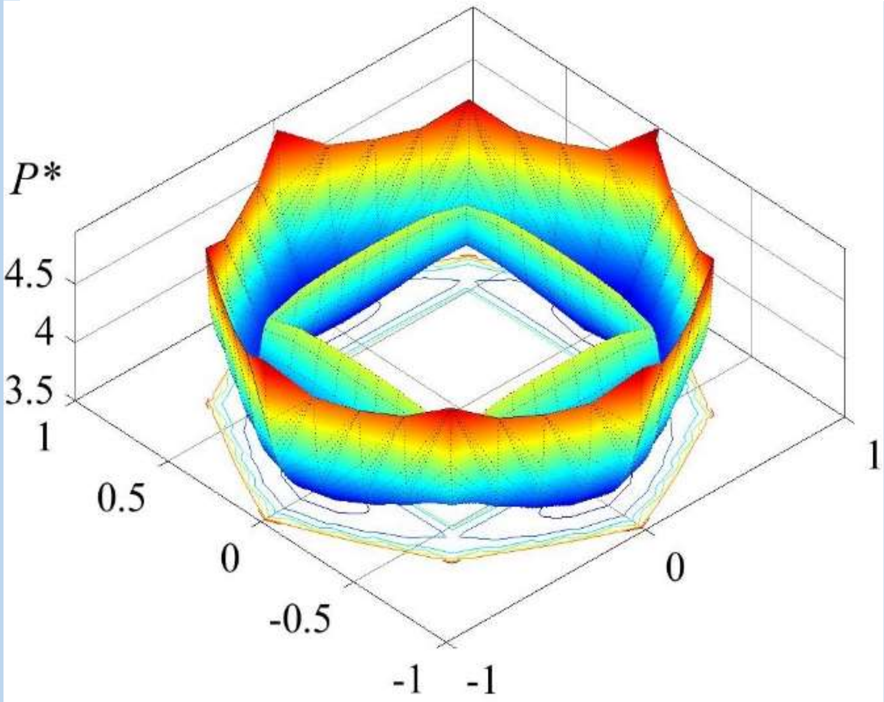
P*
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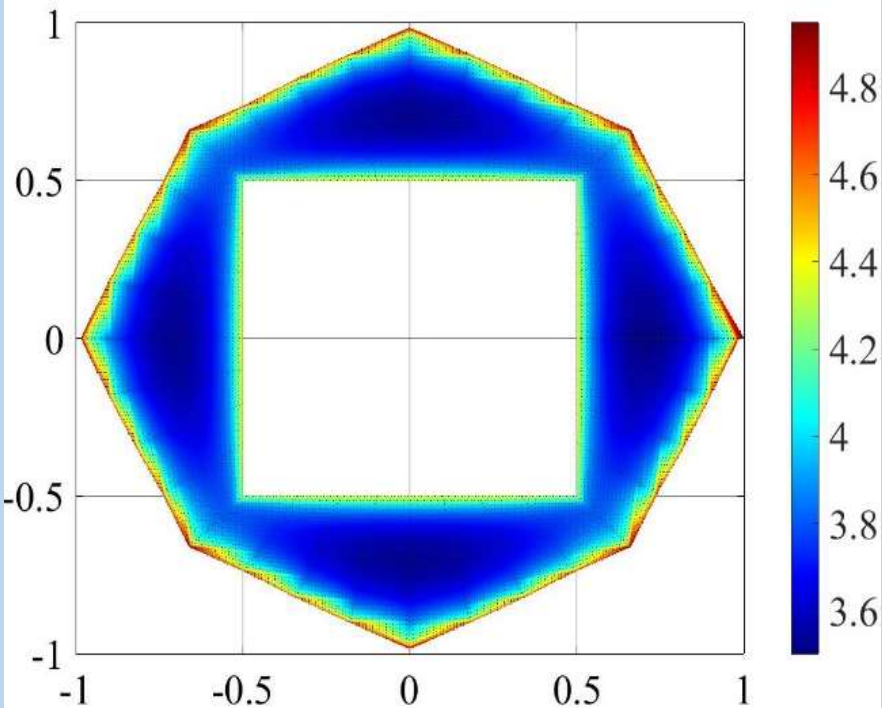
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3.6

Conclusion

The problems of indentation of punches with a flat base bounded by doubly connected close to polygonal contact domains was considered. A study was carried out and an analytical solution was obtained for a three-dimensional contact problem for a non-circular ring domain using the small parameter method, in particular, the problem of indentation of a cylindrical doubly connected punch bounded by similar lines close to triangles in the plan. The method of representing the integrals of the type of the simple layer potential for circular ring domains was used. Obviously, development of an analytical method leads to an increase in the accuracy of the obtained solutions.

We also considered a numerical-analytical method that used the expansion of the potential of a simple layer to reduce two-dimensional integral equations to one-dimensional ones, which could be considered as a solution to problems taking into account the roughness of an elastic half-space. The distribution functions of normal pressures and the value of the punch settling were found.

This approach allows to obtain an approximate solution of the problem in an analytical form and to simplify the scheme of numerical calculations as much as possible. The resulted formulae are convenient for engineering calculations. The result of the obtained analytical solution coincided with the result of the numerically analytical solution up to the fourth decimal place.

Thank you for your attention!

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