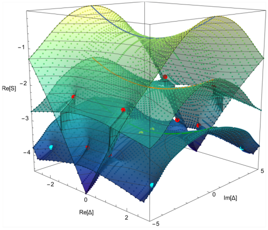
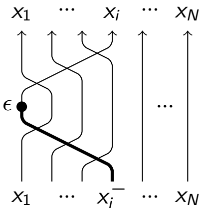
Non-locality and long-range integrability

Rob Klabbers





KMPB-Ukraine Workshop 2025

Research overview

AdS/CFT

- Spectral problem of $\mathrm{AdS}_5\times$ CFT_4:

$$\text{planar } \mathcal{N} = 4 \text{ SYM theory} \leftrightarrow \mathrm{AdS}_5 \times S^5 \text{ superstring theory}$$

- Quantum Spectral Curve for η-deformed $\mathrm{AdS}_5 \times S^5$ super string theory
- Structure of 4-point functions in supergravity limit
- (Conformal) Regge theory to study 'horizontal trajectories'.

Long-range integrability

- Special Schrödinger operators (Calogero-Sutherland) and their deformations (Ruijsenaars-Schneider)
- related PDEs; similar to KdV, Heisenberg ferromagnet and Landau-Lifshitz equation
- and related spin chains: Inozemtsev, Haldane-Shastry and q-deformations

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Long-range integrability

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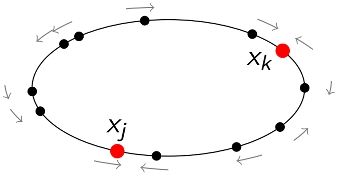
Part I

Long-range integrability

Calogero-Sutherland models

Schrödinger operators

$$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + g(g\pm 1)\sum_{j<k}^{N} V(x_j - x_k)$$



- g coupling constant
- the potentials

$$V(x)\sim\frac{1}{x^2},\quad V(x)\sim\frac{1}{\sin^2 x},\quad V(x)\sim\frac{1}{\sinh^2 x},\quad V(x)\sim\wp(x)$$

  define the Calogero-Sutherland (CS) models.
- are quantum-integrable

Physics (trig)

Condensed matter

- $1/x^2$-interactions are the critical case for long-range order [Thouless, 1969] [Hauke, Tagliacozzo, 2013]
- For $g > 1$, the excitations behave as *anyons*, obeying a generalised Pauli exclusion principle. [Haldane, 1992]

High-energy

- Second quantisation $\subset$ standard chiral CFT [Azuma, Iso, 1994] [Carey, Langmann, 1999]
- Non-polynomial CS-eigenfunctions $\leftrightarrow$ correlators in a CFT with W-symmetry [Estienne,Pasquier, Santachiara, Serban, 2012]
- Bethe/Gauge correspondence relates this model to $\mathcal{N} = 2$ susy gauge theory [Nekrasov, Shatashili, 2009]

Integrability

Trigonometric:

- Cherednik-Dunkl operators [Dunkl, 1989] [Cherednik, 1991]

$$H \sim \frac{1}{2}\sum_{j=1}^{N} d_j^2\,, \quad d_j := \partial_j - \frac{g}{2}\sum_{k\neq j}^{N} \cot \tfrac{\pi}{N}(x_j - x_k)(1 - s_{jk}) - \ldots$$

- s_{jk} permutes coordinates $x_j \leftrightarrow x_k$
- diagonalise d_j instead $\Rightarrow$ Jack-polynomials
- d_j form a degenerate affine Hecke algebra
- and induce a Yangian action by affine Schur-Weyl duality [Drinfeld, 1986]

What about elliptic?

- elliptic d_j [Buchstaber, Felder, Veselov, 1994]
- KZB-equations $\leftrightarrow$ eigenfunctions [Felder, Varchenko, 1995]
- Second quantisation sits in *finite-temperature* QFT of anyons [Langmann, 2001]
- but no algebraic structure is known

Why elliptic?

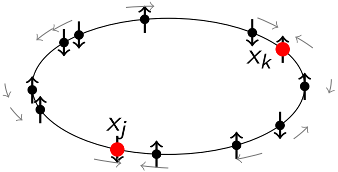
- novel quantum algebras, e.g. elliptic quantum groups, (elliptic) quantum toroidal algebras
- would make the anyonic QFT tractable
- forms the *intermediate* interaction-range, i.e. connects to the short-range regime

Why is elliptic intermediate range?

Quantum many-body system (QMBS)

Extend to spin-CS:

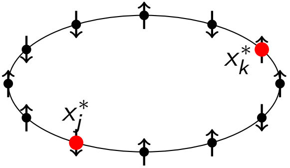
$$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{j<k}^{N} g(g \pm P_{jk})V(x_j - x_k)$$



- P_{jk} exchanges spins $v_j \otimes v_k \mapsto v_k \otimes v_j \in \mathbb{C}^r \otimes \mathbb{C}^r$
- then freeze '$T \to 0$' [Polychronakos, 1993], or rather take special classical limit of a hybrid system [Lyashik, Reshetikhin, Sechin, 2024] [Chalykh, 2024]

Spin chain

$$H = \sum_{j<k}^{N} V(x_j^* - x_k^*)(1 - P_{jk})$$



Landscape

$\frac{\text{QMBS}}{\text{spin chain}}$	*nearest-neighbour*	*elliptic*	*trig*

$$\frac{?}{\text{Heisenberg XXX}} \longleftarrow \frac{\text{ell. CS}}{\text{Inozemtsev}} \longrightarrow \frac{\text{trig. CS}}{\text{Haldane-Shastry}}$$

[Inozemtsev, 1990-1997] [Haldane (1988), Shastry (1988)]

potential $\longrightarrow$

interaction-length

Two paradigms

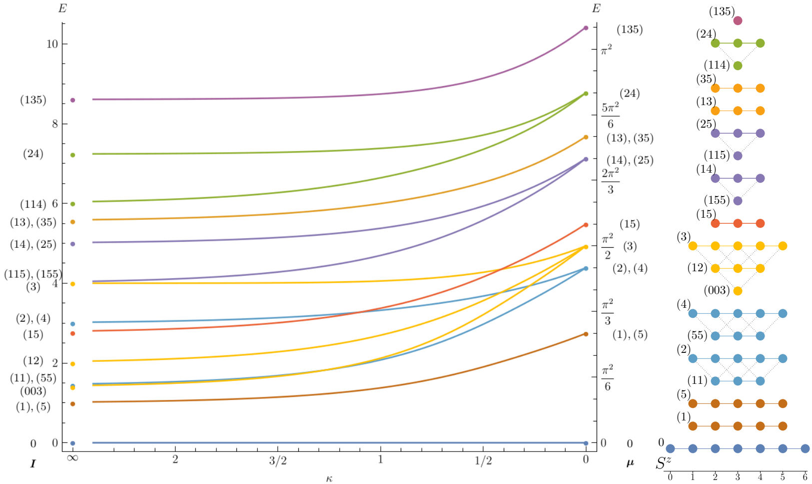
	$\sum_{j=1}^{N}(1 - P_{j,j+1})$	$\sum_{j<k} \frac{1 - P_{jk}}{\sin^2 \frac{\pi}{N}(j-k)}$
	Heisenberg	**Haldane-Shastry**
Range	Short-range	Long-range $(1/r^2)$
Integrable	(Algebraic) Bethe ansatz	by freezing a QMBS
Yangian $Y(sl_2)$	does not commute with H	does commute with H
Spectrum	solving Bethe equations	consists of irreps of $Y(sl_2)$

Connecting them would

- allow us to study the effect of interaction range
- unify two paradigms of quantum integrability

Elliptic spin chain

- 'Extended' Bethe ansatz [Inozemtsev 90-97, **RK**, Lamers 2020]
- Thermodynamic Bethe ansatz [**RK**, 2016]
- Some control over spectrum:



Landscape II

$\frac{\text{QMBS}}{\text{spin chain}}$	*nearest-neighbour*	*elliptic*	*trig*
elliptic	$\frac{?}{\text{Heisenberg XYZ}}$		
trig	$\frac{?}{\text{Heisenberg XXZ}}$	$\frac{\textit{ell.RS}}{q-\text{Inozemtsev}}$ [**RK** Lamers, 2024]	$\frac{\text{trig. RS}}{q-\text{Haldane-Shastry}}$ [Uglov, 97] [Lamers, 2018]
rational	$\frac{?}{\text{Heisenberg XXX}}$	$\frac{\text{ell. CS}}{\text{Inozemtsev}}$ [Inozemtsev, 1990-1997]	$\frac{\text{trig. CS}}{\text{Haldane-Shastry}}$ [Haldane (1988), Shastry (1988)]

Spin interaction $\longrightarrow$

potential $\longrightarrow$

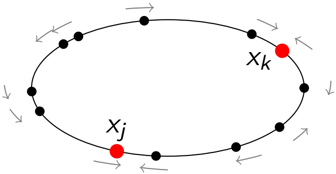
interaction-length

RS = Ruijsenaars-Schneider

Elliptic Ruijsenaars [Ruijsenaars, 1986]

Finite-difference operators

$$D_n = \sum_{I \subset \{1,\ldots,N\} |I|=n} A_I(\boldsymbol{x}) \prod_{i \in I} \Gamma_i$$



$$\Gamma_i = \mathrm{e}^{-\mathrm{i}\hbar\epsilon\partial_i} : x_i \mapsto x_i - \mathrm{i}\hbar\epsilon\,, \quad A_I(\boldsymbol{x}) = \prod_{j \notin I \in i}^{N} \frac{\theta(x_i - x_j + \eta)}{\theta(x_i - x_j)}$$

- has $[D_n, D_m] = 0$
- $D_1 + D_N^{-1} D_{N-1} \to H_{\mathrm{CS}}^{\mathrm{ell}}$

Elliptic spin Ruijsenaars

Choose an elliptic R-matrix (for some rank $r \geq 2$)

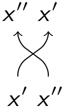
face Felder's dynamical R-matrix [Felder, 1994], satisfies dynamical Yang-Baxter equation

vertex Baxter-Belavin's symmetric R-matrix [Baxter, 1972, Belavin 1981] or its susy version satisfies quantum Yang-Baxter equation

then for each $P_i^{\text{tot}} = s_{i,i+1}\check{R}_{i,i+1}(x_i - x_{i+1})$ forms an S_N-rep on

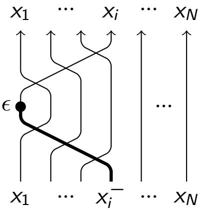
$$\mathcal{H} = \text{Fun}(\boldsymbol{x}) \otimes V^{\otimes N}\,, \quad V \cong \mathbb{C}^r$$

Elliptic spin Ruijsenaars

Represent $\check{R}(x) =$ 

then the first Hamiltonian is

$$\widetilde{D}_1 = \sum_{i=1}^{N} A_i(\boldsymbol{x}) \times$$



$$= \Gamma_i = \mathrm{e}^{-\mathrm{i}\hbar\epsilon\partial_i}\,, \qquad x_i^- \equiv x_i - \mathrm{i}\,\hbar\,\epsilon$$

$$= \sum_{i=1}^{N} A_i(\boldsymbol{x})\; \check{R}_{i-1,i}(x_{i,i-1}) \cdots \check{R}_{12}(x_{i1})\; \Gamma_i\; \check{R}_{12}(x_{1i}) \cdots \check{R}_{i-1,i}(x_{i-1,i})$$

Properties

Face [**RK**,Lamers,2024]

- $[\tilde{D}_n, \tilde{D}_m] = 0$
- fits in the landscape
- Prime candidate for algebraic structure: elliptic quantum toroidal algebra [Konno, Oshima, 2023-2024]

Vertex [Matushko, Zotov, 2022-2023]

- $[\tilde{D}_n, \tilde{D}_m] = 0$
- generates a twisted landscape
- algebraic structure (?)

contact *elliptic* *trigonometric*

$$\begin{tikzcd} H^{\text{bt}}_{\mathrm{XX}} \arrow[d] & H_{MZ} \arrow[l] \arrow[r] \arrow[d] & H_{MZ} \arrow[d] \\ H^{\text{tw}}_{\mathrm{XX}} & H_{HW} \arrow[l] \arrow[r] & H_{FK} \end{tikzcd}$$

[**RK**, Lamers, 2024] [Fukui, Kawakami, 1992]

[Hikami Wadati, 1993]

Discussion

Closing in on mechanisms of elliptic integrability

- elliptic Hecke algebra and Macdonald theory
- Is the whole vertex landscape a twisted version of the face landscape?
- Do other R-matrices also work?

More broadly:

- dynamical R-matrices
- quantum algebras
- elliptic functions
- higher genus theta functions
- boundary Yang-Baxter equation
- orthogonal polynomials/functions
- Hecke algebras

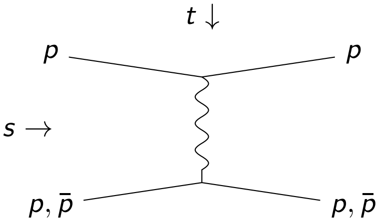
- Spin chains
- Quantum many-body systems
- CFT (e.g. Louiville, WZW)
- Fractional/para- statistics
- (Deformed) sigma models

Part II

Conformal Regge theory

Regge theory

Proton-(anti-)proton scattering:



- When $s \gg t$, Regge theory predicts that then [Regge, 1959]

$$A(t) \sim s^{\alpha(t)}$$

- $\alpha(t)$ is a Regge trajectory, $\alpha(0)$ is the *intercept*
- t-channel exchange of a resonance with spin $\alpha(t) \sim S(t)$
- leading contribution is the Pomeron
- but in $A_{pp} - A_{p\overline{p}}$ that disappears: the Odderon becomes dominant
- Characterising it is hard, ongoing discussion

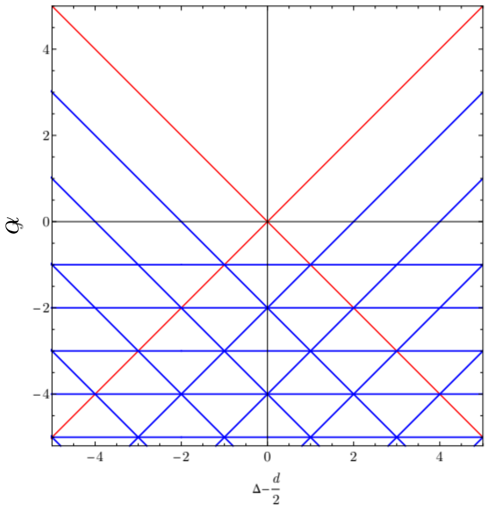
Conformal Regge theory

- $A \to \langle \mathcal{O}_1 \ldots \mathcal{O}_N \rangle$.
- $t \to \Delta$
- Regge limit $\to$ a certain light-cone limit [Costa, Goncalves, Penedones, 2012]
- OPE becomes

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \sim 1 - f_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}} f_{\mathcal{O}_3 \mathcal{O}_4 \mathcal{O}} s^{\alpha(0)+\alpha' \Delta} + \ldots$$

  where s measures boost between operators.
- but the details are very subtle, requiring analytic continuation of CFT data (Lorentzian inversion formula [Caron-Huot, 2016]). But here the math actually works!
- Regge trajectories $\to$ Light-ray operators: [Kravchuk, Simons-Duffin 2018] [Balitsky, Kazakov, Sobko, 2013]
  - non-local for general spin α
  - for $\alpha \in \mathbb{N}$ null-line integrals over local operators
- Should give rise to horizontal trajectories: $\alpha(\Delta) = \alpha_{\mathrm{H}} < 0$. [Caron-Huot, Kologlu, Kravchuk, Meltzer, Simons-Duffin, 2023]

Chew-Frautschi plot (in free Lorentzian CFT)



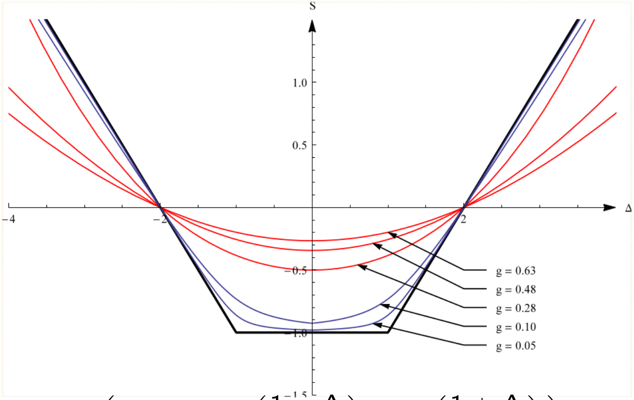
- Local operators
- Shadow operators
- Light-ray operators
- Horizontal trajectories

[Caron-Huot, Kologlu, Kravchuk, Meltzer, Simons-Duffin, 2023]

Pomeron in (planar) $\mathcal{N} = 4$ SYM

Trajectory with local operators $\mathcal{O}_S = \mathrm{Tr}(ZD^S Z) + \text{perm}$
Cartan charges: $(J_1, J_2, J_3|\Delta, S, S_2) = (2, 0, 0|2 + S + \gamma, S, 0)$

[Gromov, Levkovich-Maslyuk, Sizov, Valatka, 2014] [Alfimov, Gromov, Kazakov, 2015]



$$\alpha_{\mathbb{P}}(\Delta) = 1 + 4\left(2\psi(1) - \psi\left(\frac{1-\Delta}{2}\right) - \psi\left(\frac{1+\Delta}{2}\right)\right) g^2 + \mathcal{O}\left(g^4\right)$$

Questions

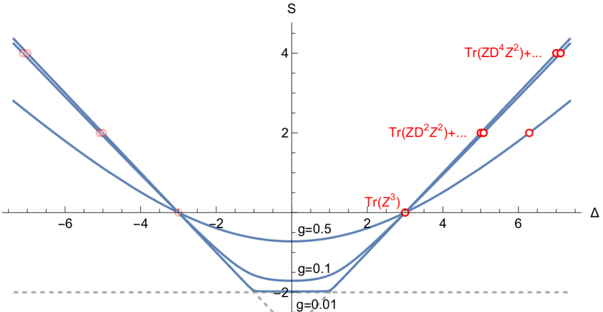
- What do horizontal trajectories really look like in interacting CFT?
- What happens in case of degeneracies?
- How do they influence the intercept of Regge trajectories?
- What is the Odderon trajectory in $\mathcal{N} = 4$ SYM?
- How to find its intercept?

We studied twist-3, $\Delta_0 - S = 3$, [**RK**, Szecsenyi, Preti, PRL 2024], where the theory is more generic.

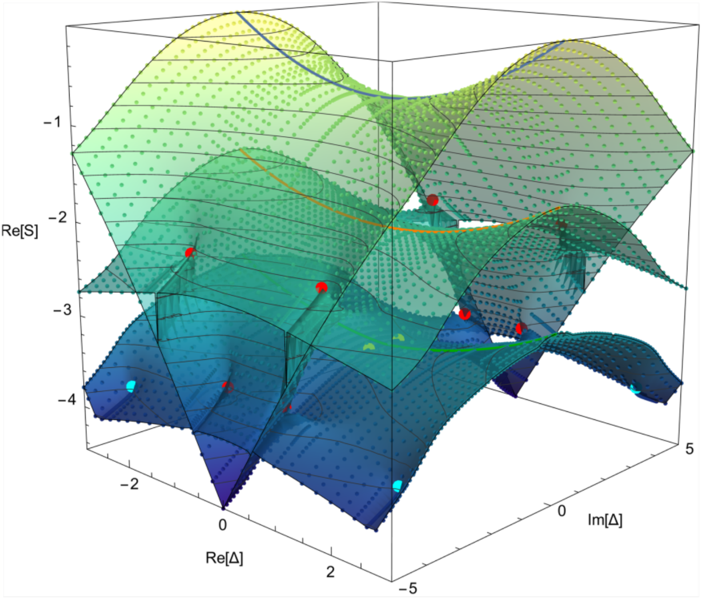
Continuation using Quantum Spectral Curve (QSC)

Spectrum of planar $\mathcal{N} = 4$ SYM (as a function of coupling g) is captured by QSC:

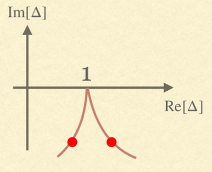
- coupled difference equations whose boundary conditions encode the Cartan charges $(J_1, J_2, J_3|\Delta, S, S_2)$ that label states [Gromov, Kazakov, Leurent, Volin, 2013]
- Ultimately just a machine: input five charges, output the sixth.*
- In principle flexible and non-perturbative, but working the machine is very technical



Riemann surface (at $g = 1/2$)

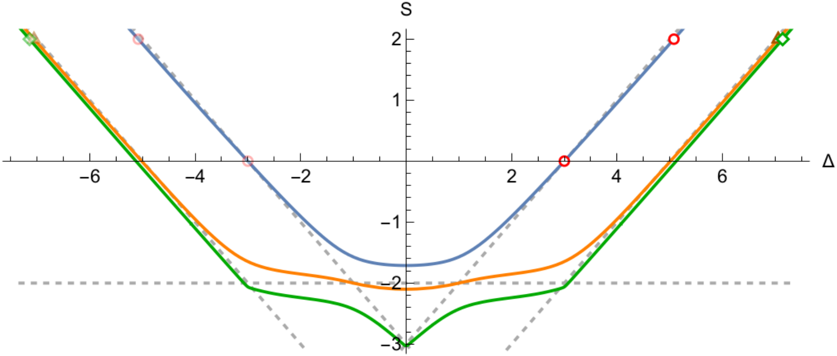


Branch points:



The three trajectories

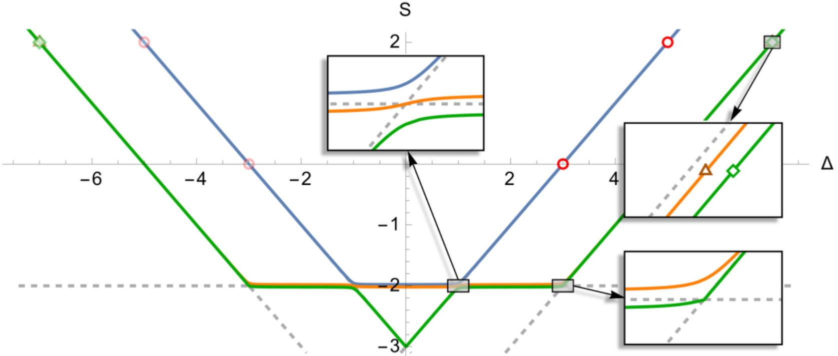
The real slice of our Riemann surface ($g = 1/10$):



We count:

- the trajectory containing the twist-3 local operators
- *two* horizontal trajectories, degenerate at zero coupling
- More branch points, more mixing (of *four* trajectories), now with twist-5 operators

At $g \approx 0$



Discussion

How is the intercept affected?

$$S(\Delta) = -2 + \sum_{n=1}^{\infty} I_n(\Delta) g^n \quad \text{including odd } n$$

and after *a lot* of hard work, we extract from QSC that

$$S(0) = -2 \pm 2g + \mathcal{O}\left(g^2\right)$$

It depends **linearly** on g! First $\mathcal{N} = 4$ SYM observable with this behaviour.

Observation: $g \to -g$ brings us from one trajectory to another!

$$S(\Delta, g) = S(\Delta, -g) \text{ when } |\Delta| < 1$$

Lessons:

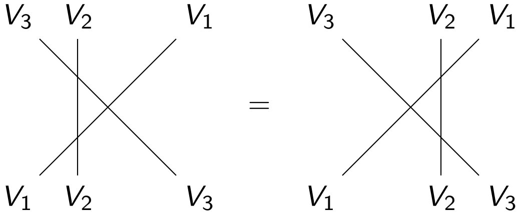
- Observables in $\mathcal{N} = 4$ can depend on *odd powers* of g, here caused by degenerate horizontal trajectories
- It shows how non-local operators form an *important and understudied* part of CFTs

Appendix

Dynamical R-matrix [Felder,1994]

Quantum Yang-Baxter equation:

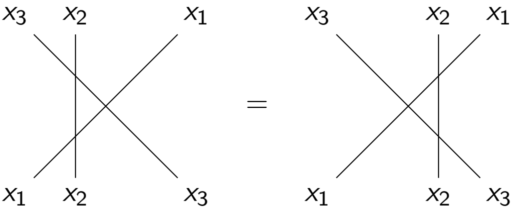
$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}$$



Dynamical R-matrix [Felder,1994]

Quantum Yang-Baxter equation:

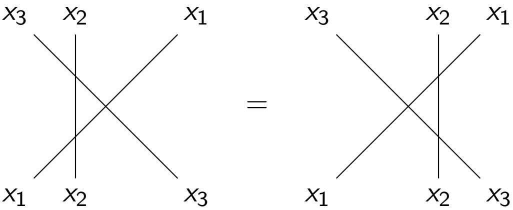
$$R_{23}(x_2-x_3)R_{13}(x_1-x_3)R_{12}(x_1-x_2) = R_{12}(x_1-x_2)R_{13}(x_1-x_3)R_{23}(x_2-x_3)$$



Dynamical R-matrix [Felder,1994]

Quantum Yang-Baxter equation: with $x_{ij} = x_i - x_j$

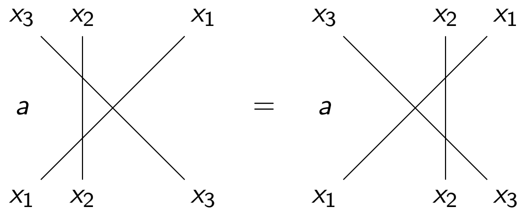
$$R_{23}(x_{23})R_{13}(x_{13})R_{12}(x_{12}) = R_{12}(x_{12})R_{13}(x_{13})R_{23}(x_{23})$$



Dynamical R-matrix [Felder,1994]

Dynamical Yang-Baxter equation:

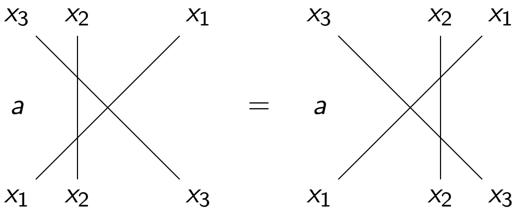
$$R_{23}(x_{23}; a)R_{13}(x_{13}; a - \sigma_2^z)R_{12}(x_{12}; a) = R_{12}(x_{12}; a - \sigma_3^z)R_{13}(x_{13}; a)R_{23}(x_{23}; a - \sigma_1^z)$$



Dynamical R-matrix [Felder,1994]

Braidlike dynamical Yang-Baxter equation: with $\check{R}_{ij} = P_{ij}R_{ij}$

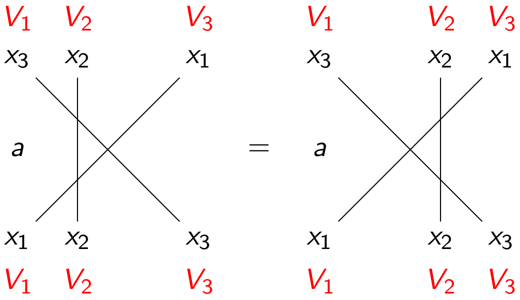
$$P_{23}\check{R}_{23}(x_{23}; a)P_{13}\check{R}_{13}(x_{13}; a-\sigma_2^z)P_{12}\check{R}_{12}(x_{12}; a)$$
$$=$$
$$P_{12}\check{R}_{12}(x_{12}; a-\sigma_3^z)P_{13}\check{R}_{13}(x_{13}; a)P_{23}\check{R}_{23}(x_{23}; a-\sigma_1^z)$$



Dynamical R-matrix [Felder,1994]

Braidlike dynamical Yang-Baxter equation:

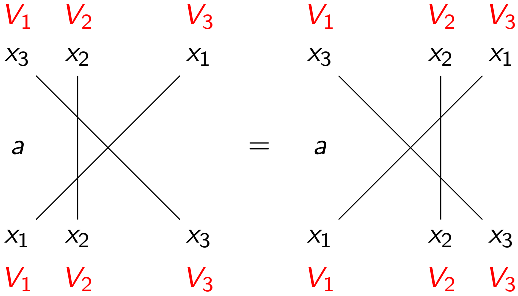
$$\check{R}_{12}(x_{23};a)\check{R}_{23}(x_{13};a-\sigma_1^z)\check{R}_{12}(x_{12};a)$$
$$=$$
$$\check{R}_{23}(x_{12};a-\sigma_1^z)\check{R}_{12}(x_{13};a)\check{R}_{23}(x_{23};a-\sigma_1^z)$$



Dynamical R-matrix [Felder,1994]

Braidlike dynamical Yang-Baxter equation:
with $P_{i,i+1}(x) = \check{R}_{i,i+1}(x; a - \sigma_1^z - \ldots - \sigma_{i-1}^z)$

$$P_{12}(x_{23})P_{23}(x_{13})P_{12}(x_{12}) = P_{23}(x_{12})P_{12}(x_{13})P_{23}(x_{23})$$



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with $P_{i,i+1}(x) = \check{R}_{i,i+1}(x; a - \sigma_1^z - \ldots - \sigma_{i-1}^z)$

$$P_{12}(x_{23})P_{23}(x_{13})P_{12}(x_{12}) = P_{23}(x_{12})P_{12}(x_{13})P_{23}(x_{23})$$

- unitarity: $P_{i,i+1}(x)P_{i,i+1}(-x) = \mathbb{I}$
- commutativity: $[P_{i,i+1}(x), P_{j,j+1}(y)] = 0$ if $|i-j| > 1$

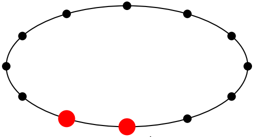
Action on spin basis: $s_i \in \{\uparrow, \downarrow\}$

$$\begin{aligned} P_{i,i+1}(x)\,|s_1, \ldots, s_N\rangle = |s_1, \ldots, s_{i-1}\rangle & \\ \otimes\, \check{R}(x, a - \textstyle\sum_{k=1}^{i-1} s_k)\,|s_i, s_{i+1}\rangle & \\ \otimes\, |s_{i+1}, \ldots, s_N\rangle\,, & \end{aligned}$$

A (familiar) starting point

Heisenberg XXX spin chain of N spin-$1/2$ particles:

$$H_{\mathrm{XXX}} = -\frac{1}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z\right)$$



- *Ubiquitous* in physics: from phase transitions of magnetic systems to anomalous dimensions in $\mathcal{N}=4$ SYM
- *Simple*: nearest-neighbour and isotropic ($SU(2)$-symmetric)
- *(quantum) integrable* due to extra mathematical structure (Bethe ansatz) $\rightarrow$ exactly solvable

Ubiquitous **because** it is simple

A (familiar) starting point

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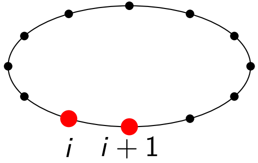
$$= \sum_{i=1}^{N}\left(\frac{1}{2} - P_{i,i+1}\right)$$



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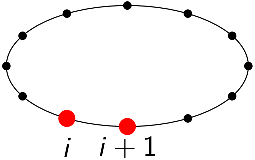
$$\sim \sum_{i=1}^{N}(1 - P_{i,i+1}) = \sum_{i=1}^{N} E_{i,i+1}$$



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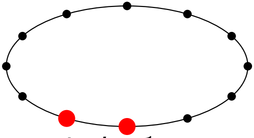
Deforming the starting point

Heisenberg XXZ spin chain of N spin-$1/2$ particles:

$$H_{\mathrm{XXZ}} = \sum_{i=1}^{N} E_{i,i+1}(q)$$



Deformed spin interaction:

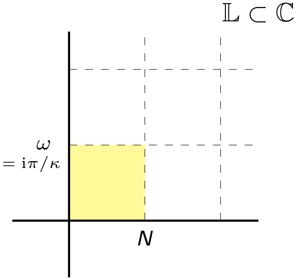
$$-2E_{jk}(q) = \sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \frac{q+q^{-1}}{2} \sigma_j^z \sigma_k^z - 1$$

- nearest neighbour, but only partially isotropic (S^z-symmetric)
- still (quantum) integrable
- Its mathematical structure is captured by the *quantum group* $U_q(\mathfrak{sl}_2)$
- from this one can see that for XXX the structure is the Yangian $Y(\mathfrak{sl}_2)$.

Inozemtsev's elliptic spin chain [Inozemtsev, 1990]

Most general solution:

$$H_{\mathrm{Ino}} = \frac{\sinh^2 \kappa}{\kappa^2} \sum_{j<k}^{N} \left(\wp(j-k) + \eta_2/\omega\right) E_{jk}$$



$$\wp(z) = \frac{1}{z^2} + \sum_{j,k\in\mathbb{Z}}{}' \left(\frac{1}{(z + jN + k\omega)^2} - \frac{1}{(jN + k\omega)^2} \right) \sim \sum_{k\in\mathbb{Z}} \frac{\kappa^2}{\sinh^2 \kappa(z + kN)}$$

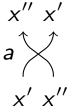
defined on $\mathbb{L}$ with lattice periods $(N, \omega = \mathrm{i}\pi/\kappa)$, where $\kappa > 0$.

$\eta_2 = 2\zeta(\omega/2)$ with $\wp(z) = -\zeta'(z)$.

Elliptic R-matrix

We need *elliptic* R-matrices: we use Felder's dynamical R-matrix [Felder, 1994]: with $x = x' - x''$ spectral, η anisotropy and a dynamical parameter

$$\check{R}(x,a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g(x,\eta\, a) & f(x,\eta\, a) & 0 \\ 0 & f(x,-\eta\, a) & g(x,-\eta\, a) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$



$$f(x,a) = \frac{\theta(\eta + a)\,\theta(x)}{\theta(a)\,\theta(x+\eta)}\,, \quad g(x,a) = \frac{\theta(x+a)\,\theta(\eta)}{\theta(a)\,\theta(x+\eta)}$$

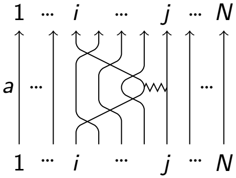
with θ the normalised odd Jacobi theta function with quasiperiods $-\omega$ and N.

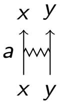
Note: our θ is periodicized sinh, not sin.

The deformed Inozemtsev spin chain, [RK Lamers, 2023]

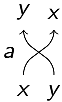
and its right chiral partner looks like

$$H^{\mathrm{R}} = \sum_{i<j}^{N} V_\eta(i-j) \times$$





$= E_{i,i+1}(x-y)$

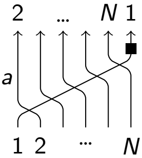


$= P_{i,i+1}(x-y)$

$$V_\eta(x) = -\frac{\rho(x+\eta)-\rho(x-\eta)}{\theta(2\eta)} \sim \frac{1}{\mathrm{sn}(x+\eta)\mathrm{sn}(x-\eta)} + \mathrm{cst}$$

Properties

- indeed limits to
  - q-Haldane-Shastry ($\kappa \to 0$ and $a \to -\mathrm{i}\infty$)
  - Inozemtsev ($\eta \to 0$ and $a \to -\mathrm{i}\infty$)
  - has a nearest-neighbour limit ($\kappa \to \infty$)
- is partially isotropic: $\left[H^{\mathrm{L/R}}, S^z\right] = 0$.
- is integrable: it belongs to a hierarchy of commuting hamiltonians and $[H^{\mathrm{L}}, H^{\mathrm{R}}] = 0$.
- is not periodic, but *twisted*: $\left[H^{\mathrm{L/R}}, G\right] = 0$

$G = a$ 

$= K_N^{-1} P_{N-1,N}(1-N) \cdots P_{12}(-1)$

  and $K_N = k_N(a - \sigma_1^z - \ldots \sigma_{N-1}^z)$ with diagonal twist $k(a) = \mathrm{e}^{\kappa \eta a \sigma^z}$. yields notion of *quasi-momentum* and deformed *magnons*.
- can be constructed for any rank

Nearest-neighbour limit towards XXZ

As $\kappa \to \infty$

$$\frac{\sinh^2 \kappa}{\kappa^2} V_\eta(x) \to \delta_{|x \bmod N|,1}$$

so only nearest neighbours survive.

Send $\kappa \to \infty$ and $\eta \to 0$ keeping $\kappa\eta = -\mathrm{i}\pi\bar{\gamma}$ fixed

$$\frac{\sinh^2 \kappa}{\kappa^2} H^{\mathrm{L/R}} \to H_{\mathrm{XXZ}} = \sum_{i=1}^{N-1} e^{\mathrm{H}}_{i,i+1} + G^{\mathrm{H}} e^{\mathrm{H}}_{1,2} G^{\mathrm{H}-1}$$

with

$$e^{\mathrm{H}}(a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\sin[\pi\bar{\gamma}(a-1)]}{\sin[\pi\bar{\gamma}a]} & -\frac{\sin[\pi\bar{\gamma}(a+1)]}{\sin[\pi\bar{\gamma}a]} & 0 \\ 0 & -\frac{\sin[\pi\bar{\gamma}(a-1)]}{\sin[\pi\bar{\gamma}a]} & \frac{\sin[\pi\bar{\gamma}(a+1)]}{\sin[\pi\bar{\gamma}a]} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $G^{\mathrm{H}} = \lim_{\kappa\to\infty} G$ built from $R^{\mathrm{H}} = 1 - \mathrm{e}^{\mathrm{i}\pi\bar{\gamma}} e^{\mathrm{H}}$.

A tale of two Temperley-Liebs

$$H_{\mathrm{XXZ}} = \sum_{i=1}^{N-1} e^{\mathrm{H}}_{i,i+1} + G^{\mathrm{H}} e^{\mathrm{H}}_{1,2} G^{\mathrm{H}-1}$$

- new, dynamical version of braid-translated XXZ spin chain [Saleur Martin, 1993]
- The $e^{\mathrm{H}}_{i,i+1}$ form a dynamical Temperley-Lieb (TL) represention, and with G^{H} this becomes affine TL:

$$\text{with } e_0 = G^{\mathrm{H}} e^{\mathrm{H}}_{1,2} G^{\mathrm{H}-1} : \quad e_i^2 = 2\cos(\pi\bar{\gamma}) e_i, \quad e_i e_{i\pm 1} e_i = e_i,$$
$$G e_i G^{-1} = e_{i-1}, \quad e_{N-1} = G^2 e_1 \dots e_{N-1}$$

On the Haldane-Shastry side (as $\kappa \to 0$ and $\eta = N\gamma$)

$$E_{i,i+1}(x) \to e_i^{\mathrm{HS}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-\pi \mathrm{i}\gamma} & -\mathrm{e}^{\pi \mathrm{i}\gamma} & 0 \\ 0 & -\mathrm{e}^{-\pi \mathrm{i}\gamma} & \mathrm{e}^{\pi \mathrm{i}\gamma} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

form another TL representation and connect to $U_q(\mathfrak{sl}_2)$.

What is the algebra of the $E_{i,i+1}(x)$?