Compact holonomy G_2 manifolds need not be formal

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Motivation

General idea: Imposing **PDEs** on compact manifolds often results in topological constraints.

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Folklore Conjecture: Compact manifolds with **special and exceptional holonomy are formal.**

Holonomy & Formality

Berger's Theorem (1955): Holonomy groups of simply connected, irreducible, compact, non-symmetric n-dimensional Riemannian manifolds are:

$SO(n)$		$\mathbb{R}$
$U(m)$ Kähler	$n=2m$	$\mathbb{C}$
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The notion of formality comes from **rational homotopy theory,** which uses the category of **commutative differential graded algebras (cDGAs)** to encode rational homotopy types.

- A **cDGA** is a commutative graded algebra endowed with a degree-1 differential that satisfies the Leibniz rule. **Examples** are $(\Omega(M), d)$ and $(H^*(M), d = 0)$.

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A manifold is **formal** if we can connect $(\Omega(M), d)$ and $(H^*(M), d = 0)$ through a chain of cDGAs (A_j, d_j) and quasi-isomorphisms of the following form:

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Sullivan's foundational results imply, in particular, that if a simply connected compact manifold is formal, $\pi_n(X) \otimes \mathbb{Q}$ can be calculated from $H^*(M, \mathbb{Q})$

G_2 manifolds (1)

- **Local model:** The standard vector cross product $\times\colon \Lambda^2\mathbb{R}^7 \to \mathbb{R}^7$
  determines a 3-form by $\varphi_0(u, v, w) = \langle u \times v, w\rangle$.

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

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- Let (M,g) be a 7-dimensional oriented Riemannian manifold; **a G_2 structure** is $\varphi \in \Omega^3(M)$ whose expression in a local orthonormal oriented frame is that of φ_0.
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- A G_2 structure (M, g, φ) is torsion-free when $\nabla\varphi = 0$.
  This implies $\mathrm{Hol}(g) \subset G_2$. In this case, if M is compact $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite.

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- Some properties of torsion-free G_2 manifolds:
    - They are Ricci-flat.
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- Compact manifolds with holonomy G_2 are difficult to construct. The four available methods rely on the understanding of Calabi-Yau/hyperkähler manifolds and require difficult analysis:

  - **Joyce (1996)**: Resolutions of orbifolds T^7/Γ, with $\Gamma \leq \mathrm{G}_2$ finite.
  - **Kovalev (2003)**: Twisted connected sum.
  - **Corti, Haskins, Nordström, Pacini (2015)**: Extension of the twisted connected sum.
  - **Joyce, Karigiannis (2021)**: Resolution of torsion-free G_2 orbifolds of the form $M/\mathbb{Z}_2$.

Main ideas of the construction

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- Each connected component of the exceptional divisor of the resolution yield to a new cohomology class. **The value of a particular triple Massey product involving those classes depends on how the different connected components of the singular locus are linked**.
- The action of Γ on T^7 was constructed to achieve a non-trivial configuration that yields a non-zero triple Massey product.

Thank you!