Mathematical structures in Feynman integrals of planar $\mathcal{N} = 4$ SYM theory

KMPB-Ukraine Workshop

Anne Spiering (Humboldt Universität zu Berlin)

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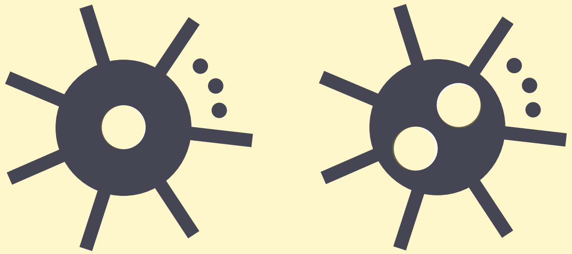
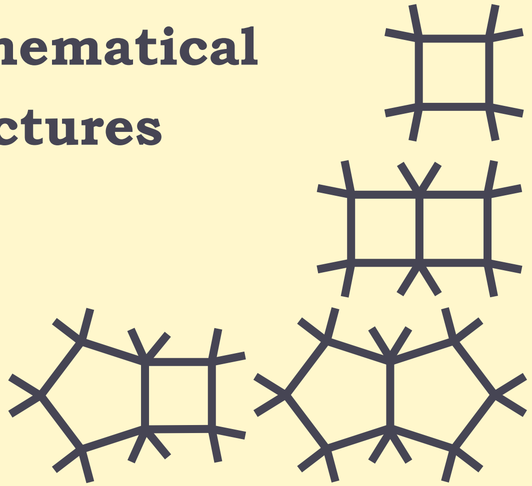
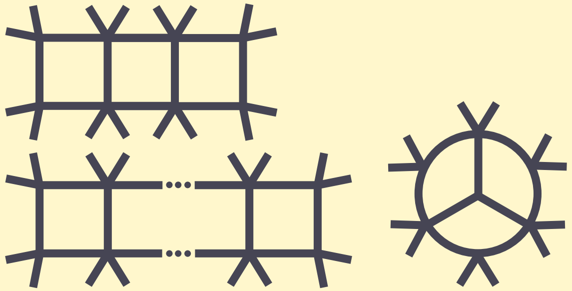
1. **Feynman integrals in planar $\mathcal{N} = 4$ SYM**



2. **Mathematical structures**



3. **Conclusions & open questions**



1. Feynman integrals in planar $\mathcal{N} = 4$ SYM theory

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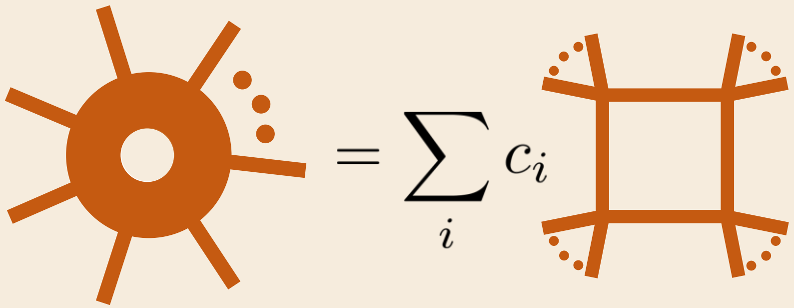
One-loop amplitudes reduce to linear combinations of **boxes**, with coefficients fixed by unitarity techniques.



[Bern, Dixon, Dunbar, Kosower '94]

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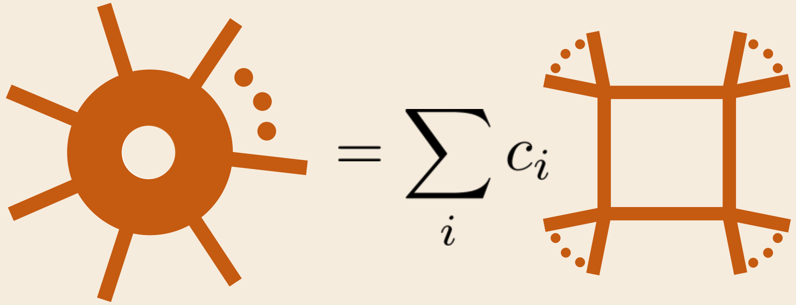
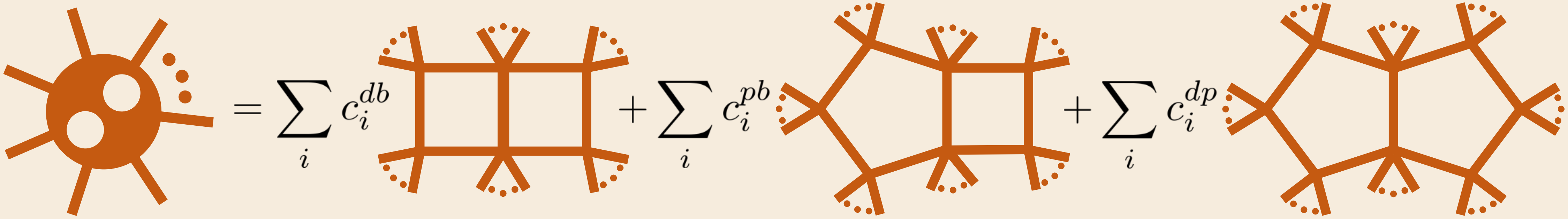


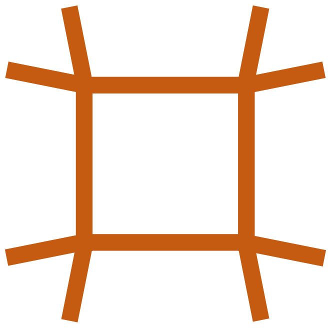
[Bern, Dixon, Dunbar, Kosower '94]

Two-loop amplitudes reduce to linear combinations of **double boxes, pentaboxes and double pentagons**, again the coefficients can be fixed via unitarity methods.



[Bourjaily, Trnka '17]

2. Mathematical structures in planar Feynman integrals



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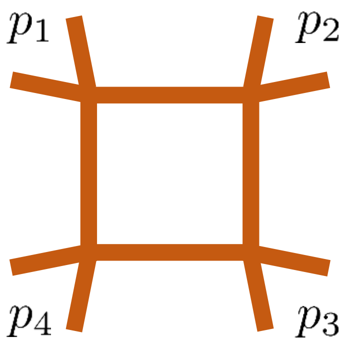

- Momenta $p_i \in \mathbb{R}^4$

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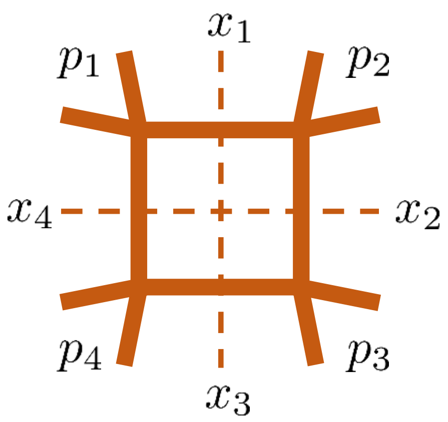

- Momenta $p_i \in \mathbb{R}^4$ and **dual momenta** $x_i \in \mathbb{R}^4$, with $p_i = x_{i+1} - x_i =: x_{i+1,i}$

  2 degrees of freedom parametrised by $z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \ (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

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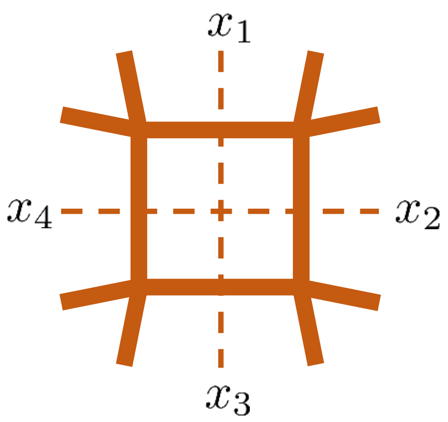

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- **Single-box integral**

$$I[\boxtimes] = \frac{1}{z-\bar{z}}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right)$$

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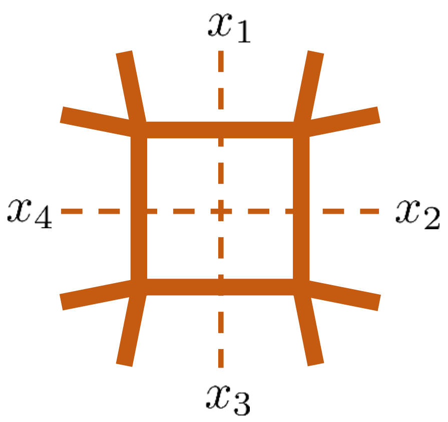

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- **Polylogarithms** form a graded Hopf algebra $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$

$$\mathcal{A}_0 = \mathbb{Q}, \; \mathcal{A}_1 = \{\log(x), \pi, \ldots\}, \; \mathcal{A}_2 = \{\mathrm{Li}_2(x), \log(x)\log(y), \zeta_2, \ldots\},$$
$$\mathcal{A}_3 = \{\mathrm{Li}_3(x), \log^3(x), \zeta_3, \ldots\}, \ldots$$

  with coproduct structure and associated **symbol** $\mathcal{S} : \mathcal{A}_n \to \mathcal{A}_1 \otimes \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_1$
  [Goncharov ‘05] [Brown ‘11]

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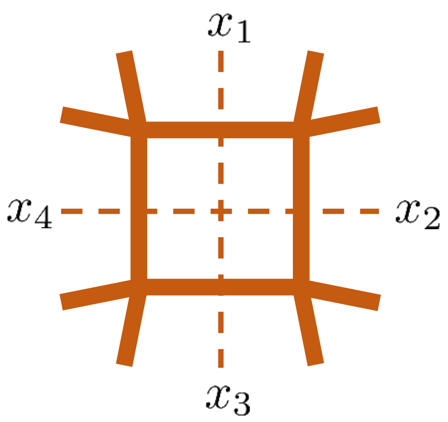

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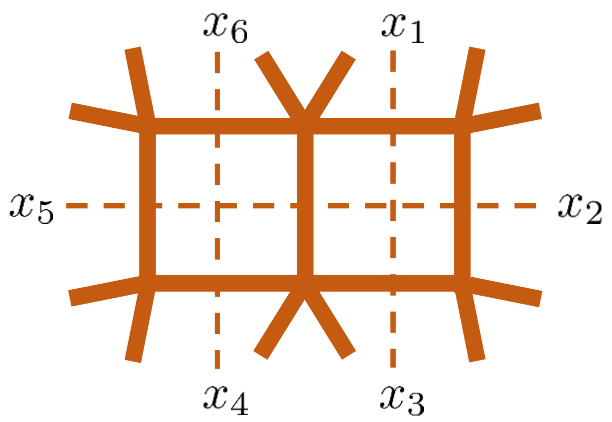
  [Goncharov '05] [Brown '11]

- **Box symbol**: $\mathcal{S}\left(I[\boxtimes]\right) = \frac{1}{z-\bar{z}}\left(\log(1-z)(1-\bar{z}) \otimes \log\frac{z}{\bar{z}} - \log z\bar{z} \otimes \log\frac{1-z}{1-\bar{z}}\right)$

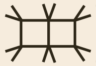
2. Mathematical structures in planar Feynman integrals

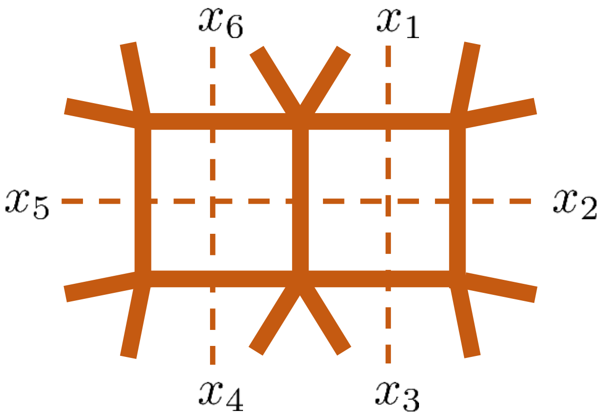


2. Mathematical structures in planar Feynman integrals

- **9 degrees of freedom**, among them $u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$

- Double-box integral is related to the **6d hexagon integral**:
  [Paulos, Spradlin, Volovich '12]

$I\Big[$  $\Big] = \int_{-\infty}^{u} \frac{dx}{y(x)}\, I\Big[$  $\Big]^{(6\mathrm{d})}\Big|_{u\to x}$



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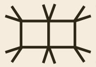
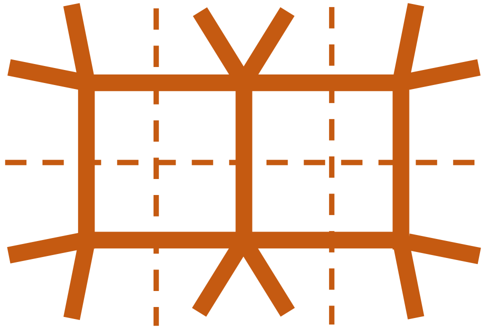
$I\Big[$  $\Big] = \int_{-\infty}^{u} \frac{dx}{y(x)}\, I\Big[$  $\Big]^{(6\mathrm{d})}\Big|_{u\to x}$

weight-3 polylogarithm
[Nandan, Paulos, Spradlin, Volovich '13]
[Ren, Spradlin, Vergu, Volovich '23]



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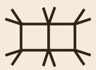
$I\Big[$  $\Big] = \int_{-\infty}^{u} \frac{dx}{y(x)}\, I\Big[$  $\Big]^{(6\mathrm{d})}\Big|_{u\to x}$

$y^2(x)$ cubic in x
double box is **elliptic**!
[Caron-Huot, Larsen '12]

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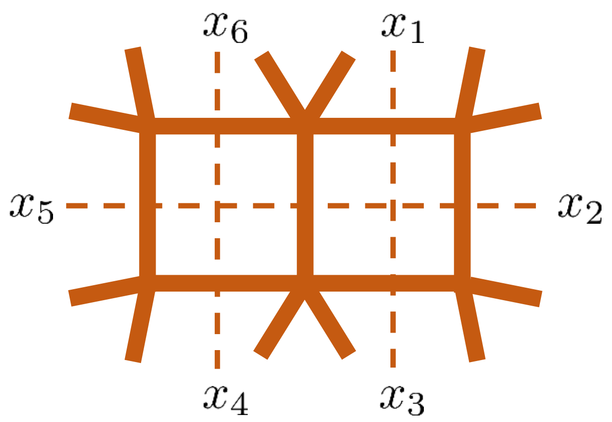

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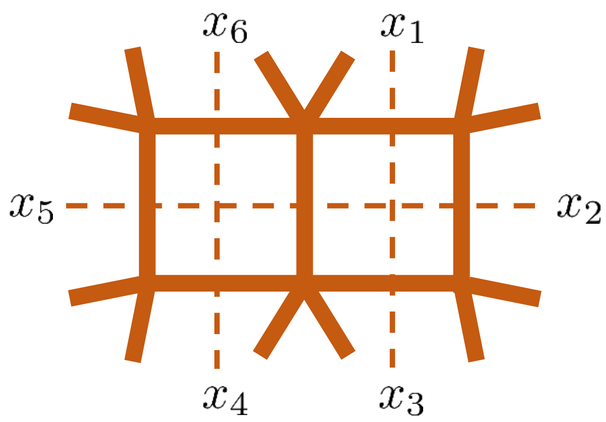
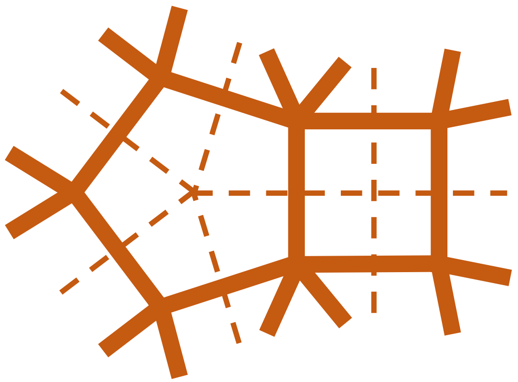


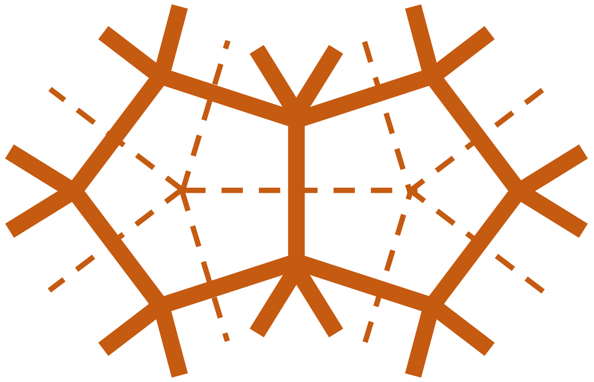
- From an **elliptic symbol bootstrap**:

$$\Delta_{2,2}\Big(I[\text{double box}]\Big) = \sum_{\substack{i,j=1\\ i<j}}^{6} I[\text{box}]_{\{i,j\}^c} \otimes \frac{2\pi i}{\omega_1} \int^{u} \frac{dx}{y(x)} \log R_{ij}(x) - (u \to \infty)$$

[Morales, **AS**, Wilhelm, Yang, Zhang '22]

2. Mathematical structures in planar Feynman integrals





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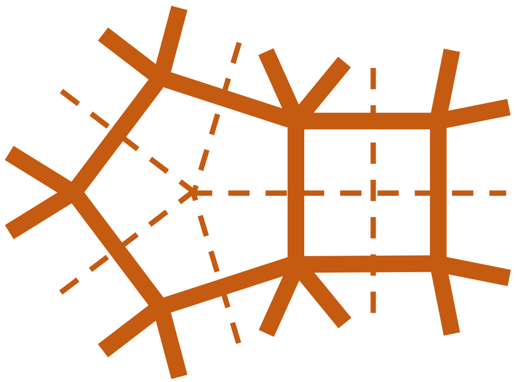
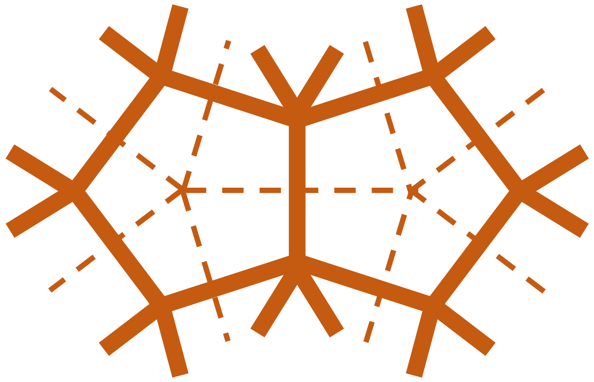


- **13 degrees of freedom**



- **17 degrees of freedom**

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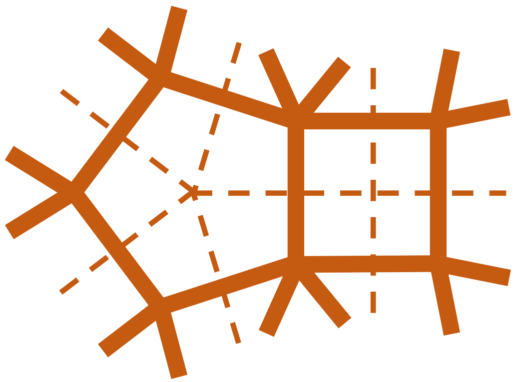

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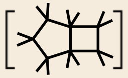
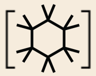
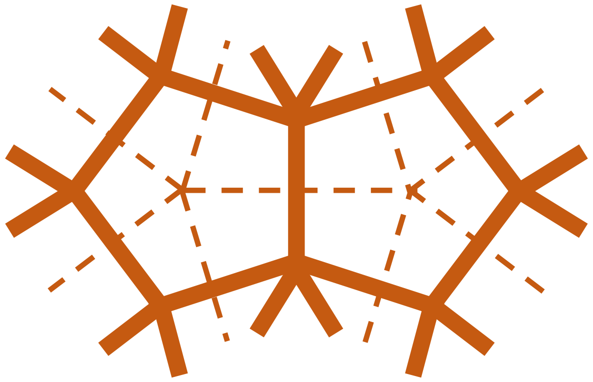
$I[$  $] = \sum_{\sigma=\pm} C^{\sigma} \sum_{i=1}^{7} \int_{-\infty}^{u} \frac{dx\, y_i(r^{\sigma})}{(x - r^{\sigma}) y_i(x)}\, I[$  $]^{(6\mathrm{d})}_{\{i\}^c} \Big|_{u \to x}$



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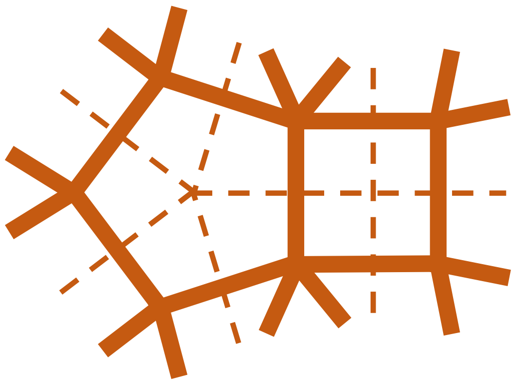

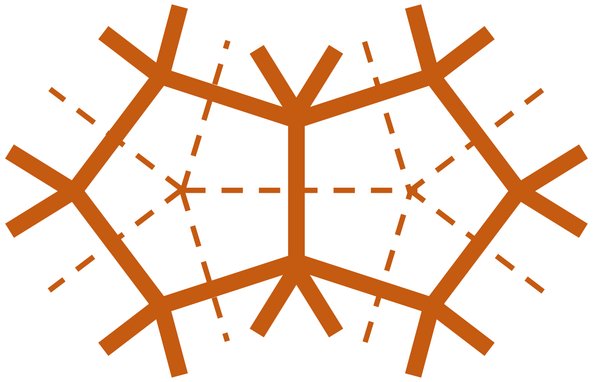

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$$I\left[\;\;\right] = \sum_{\sigma=\pm} C^{\sigma} \sum_{i=1}^{7} \int_{-\infty}^{u} \frac{dx\, y_i(r^{\sigma})}{(x - r^{\sigma})\, y_i(x)}\, I\left[\;\;\right]^{(6\mathrm{d})}_{\{i\}^c}\Big|_{u \to x}$$

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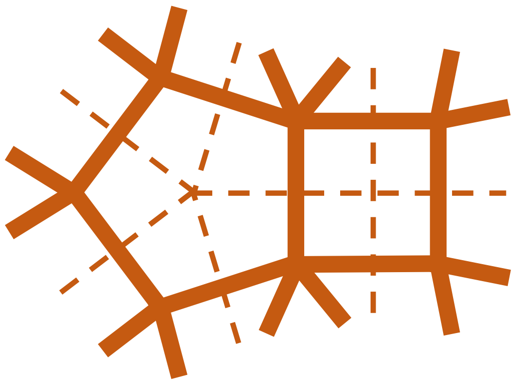

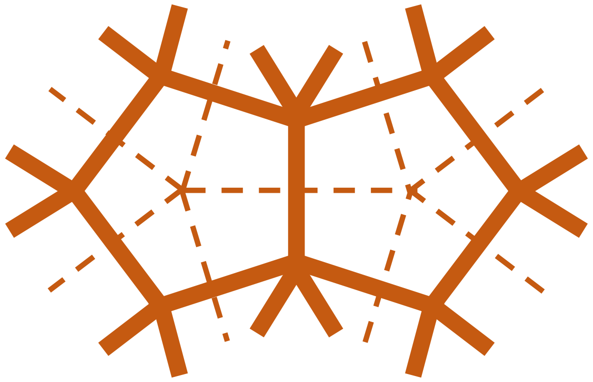

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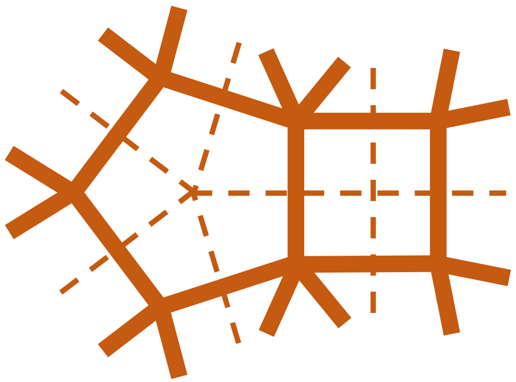

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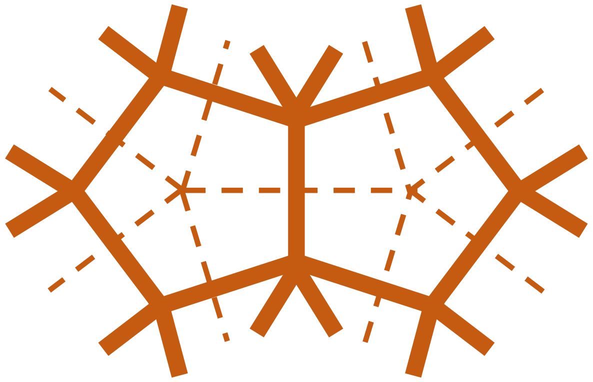

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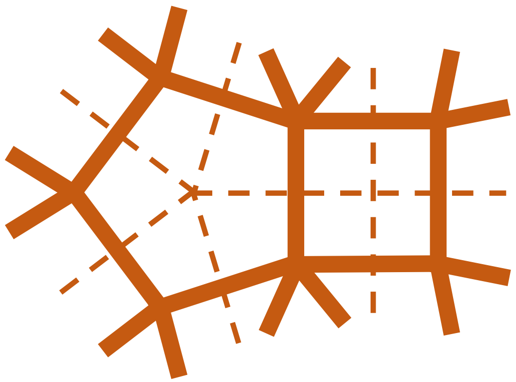

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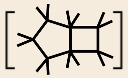
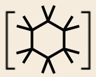
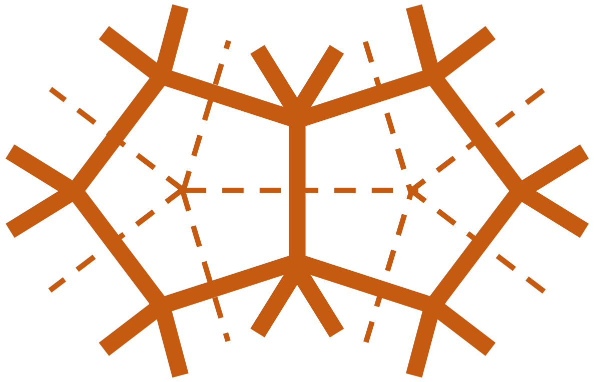
$I\Big[$  $\Big] = \sum_{\sigma=\pm} C^{\sigma} \sum_{i=1}^{7} \int_{-\infty}^{u} \frac{dx\, y_i(r^{\sigma})}{(x-r^{\sigma})\, y_i(x)}\, I\Big[$  $\Big]^{(6\mathrm{d})}_{\{i\}^c}\Big|_{u\to x}$

- **4 elliptic curves**



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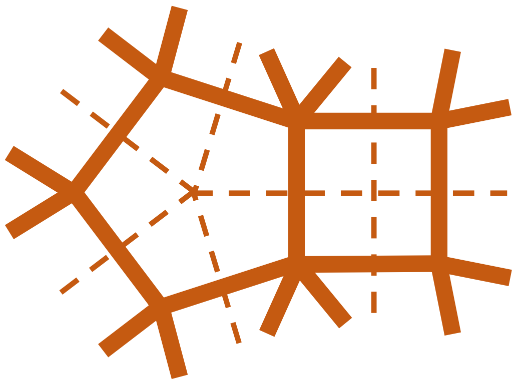

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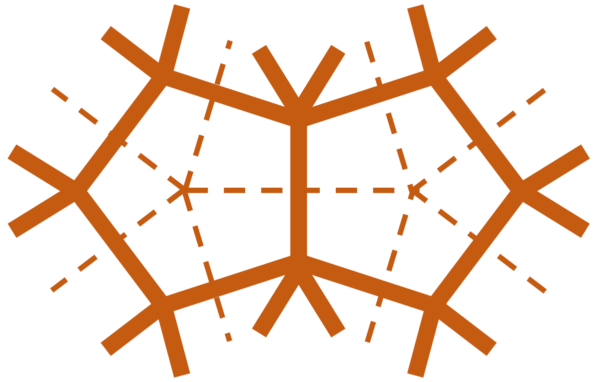

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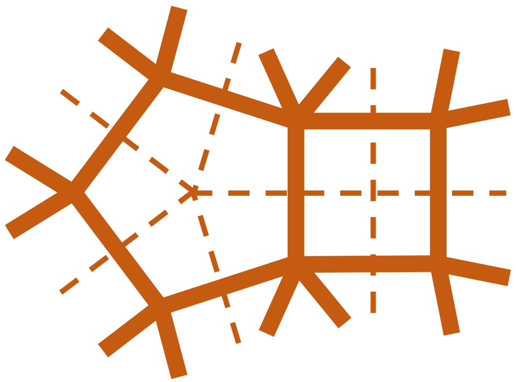

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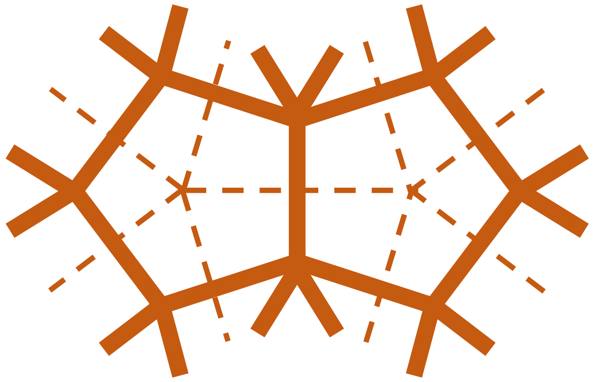

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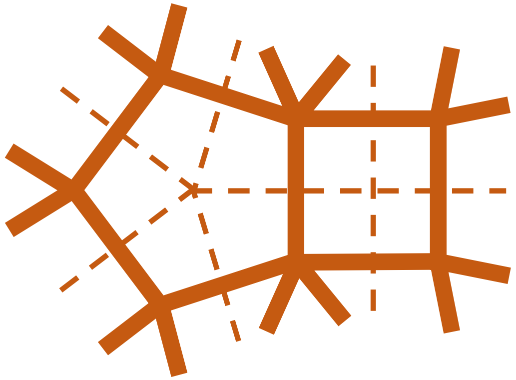
$$I[\,\cdot\,] \cong \sum_{\substack{i,j=1\\ i<j}}^{8} \int_{-\infty}^{u} dx \begin{pmatrix}\text{elliptic/algebraic}\\ \text{integration kernel}\end{pmatrix}_{ij} I[\,\cdot\,]^{(6\mathrm{d})}_{\{i,j\}^c}\Big|_{u\to x}$$

- **16 elliptic curves**

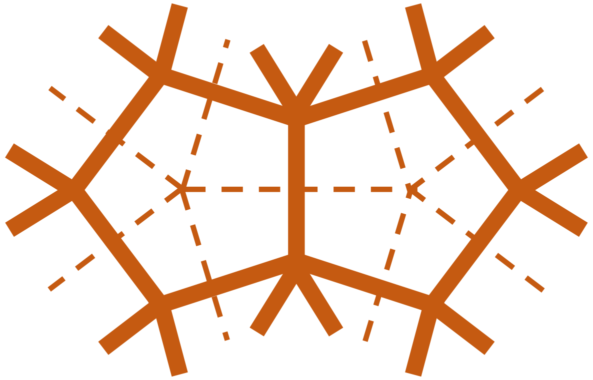
2. Mathematical structures in planar Feynman integrals



- **13 degrees of freedom**
- Relation to **hexagon** integral: [AS, Wilhelm, Zhang '24]

$$I[\dots] = \sum_{\sigma=\pm} C^{\sigma} \sum_{i=1}^{7} \int_{-\infty}^{u} \frac{dx\, y_i(r^{\sigma})}{(x-r^{\sigma})y_i(x)}\, I[\dots]^{(6d)}_{\{i\}^c}\Big|_{u\to x}$$

- **4 elliptic curves**



- **17 degrees of freedom**
- Relation to **hexagon** integral: [AS, Wilhelm, Zhang '24]

$$I[\dots] \cong \sum_{\substack{i,j=1\\ i<j}}^{8} \int_{-\infty}^{u} dx \begin{pmatrix}\text{elliptic/algebraic}\\ \text{integration kernel}\end{pmatrix}_{ij} I[\dots]^{(6d)}_{\{i,j\}^c}\Big|_{u\to x}$$

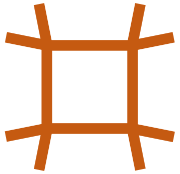
- **16 elliptic curves**

- **Elliptic symbol**: [AS, Wilhelm, Zhang '24]

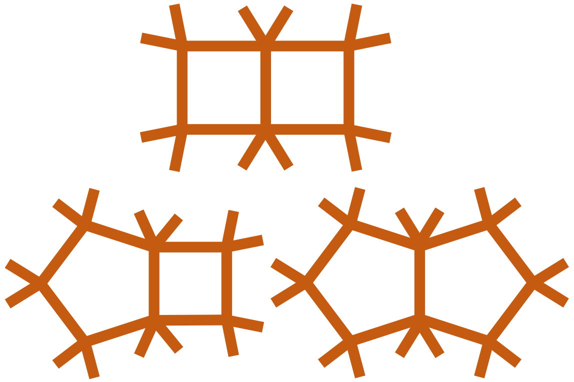
$$\Delta_{2,2}\Big(I[\dots](c)\Big) \cong \sum_{i<j<k} I[\dots]_{\{i,j,k\}^c} \otimes \left[(-1)^k \int^{u} \frac{dx\, y(c)}{(x-c)y(x)} \log R^k_{ij}(x) + (i\leftrightarrow j \leftrightarrow k)\right] - (u\to\infty)$$

3. Conclusions & open questions

1-loop



2-loop



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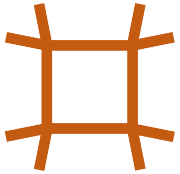
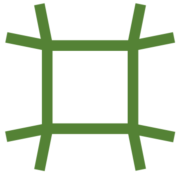
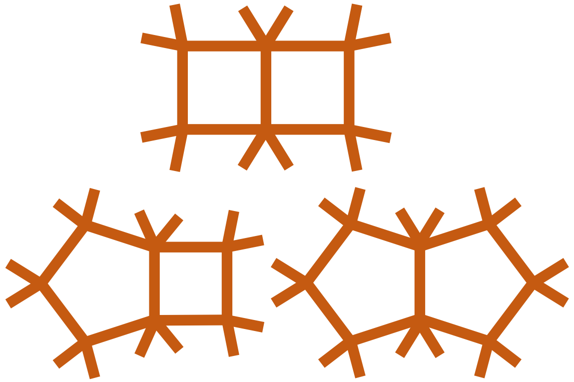
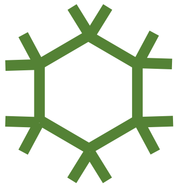




2-loop



one-fold elliptic
integrals over



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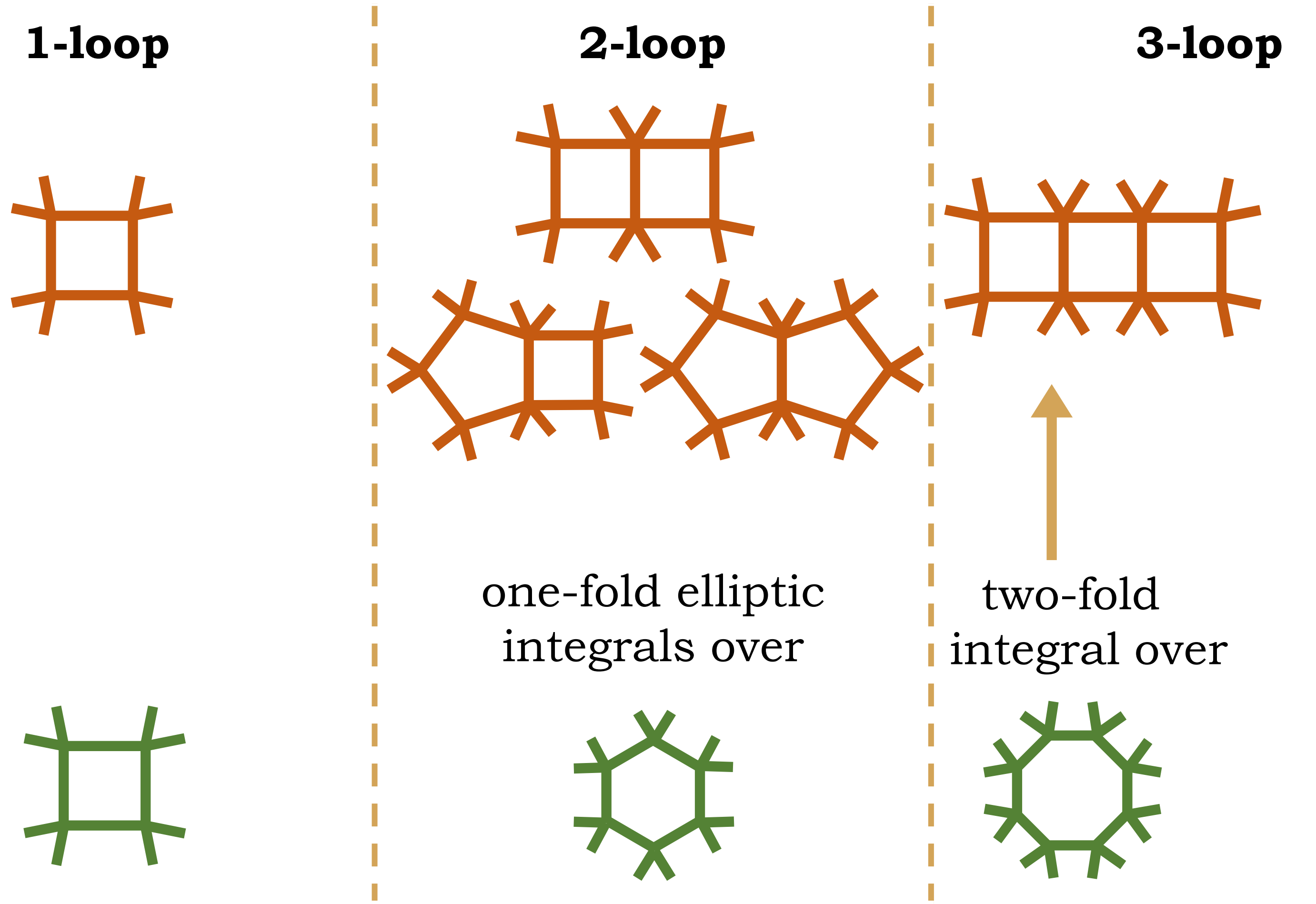

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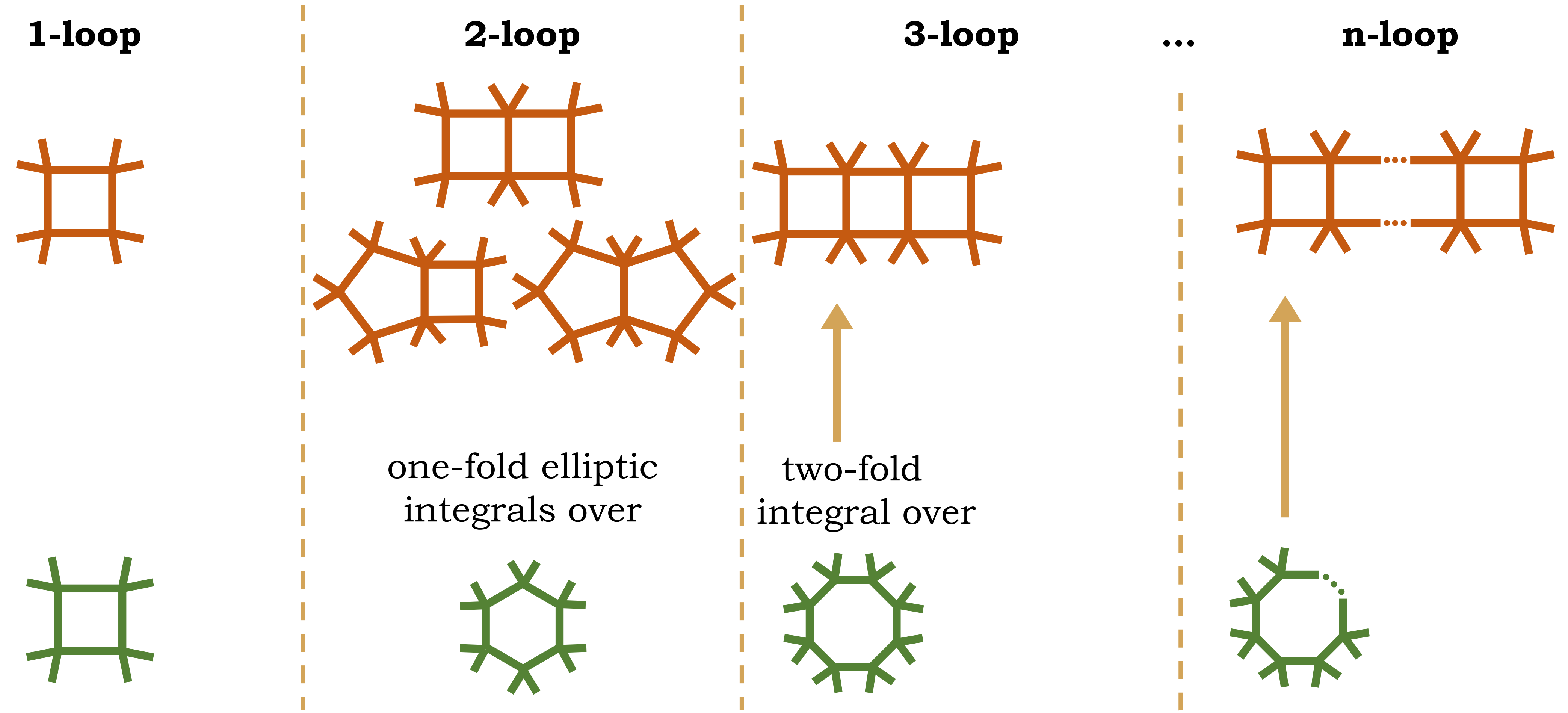

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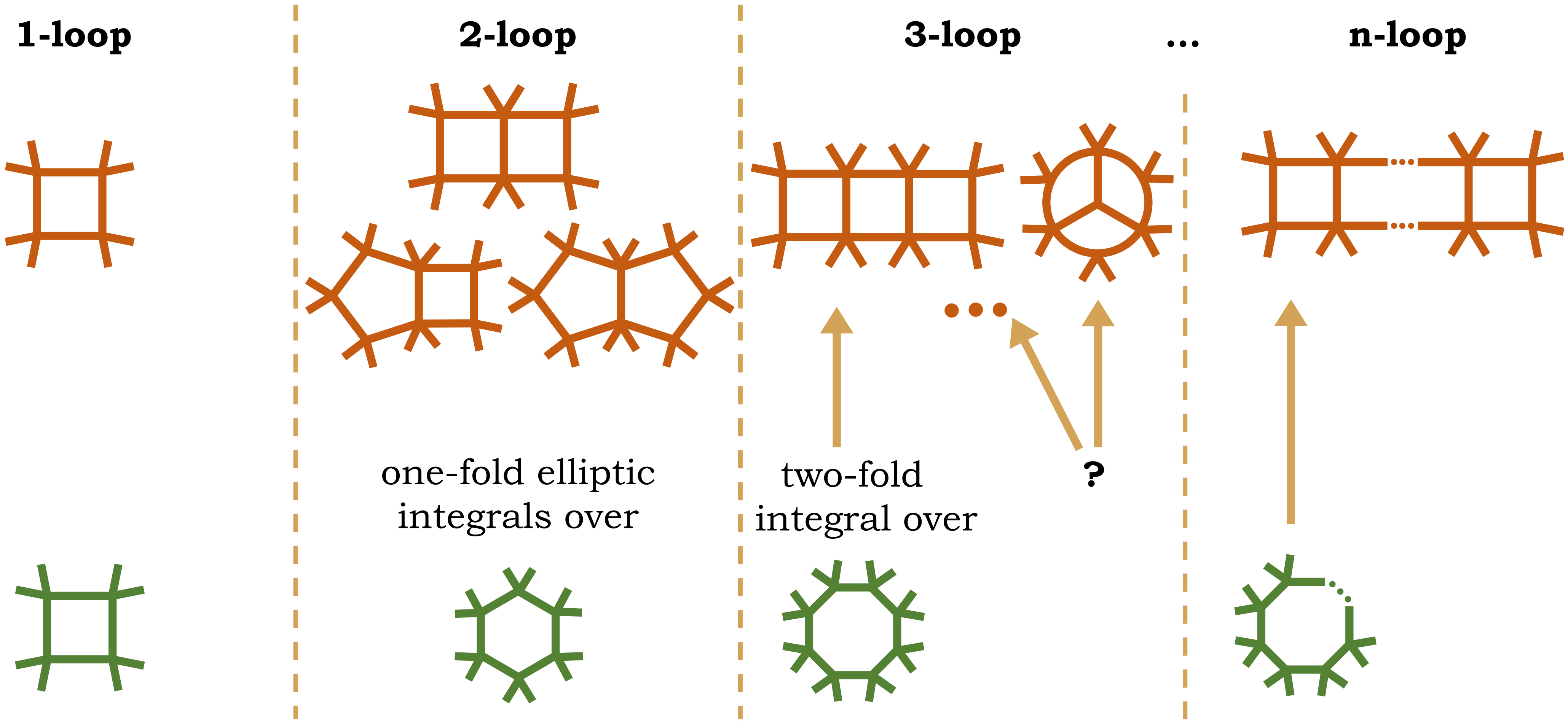
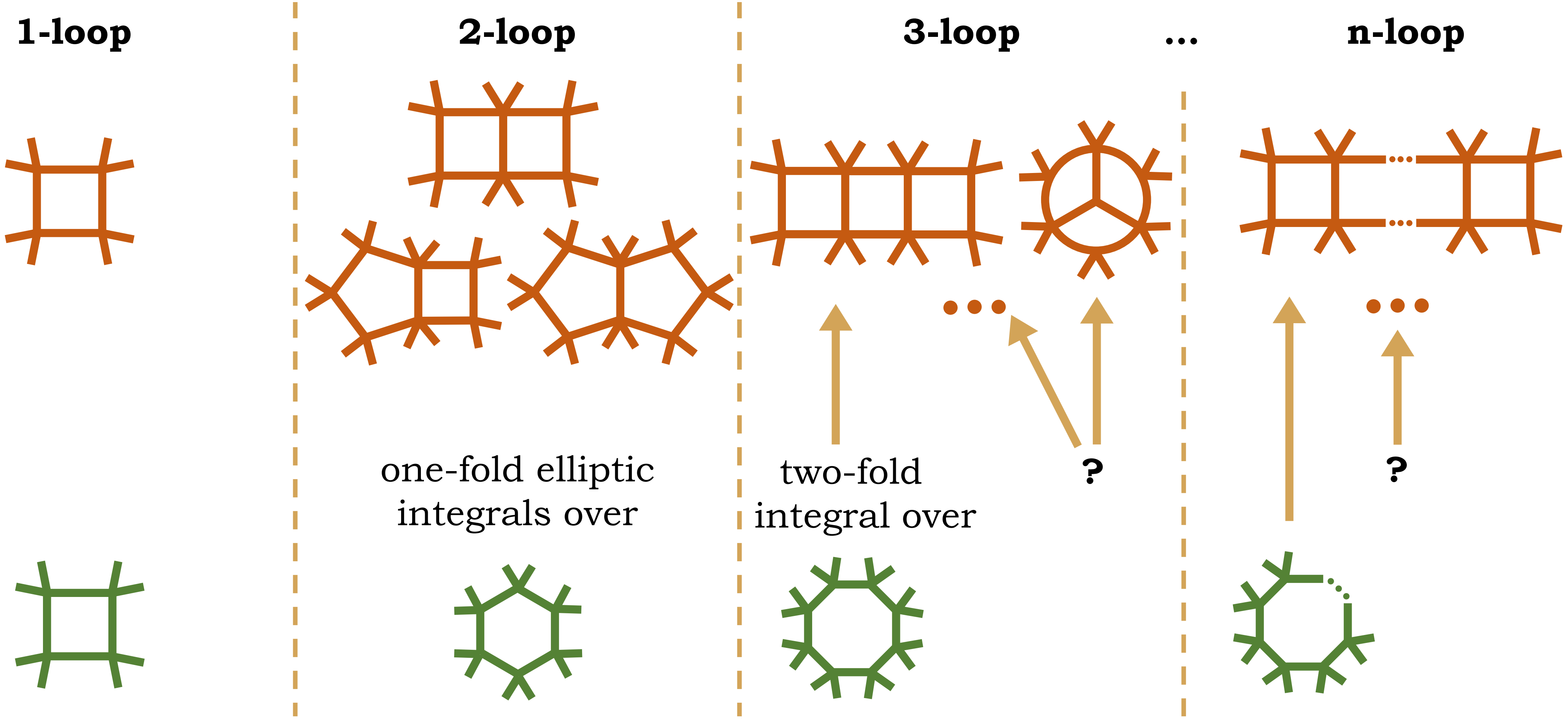


3. Conclusions & open questions

What is the n-loop perturbative structure of Feynman integrals in planar $\mathcal{N} = 4$ SYM theory?



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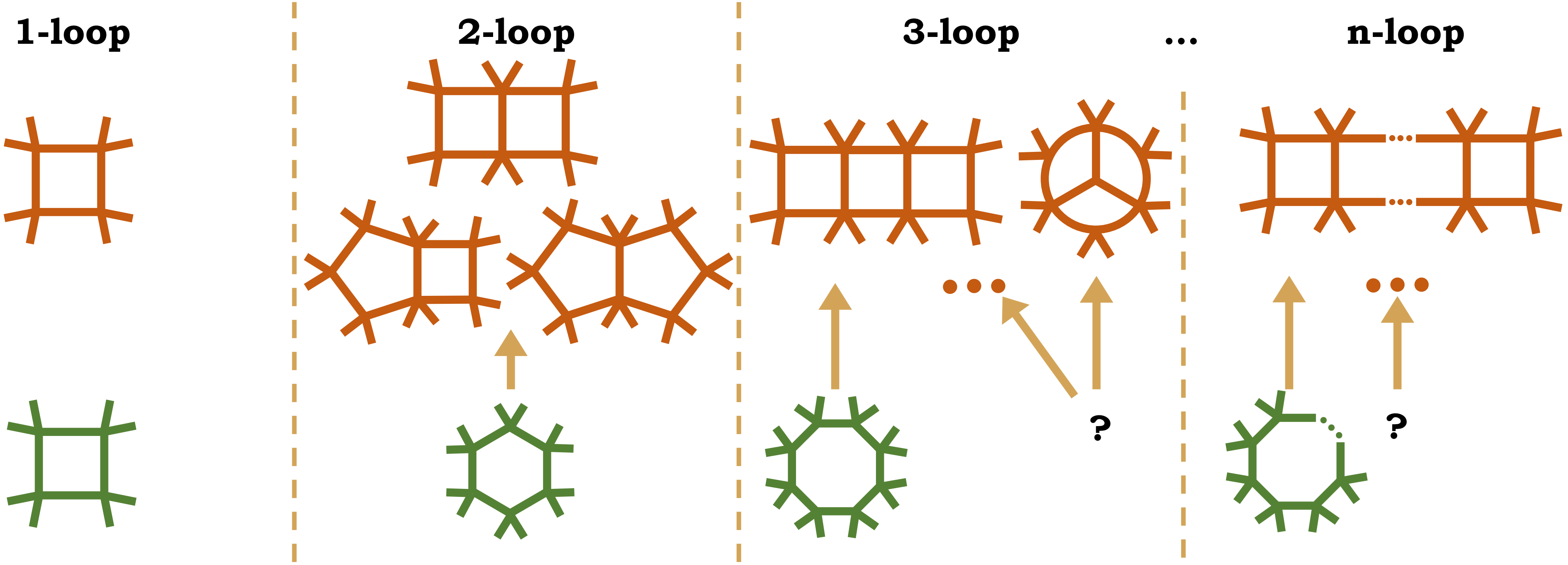

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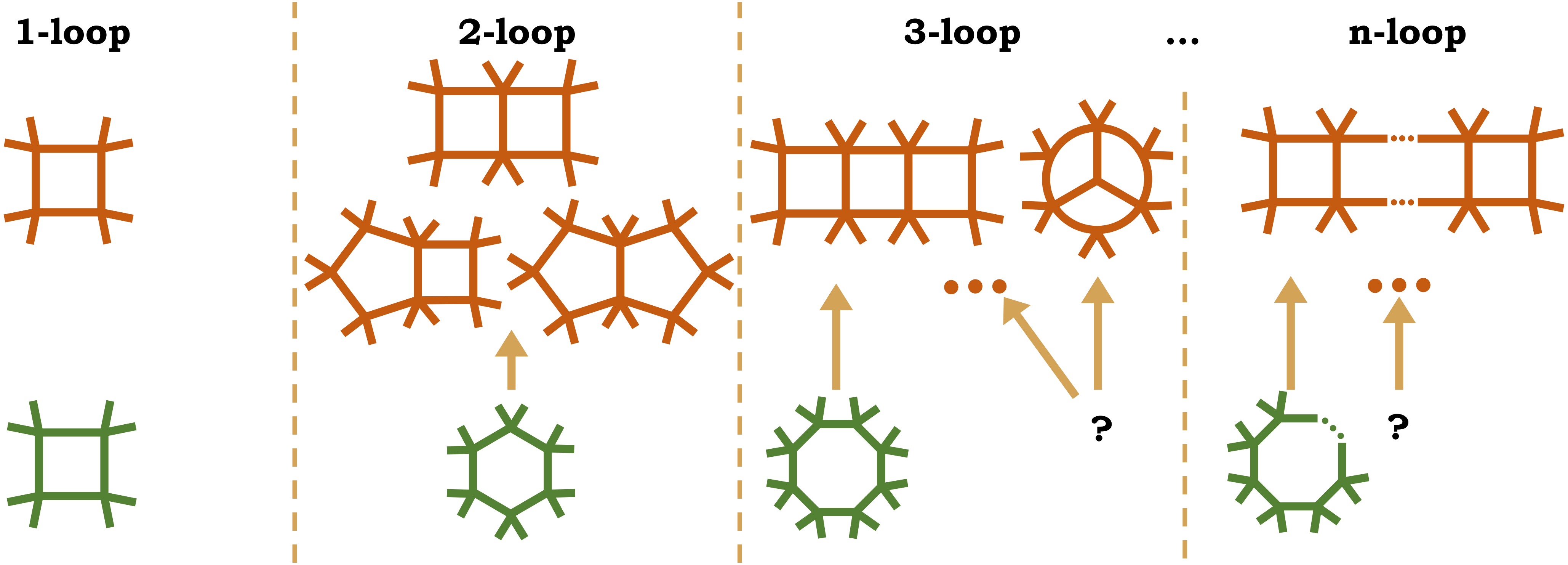

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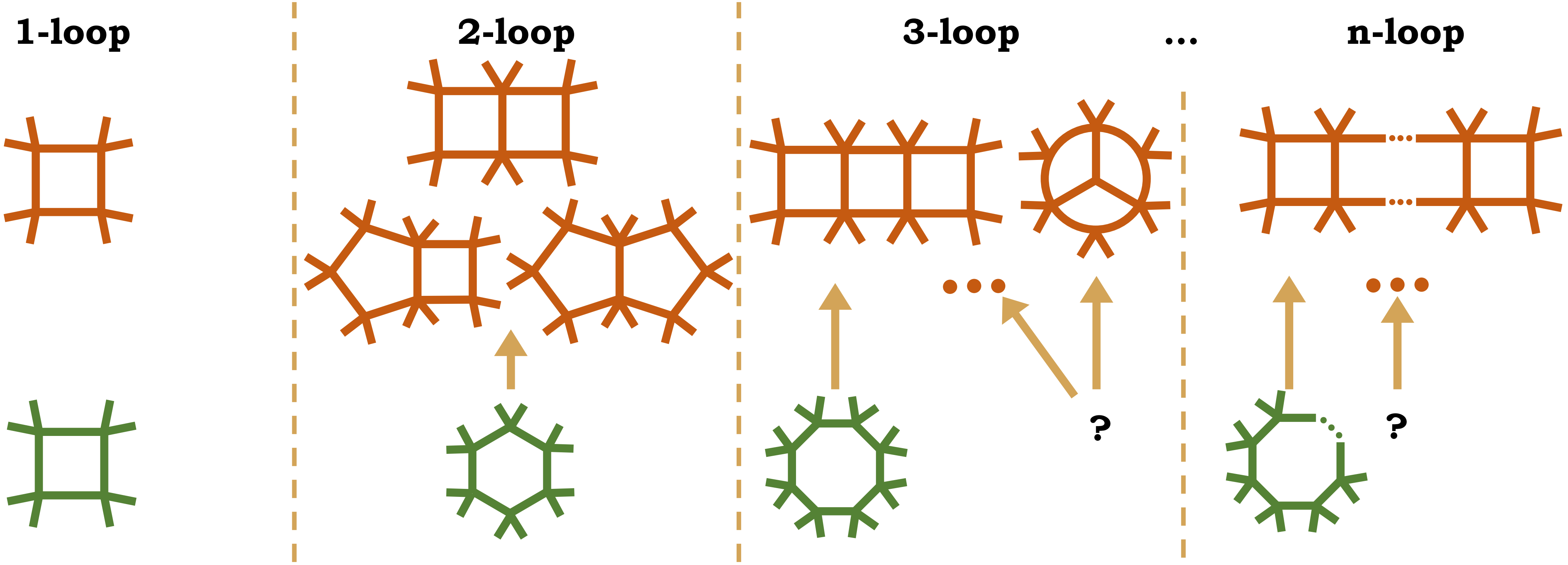

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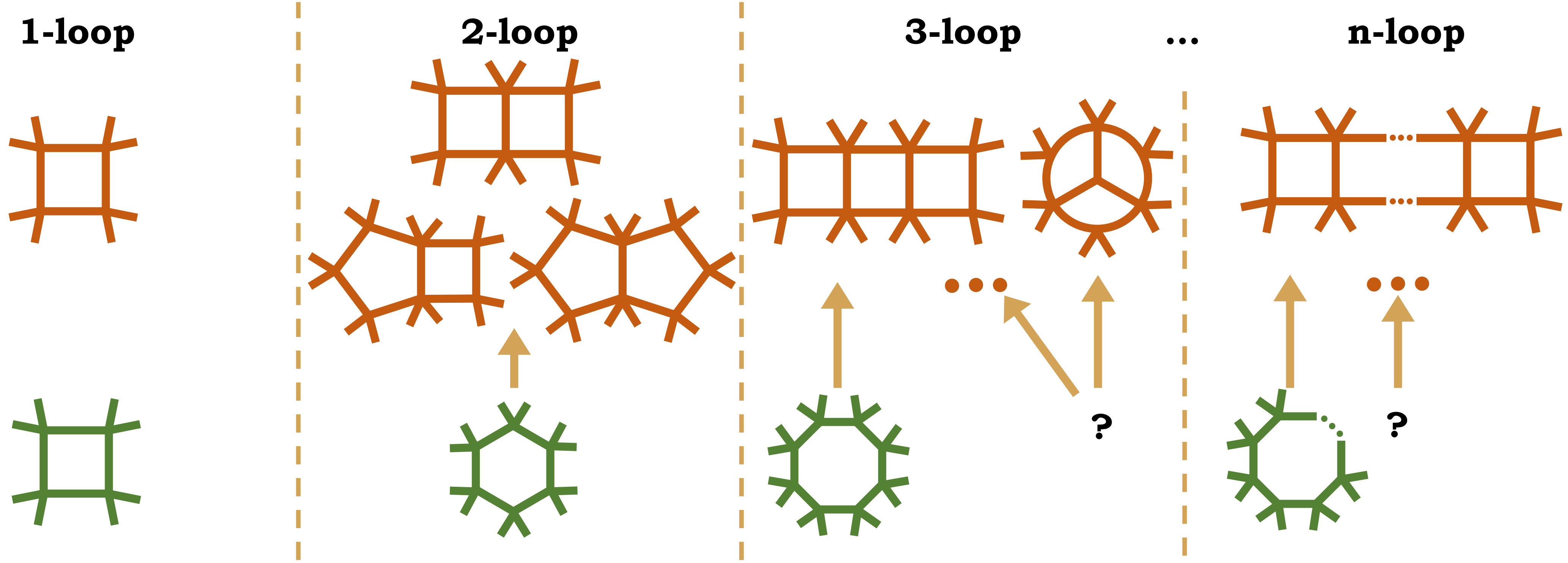

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Thank you!