It is possible to write $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda$ components of cov. derivative

Take a diferential form $\omega = \omega_\mu dx^\mu$, $\nabla_\mu \omega_\nu dx^\nu = (\partial_\mu \omega_\nu + \tilde{\Gamma}^\lambda_{\mu\nu}\, \omega_\lambda) dx^\nu$

$\tilde{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu}$ (Show this by using $\nabla_\mu (\omega_\lambda V^\lambda) = \partial_\mu (\omega_\lambda V^\lambda)$)

* $\nabla_\mu f = \partial_\mu f$; acts on scalars as partial derivatives.
* It commutes with contractions $\nabla_\mu T^\lambda{}_{\lambda\rho} = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$

Covariant derivative acts on a general (k,l) tensor in the following way

$$\nabla_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \partial_\sigma T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} + \dots$$

$$- \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1\mu_2\dots\mu_k}{}_{\lambda\nu_2\dots\nu_l} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\lambda\dots\nu_l} - \dots$$

<u>Ex:</u> For a $(1,1)$ tensor T $\nabla_\mu T_\nu{}^\rho = \partial_\mu T_\nu^\rho + \Gamma^\rho_{\mu\sigma} T^\sigma_\nu - \Gamma^\sigma_{\mu\nu} T^\rho_\sigma$

Transformation law: $\nabla_{\mu'} V^{\alpha'} = \dfrac{\partial x^\mu}{\partial x^{\mu'}} \dfrac{\partial x^{\alpha'}}{\partial x^\alpha} \nabla_\mu V^\alpha$

Write LHS of the above eqn. by using definition

1) $\nabla_{\mu'} V^{\alpha'} = \partial_{\mu'} V^{\alpha'} + \Gamma^{\alpha'}_{\mu'\lambda'} V^{\lambda'}$ $\rightarrow$ then apply $\partial_{\mu'} V^{\alpha'} = \dfrac{\partial x^\mu}{\partial x^{\mu'}} \dfrac{\partial x^{\alpha'}}{\partial x^\alpha} \partial_\mu V^\alpha$

2) Expand RHS by using definition

Exercise: Show that Christoffel symbols don't transform as tensors.

<u>Definition:</u> We denote the vector field defined by $X \circ Y - Y \circ X$ with $[X,Y]$, that is $[X,Y](f) = X(Y(f)) - Y(X(f))$ and call it the Lie bracket of X and Y. Lie bracket defines a map

$$[\cdot,\cdot] : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

Take a chart coordinate and $X = X^\mu \partial_\mu$, $Y = Y^\nu \partial_\nu$ then we have

$$[X,Y] = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu = X^\mu \partial_\mu Y^\nu \partial_\nu - Y^\mu \partial_\mu X^\nu \partial_\nu$$

Definition: For each connection ∇ on TM we define the torsion associated to ∇ as the tensor $\mathcal{T} \in C^\infty(TM \otimes \Lambda^2 T^*M)$ (2,1) tensor by

$$\mathcal{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

At each point $p \in M$ $\mathcal{T}(X,Y)$ is a bilinear function $T_pM \times T_pM \to T_pM$

$$\mathcal{T}(X,Y) = X^a(\partial_a Y^c + \Gamma^c_{ab} Y^b)e_c - Y^a(\partial_a X^c + \Gamma^c_{ab} X^b)e_c$$
$$- (X^a \partial_a Y_c - Y^a \partial_a X^c)e_c$$
$$= X^a Y^b \Gamma^c_{ab} e_c - Y^a X^b \Gamma^c_{ab} e_c = X^a Y^b (\Gamma^c_{ab} - \Gamma^c_{ba}) e_c$$

Thus, we get the torsion tensor in local coordinates

$$\mathcal{T} = \mathcal{T}^c_{ab}\, dx^a \otimes dx^b \otimes \partial_c = (\Gamma^c_{ab} - \Gamma^c_{ba})\, dx^a \otimes dx^b \otimes \partial_c$$

A connection ∇ is symmetric if its torsion vanishes identically:

$$\nabla_Y X - \nabla_X Y = [X,Y] \quad \text{(torsion-free condition)}$$

We say that ∇ is compatible with metric g if $\boxed{\nabla_\rho g_{\mu\nu} = 0}$

Fundamental Theorem of Riemannian Geometry

Let (M,g) be a Riemannian manifold. There exists a unique linear connection ∇ on M that is compatible with g and torsion-free.

Proof: We have two conditions given i) metric compatibility, ii) Torsion-free.

Let's use them. $\nabla_\rho g_{\mu\nu} = 0 \Leftrightarrow X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

Take $X \to \partial_\rho$, $Y \to \partial_\mu$, $Z \to \partial_\nu$. $\partial_\rho g(\partial_\mu, \partial_\nu) = g(\Gamma^\lambda_{\rho\mu} \partial_\lambda, \partial_\nu) + g(\partial_\mu, \Gamma^\lambda_{\rho\nu} \partial_\lambda)$

$$\Rightarrow \partial_\rho g_{\mu\nu} = \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} + \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} \Rightarrow \underbrace{\partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda}}_{\nabla_\rho g_{\mu\nu}} = 0$$

① $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

② $Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$

③ $Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$

Use torsion-free condition
$\nabla_Y X - \nabla_X Y = [X,Y]$, also g is symmetric

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) = 2g(\nabla_X Y, Z) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$$

$$2g(\nabla_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) - X(g(Y,Z)) - Y(g(Z,X)) + Z(g(X,Y)) \quad (1)$$

This equation is valid for all vector fields X, Y, Z.

$$2g(\tilde{\nabla}_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) - X(g(Y,Z)) - Y(g(Z,X)) + Z(g(X,Y)) \quad (2)$$

Now let's assume that there's another connection $\tilde{\nabla}$ satisfying this equation.

Subtract (2) from (1)

$$2g(\nabla_X Y - \tilde{\nabla}_X Y, Z) = 0 \Rightarrow \nabla_X Y - \tilde{\nabla}_X Y = 0 \Rightarrow \nabla = \tilde{\nabla} \;\blacksquare.$$

<u>Exercise:</u> Obtain Christoffel symbol in terms of metric. Use $\nabla_\rho g_{\mu\nu} = 0$

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0$$ (cycle the indices $\rho \to \mu \to \nu \to \rho$).

$$\vdots$$

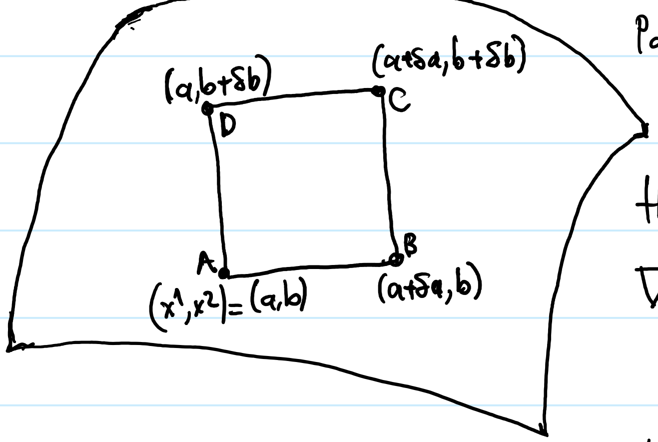
$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

<u>Definition:</u> Let (M,g) be a Riemannian manifold, and let ∇ be any connection on M. Following equation defines a map $R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \text{for all } X,Y,Z \in \mathfrak{X}(M).$$

index notation $R^l_{jhi} = \partial_i \Gamma^l_{hj} - \partial_h \Gamma^l_{ij} + \Gamma^l_{im}\Gamma^m_{hj} - \Gamma^l_{hm}\Gamma^m_{ij}$.

Idea for derivation of Riemann curvature tensor.



Parallel transport a vector V along path A-B-C and then along path A-D-C. Then take the difference of these two transported vectors.

$$\nabla_{e_1} V = \frac{\partial}{\partial x^1} V^\alpha + \Gamma^\alpha_{\mu 1} V^\mu = 0 \quad \text{(Condition for parallel transport)}$$

$$\Rightarrow \frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1} V^\mu$$

Integrating it from A to B: $V^\alpha(B) - V^\alpha(A) = \int_a^{a+\delta a} \frac{\partial V^\alpha}{\partial x^1} dx^1 = -\int_a^{a+\delta a} V^\mu \Gamma^\alpha_{\mu 1} dx^1$

Complete the other paths and write $\delta V^\alpha = V^\alpha(C)_{\text{path2}} - V^\alpha(C)_{\text{path1}}$

$$\delta V^\alpha \simeq \delta a \delta b \left[\frac{\partial \Gamma^\alpha_{\mu 1}}{\partial x^2} - \frac{\partial \Gamma^\alpha_{\mu 2}}{\partial x^1} + \Gamma^\nu_{\mu 1}\Gamma^\alpha_{\nu 2} - \Gamma^\nu_{\mu 2}\Gamma^\alpha_{\nu 1}\right] V^\mu$$

$$R^{\alpha}_{\mu 2 1} = \frac{\partial \Gamma^{\alpha}_{\mu 1}}{\partial x^2} - \frac{\partial \Gamma^{\alpha}_{\mu 2}}{\partial x^1} + \Gamma^{\nu}_{\mu 1} \Gamma^{\alpha}_{\nu 2} - \Gamma^{\nu}_{\mu 2} \Gamma^{\alpha}_{\nu 1}$$

We considered only x^1, x^2 coord.s but this can be generalized to x^β, x^γ general coord.s and obtain the general form as

$$R^{\alpha}_{\mu \gamma \beta} = \frac{\partial \Gamma^{\alpha}_{\mu \beta}}{\partial x^\gamma} - \frac{\partial \Gamma^{\alpha}_{\mu \gamma}}{\partial x^\beta} + \Gamma^{\nu}_{\mu \beta} \Gamma^{\alpha}_{\nu \gamma} - \Gamma^{\nu}_{\mu \gamma} \Gamma^{\alpha}_{\nu \beta}$$

Some properties of Riemann tensor

1. Anti-symmetry in the last two indices $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$
2. Anti-symmetry in the first two indices (for the fully covariant form)

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

3. $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$ (1st Bianchi Identity)
4. $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$ (2nd Bianchi Identity)
5. Contractions: i) Ricci tensor is obtained by contracting the first and third indices of the Riemann tensor. $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\beta} g_{\alpha\beta} R^{\alpha}{}_{\mu\lambda\nu}$

   ii) Ricci scalar (scalar curvature) is the trace of the Ricci tensor.

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

6. If Riemann tensor identically vanishes everywhere ($R^{\rho}{}_{\sigma\mu\nu} = 0$) on the manifold, the manifold is flat (Euclidean or Minkowski space).

   $R_{\mu\nu} = 0$ condition alone doesn't imply flatness of the manifold.

   If $R_{\mu\nu}$ vanishes the manifold is called Ricci-flat.

7. On an n-dimensional manifold, the Riemann tensor has $\frac{n^2(n^2-1)}{12}$ independent components due to its symmetries.

8. Geodesic deviation equation: The Riemann tensor governs the geodesic deviation equation, which describes how nearby geodesics (paths of freely falling particles) converge or diverge due to curvature.

$$\frac{D^2 \xi^\rho}{d\tau^2} = -R^\rho{}_{\sigma\mu\nu} u^\sigma u^\mu \xi^\nu$$

where ξ^ρ is the separation vector between geodesics τ is the proper time. and $u^\mu = \frac{\partial x^\mu}{\partial \tau}$ is the four-velocity of the object.

<u>1st Bianchi Identity:</u>

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Recall the Riemann curvature definition,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$ apply this to the above equation

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y$$

$$= \nabla_X \overbrace{(\nabla_Y Z - \nabla_Z Y)}^{[Y,Z]} - \nabla_{[Y,Z]} X + \nabla_Y \overbrace{(\nabla_Z X - \nabla_X Z)}^{[Z,X]} - \nabla_{[Z,X]} Y + \nabla_Z \overbrace{(\nabla_X Y - \nabla_Y X)}^{[X,Y]} - \nabla_{[X,Y]} Z$$

$$= \underbrace{\nabla_X [Y,Z] - \nabla_{[Y,Z]} X} + \underbrace{\nabla_Y [Z,X] - \nabla_{[Z,X]} Y} + \underbrace{\nabla_Z [X,Y] - \nabla_{[X,Y]} Z},$$

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$ (Jacobi identity)

sum these

$$\left\{ \begin{aligned} [X,[Y,Z]] &= X \circ (Y \circ Z - Z \circ Y) - (Y \circ Z - Z \circ Y) \circ X \\ &= \cancel{X \circ Y \circ Z} - \cancel{X \circ Z \circ Y} - \cancel{Y \circ Z \circ X} + \cancel{Z \circ Y \circ X} \\ [Y,[Z,X]] &= \cancel{Y \circ Z \circ X} - \cancel{Y \circ X \circ Z} - \cancel{Z \circ X \circ Y} + \cancel{X \circ Z \circ Y} \\ [Z,[X,Y]] &= \cancel{Z \circ X \circ Y} - \cancel{Z \circ Y \circ X} - \cancel{X \circ Y \circ Z} + \cancel{Y \circ X \circ Z} \end{aligned} \right.$$

$$\Rightarrow [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \longrightarrow$$ Jacobi identity

Einstein Field Equations with cosmological constant is given

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu} \quad (\text{RHS}=0 \text{ for vacuum solutions})$$

Einstein tensor is defined as $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$

If you use contracted second Bianchi identity you will get

$$\nabla^{\mu}G_{\mu\nu}=0$$

This guarantees the local conservation of energy and momentum.

Vacuum EFEs can be used to define Einstein manifolds. ($\dim M=n$)

$$g^{\mu\nu}\rightarrow/ \; R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=0 \quad (\text{apply } g^{\mu\nu}) \qquad \boxed{g^{\mu\nu}g_{\mu\nu}=n}$$

$$R-\frac{R}{2}n+\Lambda n=0 \Rightarrow R\left(\frac{n}{2}-1\right)=\Lambda n \Rightarrow R=\frac{2n}{n-2}\Lambda$$

$$R_{\mu\nu}-\frac{n}{n-2}\Lambda g_{\mu\nu}+\Lambda g_{\mu\nu}=0 \Rightarrow R_{\mu\nu}=\frac{2\Lambda}{n-2}g_{\mu\nu}=\lambda g_{\mu\nu}$$

(Obviously λ is constant).

<u>Definition</u>: An Einstein manifold is a (pseudo)Riemannian manifold (M,g) whose Ricci curvature tensor is proportional to its metric tensor.

$$\mathrm{Ric}=\lambda g \quad \text{or} \quad R_{\mu\nu}=\lambda g_{\mu\nu} \qquad \lambda\in\mathbb{R}$$

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