It does not change under a coordinate transformation. The determinant of an $n \times n$ matrix $M^{\mu}_{\ \mu'}$ can be written by using Levi-Civita symbol as

(Ex: Verify this for 2x2 matrix)

$$\tilde{\varepsilon}_{\mu'_1\mu'_2\ldots\mu'_n}|M| = \tilde{\varepsilon}_{\mu_1\mu_2\ldots\mu_n} M^{\mu_1}_{\ \mu'_1} M^{\mu_2}_{\ \mu'_2} \ldots M^{\mu_n}_{\ \mu'_n} \qquad (2.5)$$

Now if we set $M^{\mu}_{\ \mu'} = \partial x^{\mu}/\partial x^{\mu'}$ in (2.5) and use the fact that $\partial x^{\mu'}/\partial x^{\mu}$ is the inverse matrix of $M^{\mu}_{\ \mu'}$ we get the following

$$\tilde{\varepsilon}_{\mu'_1\mu'_2\ldots\mu'_n} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \tilde{\varepsilon}_{\mu_1\mu_2\ldots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \frac{\partial x^{\mu_2}}{\partial x^{\mu'_2}} \ldots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}} \qquad (2.6)$$

Hence, the Levi-Civita symbol transforms in a similar way to the tensor transformation law (2.3) with the additional determinant coefficient. Objects transforming in this way are called as <u>tensor densities.</u> More precisely, a tensor density of weight w transforms under a general coordinate transformation $x^{\mu} \rightarrow x'^{\mu}$ as:

$$T'^{\mu_1\mu_2\ldots}_{\ \ \nu_1\nu_2\ldots} = \left|\frac{\partial x'}{\partial x}\right|^{w} \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{\mu_2}}{\partial x^{\alpha_2}} \ldots \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \frac{\partial x^{\beta_2}}{\partial x'^{\nu_2}} T^{\alpha_1\alpha_2\ldots}_{\ \ \beta_1\beta_2\ldots} \qquad (2.7)$$

Exercise: For a general space with coordinates x^{μ} and a metric tensor $g_{\mu\nu}$ the line element ds^2 is given as $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$. Show that the line element (ds^2) is invariant under general coordinate transformation.

Metric tensor transforms as $g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}$

taking determinant of both sides we get

$$|g'| = \left|\frac{\partial x'}{\partial x}\right|^{-2} |g| \qquad (2.8)$$

Thus, we observe that metric determinant transforms as a scalar density of weight -2. Also according to (2.6) Levi-Civita symbol is a (0,2) tensor density of weight $+1$. However, we generally prefer working with tensors rather than densities and it is possible to convert densities into tensors by multiplying $|g|^{w/2}$. Then we can write the Levi-Civita tensor as

$$\varepsilon_{\mu_1\mu_2\ldots\mu_n} = \sqrt{|g|}\,\tilde{\varepsilon}_{\mu_1\mu_2\ldots\mu_n} \qquad (2.9)$$

For a tensor density $T'^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m}$ of weight w one can write the following by using (2.8)

$$\left(\sqrt{|g'|}\right)^w T'^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\rho_n}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\sigma_m}}{\partial x'^{\nu_m}}\left(\sqrt{|g|}\right)^w T^{\rho_1\rho_2\ldots\rho_n}{}_{\sigma_1\sigma_2\ldots\sigma_m}$$

(which obviously transforms as a tensor.)

Differential Forms

A p-form is a totally skew-symmetric covariant tensor of rank p.

Let $\Lambda^p(x)$ be the set of p-forms at X and let $C^\infty(\Lambda^p)$ be the space of smooth p-forms. $\Lambda^p(x)$ is a vector space and has a basis:

$$\{dx^{m_1}\wedge dx^{m_2}\wedge\ldots\wedge dx^{m_p}\},\quad m_1<m_2<\ldots<m_p$$

where $dx^{m_1}\wedge dx^{m_2}\wedge\ldots\wedge dx^{m_p} := \frac{1}{p!}\left[\begin{array}{l}\text{sum of even permutations of } dx^{m_1}\otimes\ldots\otimes dx^{m_p}\\ -\text{sum of odd permutations of } dx^{m_1}\otimes\ldots\otimes dx^{m_p}\end{array}\right]$

it is called wedge product. For example, $dx\wedge dy := \frac{1}{2}\, dx\otimes dy - dy\otimes dx$. ($dx\wedge dy = -dy\wedge dx$)

A zero-form is a function and a 1-form is covariant vector (as mentioned earlier)

Suppose $\alpha\in\Lambda^p$ and $\beta\in\Lambda^q$ and define $I=(i_1,i_2,\ldots,i_p)$ with $i_1<i_2<\ldots<i_p$, $J=(j_1,j_2,\ldots,j_q)$ with $j_1<j_2<\ldots<j_q$. Moreover, introduce $dx^I = dx^{i_1}\wedge dx^{i_2}\ldots\wedge dx^{i_p}$.

Then we can write

$$\alpha = \frac{1}{p!}\alpha_{i_1i_2\ldots i_p}\, dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_p} = \frac{1}{p!}\alpha_I\, dx^I$$

$$\beta = \frac{1}{q!}\beta_{j_1j_2\ldots j_q}\, dx^{j_1}\wedge dx^{j_2}\wedge\ldots\wedge dx^{j_q} = \frac{1}{q!}\beta_J\, dx^J$$

Hence, $(\alpha\wedge\beta)\in\Lambda^{p+q}$ and

$$\alpha\wedge\beta = \frac{1}{p!\,q!}\alpha_{i_1i_2\ldots i_p}\beta_{j_1j_2\ldots j_q}\, dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_p}\wedge dx^{j_1}\wedge dx^{j_2}\wedge\ldots\wedge dx^{j_q}$$

$$= \frac{1}{p!\,q!}\alpha_I\beta_J\, dx^I\wedge dx^J$$

<u>Proposition:</u> $\alpha\wedge\beta = (-1)^{pq}\beta\wedge\alpha$ (Check it by using the given definitions above)

Corollary: $\alpha\wedge\alpha = 0$ if α is a form of odd degree

One can build the following table for the vector spaces of forms for a 3-dim manifold.

vector space	p-form	basis	$\dim \Lambda^p$
Λ^0	ω_0	$\{1\}$	1
Λ^1	ω_1	$\{dx^1, dx^2, dx^3\}$	3
Λ^2	ω_2	$\{dx^1\wedge dx^2, dx^2\wedge dx^3, dx^3\wedge dx^1\}$	3
Λ^3	ω_3	$\{dx^1\wedge dx^2\wedge dx^3\}$	1

More generally, we have $\dim \Lambda^p = \binom{n}{p} = \frac{n!}{(n-p)!\,p!} = \dim \Lambda^{n-p}$ $(n = \dim M)$.

Proposition: Let $\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \dots \oplus \Lambda^n$ be the space consisting of forms of all degrees. Then $\dim \Lambda^* = 2^n$.

Λ^* is closed under wedge product and it is a graded algebra or Cartan's exterior algebra.

Define an operator $\mathbf{d}$, called exterior derivative, which maps p-forms to (p+1)-forms, $\mathbf{d}: \Lambda^p \longrightarrow \Lambda^{p+1}$. For a k-form $\beta = \frac{1}{k!}\beta_{i_1 i_2 \cdots i_k} dx^{i_1}\wedge\dots dx^{i_k}$

$$\mathbf{d}\beta = \frac{1}{k!}\frac{\partial \beta_{i_1 i_2\cdots i_k}}{\partial x^j} dx^j \wedge dx^{i_1}\wedge \dots dx^{i_k} \in \Lambda^{p+1}$$

Suppose $\alpha \in \Lambda^k$, $\beta \in \Lambda^l$ and $a, b \in \mathbb{R}$. Then d satisfies the folowing properties.

1. $d(a\alpha + b\beta) = a(d\alpha) + b(d\beta)$
2. $d(\alpha\wedge\beta) = (d\alpha\wedge\beta) + (-1)^k(\alpha\wedge d\beta)$
3. $d(d\alpha) = 0$ (nilpotency)

Show these as an exercise.

We say a p-form ω is <u>closed</u> if $d\omega = 0$, and <u>exact</u> if $\omega = d\alpha$ for some (p-1)-form α. A vector space of closed p-forms on a manifold is denoted by $Z^p(M)$ and vector space of exact forms is denoted by $B^p(M)$. Define another vector space consisting of elements called cohomology classes, as the closed forms modulo the exact forms.

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}$$

Two closed forms define the same cohomology class if they differ by an exact form.

<u>Ex:</u> $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ Check if ω is closed and/or exact?

Recall that $\dim \Lambda^p = \dim \Lambda^{n-p}$. There is indeed a duality between these spaces, an isomorphism given by Hodge $*$ operation such that

$*: \Lambda^p \longrightarrow \Lambda^{n-p}$. It acts on p-forms in the following way:

Take $\alpha = \frac{1}{p!}\alpha_{i_1\dots i_p} dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_p}$

$$*\alpha = \frac{1}{p!(n-p)!}\alpha_{i_1\dots i_p}\,\varepsilon^{i_1\dots i_p}{}_{i_{p+1}\dots i_n}\,dx^{i_{p+1}}\wedge\dots\wedge dx^{i_n}$$

where $\varepsilon^{i_1\dots i_p}{}_{i_{p+1}\dots i_n} = g^{i_1j_1}\dots g^{i_pj_p}\,\varepsilon_{j_1\dots j_p i_{p+1}\dots i_n}$ (raise the indices by metric)

$**\alpha = (-1)^{s+p(n-p)}\alpha$, s: the number of minus signs in the eigenvalues of the metric.

Example: Consider $\alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz$ in $\mathbb{R}^3$ with coordinates (x,y,z)

$$*\alpha = \alpha_x\, dy\wedge dz + \alpha_y\, dz\wedge dx + \alpha_z\, dx\wedge dy$$

$$*1 = dx\wedge dy\wedge dz \quad , \quad *(dx\wedge dy\wedge dz) = 1 \; , \; *(dx\wedge dy) = dz$$

Interior product (or contracted multiplication) is defined as an operator acting on p-forms such that $i_X : \Lambda^p \to \Lambda^{p-1}$ with a vector field $X\in T(M)$. If f is a function $i_X f = 0$. For a general p-form we have

$$i_X\omega = \frac{1}{(p-1)!}X^\nu\,\omega_{\nu\mu_2\dots\mu_p}\,dx^{\mu_2}\wedge\dots\wedge dx^{\mu_p}$$

For a 1-form: $i_X\omega = \omega_\mu X^\mu \equiv \omega(X)$

2-form: $i_X\omega = \frac{1}{2!}\omega_{\mu\nu}(X^\mu dx^\nu - X^\nu dx^\mu)$

Applying interior product twice to a 2-form

$$i_Y i_X\omega = \frac{1}{2}\omega_{\mu\nu}\, i_Y i_X(dx^\mu\wedge dx^\nu) = \frac{1}{2}\omega_{\mu\nu}(X^\mu Y^\nu - Y^\mu X^\nu)$$

$$= \omega_{\mu\nu}X^\mu Y^\nu = \omega(X,Y)$$

Riemannian Geometry

<u>Definition 3.1</u> Let M be an n-dim. smooth manifold. For any open subset $U \subseteq M$, an n-tuple of vector fields $(X_1, \ldots, X_n)$ over U is called a frame over U if and only if $(X_1(p), \ldots, X_n(p))$ is a basis of the tangent space T_pM, for every $p \in U$. If $U = M$ then the X_i are global sections and $(X_1, \ldots, X_n)$ is called a <u>frame of M</u>.

<u>Definition 3.2</u>: Given a smooth n-dim. manifold M, a Riemannian metric on M (or TM) is a family of inner products $(\langle \cdot, \cdot \rangle_p)_{p \in M}$ on each tangent space T_pM, such that $\langle \cdot, \cdot \rangle_p$ depends smoothly on p, which means that for every chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, for every frame $(X_1, \ldots, X_n)$ on U_α, the maps $p \mapsto \langle X_i(p), X_j(p) \rangle_p$, $p \in U_\alpha$, $1 \leq i, j \leq n$ are smooth. A smooth manifold M with a Riemannian metric is called a Riemannian manifold.

(A Riemannian metric is a family of smoothly varying inner products on the tangent spaces of a smooth manifold.)

A Riemannian metric g is a tensor field of type $(0,2)$ on M with the subsequent properties at each point $p \in M$ $\qquad (g_p : T_pM \times T_pM \to \mathbb{R})$

i) $g_p(X, Y) = g_p(Y, X)$ (symmetric)

ii) $g_p(X, X) \geq 0$ equality only for $X = 0$ (positive definite)

if instead of ii) we have

iii) $g_p(X, Y) = 0 \ \forall X \in T_pM \Rightarrow Y = 0$ (non-degenerate)

Then the metric g_p is called as pseudo-Riemannian metric. For this type of metric $g_p(X, X)$ can be positive, negative or zero for non-zero vectors. Signature of a pseudo-Riemannian metric is $\text{sign}(p, q)$ where p is the number of positive eigenvalues, and q is the number of negative eigenvalues.

$q = 0 \Rightarrow$ Riemannian metric, $(p, q) = (1, n-1)$ or $(n-1, 1) \Rightarrow$ Lorentzian metric.

The metric is bilinear in its arguments. (recall the properties of tensors)

$$g\left(\frac{\partial}{\partial x^m}, \frac{\partial}{x^n}\right) := g_{mn}, \quad g(X,Y) = g\left(X^m \frac{\partial}{\partial x^m}, Y^n \frac{\partial}{\partial x^n}\right) = g_{mn} X^m Y^n$$

The isomorphism between vectors and forms is provided by the metric

$$\tilde{X} = \tilde{X}_\mu dx^\mu = (g_{\mu\nu} X^\nu)\, dx^\mu, \quad X = X^\mu \frac{\partial}{\partial x^\mu} = (g^{\mu\nu} \tilde{X}_\nu) \frac{\partial}{\partial x^\mu}$$

$$g(X,Y) = \tilde{X}(Y) = \tilde{X}_\mu dx^\mu\, Y^\nu \frac{\partial}{\partial x^\nu} = \tilde{X}_\mu \delta^\mu_\nu Y^\nu = \tilde{X}_\mu Y^\mu = g_{\mu\nu} X^\nu Y^\mu$$

Hence, we raise and lower the indices of general tensors by using the metric tensor. For example, consider a mixed tensor $T^\mu{}_\nu = g^{\mu\sigma} T_{\sigma\nu}$

The norm $\|X\|$ (or length) of a vector $X \in T_pM$ is defined by

$$\|X\|^2 = g(X,X) = g_{\mu\nu} X^\mu X^\nu$$

Also, the infinitesimal distance squared (line element) is obtained by inserting the infinitesimal displacement $dx^\mu\, \partial/\partial x^\mu$ into the metric

$$ds^2 = g\left(dx^\mu \frac{\partial}{\partial x^\mu}, dx^\nu \frac{\partial}{\partial x^\nu}\right) = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) dx^\mu dx^\nu = g_{\mu\nu}\, dx^\mu dx^\nu$$

(often this is called a metric, but that's not correct!)

Let (M,g) be a (pseudo) Riemannian manifold, the diffeomorphism $f: M \to M$ is an <u>**isometry**</u> if it preserves the metric: $f^* g_{f(p)} = g_p$ which means

$$g_{f(p)}(f_* X, f_* Y) = g_p(X,Y) \quad \text{for } X, Y \in T_pM$$

In coordinate language, x is the coordinate of p and y of $f(p)$, we have

$$g_{\alpha\beta}(f(x)) \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} = g_{\mu\nu}(x).$$

Isometries form a group, called the isometry group, preserving the distance between points. For example, we have the isometry for $\mathbb{R}^n$

$$f: x \mapsto Ax + b \quad \text{with } A \in O(m),\ b \in \mathbb{R}^n. \qquad (A^T A = I)$$

Isometry group of $\mathbb{R}^n$ is represented by $\mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ ↳ semi-direct product

If also orientations to be preserved then $\mathrm{Iso}(\mathbb{R}^n) = SO(n) \ltimes \mathbb{R}^n$

(Check the last page for extra explanation)

Let M be a smooth manifold. A connection on M is a $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

where we write $\nabla_X Y$ for $\nabla(X,Y)$ such that the following conditions hold.

i) $\nabla_{fX} Y = f \nabla_X Y$

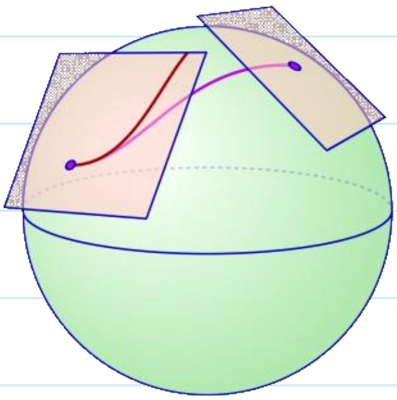
ii) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$,

for all $X, Y \in \mathfrak{X}(M)$ and all $f \in C^\infty(M)$. The vector field $\nabla_X Y$ is called the covariant derivative of Y with respect to X.

Let's consider a local frame $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ on $U \subseteq M$. We have the relation

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad \text{or} \quad \nabla_{e_i} e_j = \Gamma^k_{ij} e_k \quad (e_i = \partial/\partial x^i)$$

for some unique smooth functions Γ^k_{ij} defined on U, called the Christoffel symbols.



* Connection permits tangent vec. fields to be differentiated as if they were functions on M with values in fixed vector space.
* A connection on a manifold does not assume that the manifold is equipped with a Riemannian metric.
* It is an object on a smooth mfd. which connects (relates) tangent spaces.

Prop: Every smooth manifold M possesses a connection.

(See J.M. Lee, Introduction to Smooth Manifolds)

Let's take two vector fields $X, Y \in \mathfrak{X}(M)$ given in terms of local frame as $X = X^i e_i$, $Y = Y^j e_j$.

dummy index

$$\nabla_X Y = \nabla_X (Y^j e_j) = (X Y^j) e_j + X^i Y^j \nabla_{e_i} e_j = X Y^j e_j + X^i Y^j \Gamma^k_{ij} e_k$$

$$= X Y^k e_k + X^i Y^j \Gamma^k_{ij} e_k = (X Y^k + X^i Y^j \Gamma^k_{ij}) e_k$$

It is possible to write $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda$ components of cov. derivative

Take a diferential form $\omega = \omega_\mu dx^\mu$, $\nabla_\mu \omega_\nu dx^\nu = (\partial_\mu \omega_\nu + \tilde{\Gamma}^\lambda_{\mu\nu} \omega_\lambda) dx^\nu$

$\tilde{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu}$ (Show this by using $\nabla_\mu (\omega_\lambda V^\lambda) = \partial_\mu(\omega_\lambda V^\lambda)$)

* $\nabla_\mu f = \partial_\mu f$; acts on scalars as partial derivatives.
* It commutes with contractions $\nabla_\mu T^\lambda{}_{\lambda\rho} = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$

Covariant derivative acts on a general (k,l) tensor in the following way

$$\nabla_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \partial_\sigma T^{\mu_1 \mu_2 \dots \mu_k}{}_{\nu_1 \nu_2 \dots \nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \dots \mu_k}{}_{\nu_1 \nu_2 \dots \nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \dots \mu_k}{}_{\nu_1 \nu_2 \dots \nu_l} + \dots$$

$$- \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1 \mu_2 \dots \mu_k}{}_{\lambda \nu_2 \dots \nu_l} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1 \mu_2 \dots \mu_k}{}_{\nu_1 \lambda \dots \nu_l} - \dots$$

Ex: For a (1,1) tensor T $\nabla_\mu T_\nu{}^\rho = \partial_\mu T_\nu^\rho + \Gamma^\rho_{\mu\sigma} T_\nu^\sigma - \Gamma^\sigma_{\mu\nu} T_\sigma^\rho$

Transformation law: $\nabla_{\mu'} V^{\alpha'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \nabla_\mu V^\alpha$

Write LHS of the above eqn. by using definition

1) $\nabla_{\mu'} V^{\alpha'} = \partial_{\mu'} V^{\alpha'} + \Gamma^{\alpha'}_{\mu'\lambda'} V^{\lambda'}$ → then apply $\partial_{\mu'} V^{\alpha'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \partial_\mu V^\alpha$

2) Expand RHS by using definition

Exercise: Show that Christoffel symbols don't transform as tensors.

Semi-direct product $O(n) \ltimes \mathbb{R}^n$ reflects the fact that translations and orthogonal transformations do not commute. Group elements are ordered pairs (A, b) where $A \in O(n)$ and $b \in \mathbb{R}^n$. The group operation is defined as:

$$(A_1, b_1) \circ (A_2, b_2) = (A_1 A_2, b_1 + A_1 b_2)$$

For $n=3$: $\mathrm{Iso}(\mathbb{R}^3)$ $\dim \mathrm{Iso}(\mathbb{R}^3) = (\dim O(3)) + (\dim \mathbb{R}^3) = 3+3=6$

A practical formula for Hodge star $*$:

$$\alpha \wedge * \alpha = \langle \alpha, \alpha \rangle dVol_g \quad , \quad dVol_g : \text{Volume form}$$