Heuristic Derivation of EFE:

* By EP

$$\boxed{\text{Gravity}} \xleftrightarrow{\text{By EP}} \boxed{\text{Spacetime curvature}}$$

$$\boxed{\text{Spacetime curvature}} \xrightarrow{\text{defined by}} \boxed{\text{Riemannian Geometry}}$$

(locally flat $\equiv$ SR holds locally)

$$\left(\text{describey by } \underline{\text{Reimann tensor}}\right) \longleftarrow \boxed{\text{Riemannian Geometry}}$$

$$R_{\mu\nu\alpha\beta} \sim \partial^2 g + (\partial g)^2$$

* Also from Poisson eqn.

$$\underset{\text{LHS}}{\boxed{\text{Spacetime curvature}}} \propto \underset{\text{RHS}}{\boxed{\text{Source Density}}}$$

$$\left.\begin{matrix} E^2 - c^2 p^2 = m^2 c^4 \\ \text{In SR} \end{matrix}\right\} \begin{matrix} \text{mass} \\ \text{energy} \\ \text{mom.} \end{matrix} \Bigg| \text{ density}$$

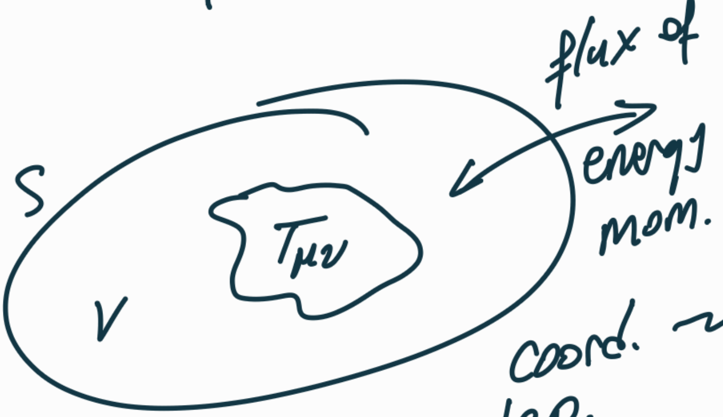
* So mass density ρ in Poisson eqn. should be generalized to some object including all these: mass/energy/mom.

$\Rightarrow$ we already have such a quantity in SR.

$$\left(\begin{array}{c} 2^{\text{nd}}\text{-rank} \\ \text{Tensor} \end{array}\right) T_{\mu\nu} = \left[\begin{array}{c|c} \begin{array}{c}\text{energy}\\ \text{density}\end{array} & \begin{array}{c}\text{energy flux}\\ \text{density}\end{array} \\ \hline \begin{array}{c}\text{mom.}\\ \text{density}\end{array} & \begin{array}{c}\text{mom. flux density}\\ \text{(stress tensor)}\end{array} \end{array}\right]$$

$\Rightarrow$ EM tensor

$\Rightarrow$ Also from SR



(like $\partial_\mu j^\mu = 0$)

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

coord. dep.

Conservation of EM.

$\Rightarrow$ In GR, we should have general coords.

$$\left(\begin{array}{l} \eta_{\mu\nu} \longrightarrow g_{\mu\nu} \\ \partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma^{\cdot}_{\cdot\mu} \end{array} \right) \Longrightarrow \overset{\text{general covariance}}{\boxed{\nabla_\mu T^{\mu\nu} = 0}}$$

$\hookrightarrow$ principle of general covariance

$\Downarrow$ Covariantly const.

* Then we have the properties of $T_{\mu\nu}$:

$$(T_{\mu\nu}) \Longrightarrow \begin{cases} - \text{vanishes in the absense of "matter"} \\ - \text{is a } 2^{nd}\text{-rank tensor} \\ - \text{is sym ; } T_{\mu\nu} = T_{\nu\mu} \\ - \text{is cov. const.} \end{cases}$$

* All these suggest that

$$\underbrace{F\left[R_{\mu\nu\alpha\beta}(g_{\mu\nu})\right]}_{\text{spacetime curv. part.}} \propto \underbrace{F\left[T_{\mu\nu}\right]}_{\text{source density part}}$$

LHS RHS

(structure of GR eqns.)

* Let's try some possibilities:

(i) The simplest possibility is

$$g_{\mu\nu} \propto T_{\mu\nu} \quad ?$$

- satisfies $\nabla_\mu T^{\mu\nu} = 0$ since $\nabla_\mu g_{\alpha\beta} = 0$
- gives $g_{\mu\nu} = 0$ in vacuum ($T_{\mu\nu} = 0$)

$\Rightarrow$ does not work! (trivial)

(ii) Another possibility is

$$R_{\mu\nu\alpha\beta} \propto T_{\mu\nu} T_{\alpha\beta} \quad ?$$

- does not satisfy $\nabla_\mu T^{\mu\nu} = 0$
- gives flat spacetime $\left(g_{\mu\nu} = \eta_{\mu\nu} \right)$ in vacuum $\left(T_{\mu\nu} = 0 \right)$

$\Longrightarrow$ So, does not work!

(iii) One last guess is

$$R_{\mu\nu} \propto T_{\mu\nu} \quad ? \qquad R_{\mu\nu} \equiv R^{\mu}{}_{\alpha\mu\beta}$$

- does not satisfy $\nabla_\mu T^{\mu\nu} = 0$ since $\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R \neq 0$

$\Rightarrow$ So does not work! — Vacuum case is OK! $R_{\mu\nu} = 0$

* With all these possibilities, we should also note that

(Poisson Eqn.) $\nabla^2 \phi = \underline{\underline{\partial_i \partial_i}} \phi = 4\pi G \rho$ (in NG)

Divergence appears

(Since GR should recover NG) $\Rightarrow$ $\dfrac{GR}{g_{\mu\nu}} \longleftrightarrow \dfrac{NG}{\phi}$

$\Rightarrow$ LHS $\equiv F[\partial_i \partial_i g_{\mu\nu}]$ in GR eqn.

$\Rightarrow$ Then we have

$$\underline{\text{LHS}}$$

$$F[R_{\mu\nu}(g_{\mu\nu})] \propto F[T_{\mu\nu}]$$

* After these reasoning, there is one particular combination of $R_{\mu\nu\alpha\beta}$ which satisfies all the properties of RHS:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = K \, T_{\mu\nu}} \quad \text{(EFE)}$$

K = proportionality const
(determining gravitational coupling)

- You can indeed show that

$$\nabla_\mu G^{\mu\nu} = 0 \rightarrow \text{Bianchi identity}$$

Exercise

* You can also add CC to LHS

$$G_{\mu\nu} + \underline{\underline{\Lambda g_{\mu\nu}}} = \kappa T_{\mu\nu}$$

$\Downarrow$ can also be derived from action principle

$$\underbrace{S = \int \sqrt{-g}\, R\, d^4x}_{\text{Einstein-Hilbert action}}$$