# Entropic Order

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This presentation is based on

★ arXiv: 2005.03676 with **Noam Chai, Soumyadeep Chaudhuri, Chang-Ha Choi , Eliezer Rabinovici, and Misha Smolkin.**

★ arXiv: 2412.09459 with **Fedor Popov.**

★ arXiv: 2503.22789 **Yiqiu Han, Xiaoyang Huang, Andrew Lucas, Fedor Popov.**

★ Work in Progress....

- General principles in thermal physics
- Lattice gas
- Ferromagnet
- Superconductivity?!

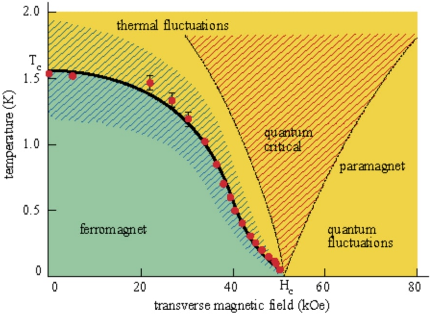
Many systems exhibit symmetry breaking at zero temperature. For instance, ferromagnets, massless QCD, anti-ferromagnets, superfluids etc. We usually think that if we heat these systems up, i.e. study instead of the vacuum the thermal state

$$e^{-\beta H}$$

then all the symmetries are restored for sufficiently small $\beta$. (I am talking about ordinary symmetries only.)

Indeed, most phase diagrams for quantum critical points look like this (phase diagram of $LiHoF_4$ as measured by Bitko and co-workers)

*Figure 1*: **Quantum criticality in a ferromagnet.**



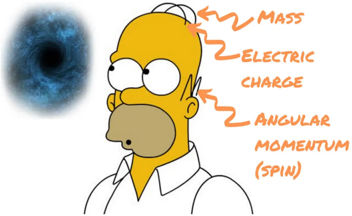
Symmetry should be restored at high enough temperature.

- At finite temperature we minimize

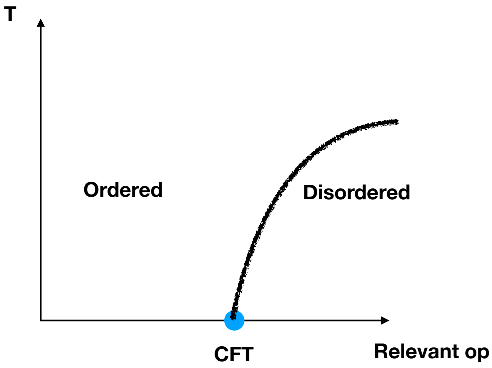
$$F = E - ST \ .$$

  At large $T$ the dominant contribution is from high entropy states and those are disordered. Or so we are taught in school.

- For very special systems (holographic ones), high temperature states are dual to a black hole in AdS. For the latter, it is widely believed that there is a no hair theorem and hence no symmetry breaking.

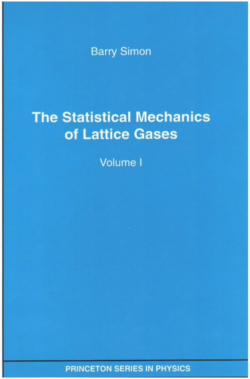


The question is therefore clear: Can there be symmetry breaking at high temperature? If so the phase diagram would have to look like the following:



In summary: experiments, the no-hair theorem, and thermodynamic arguments all suggest that the expectation values of order parameters must vanish at high temperature and more generally all the sites decouple from each other.

These results can be proven mathematically under various assumptions. It is sometimes referred to as the "uniqueness of the Gibbs state." Or "Dobrushin theorem." A quantum version is the "heat death of entanglement" [Martin Kliesch, Christian Gogolin, Michael Kastoryano, Arnau Riera, and Jens Eisert].



In the classical version one proves that sites are uncorrelated at high temperature

$$\beta < \beta_c,\ |x-y| \gg \beta : \quad \langle U(x)V(y)\rangle_\beta \to \langle U(x)\rangle_\beta \langle V(y)\rangle_\beta$$

for any operators $U, V$

In the quantum version one further proves that entanglement cannot survive heat

$$\beta < \beta_c \ , \quad \frac{e^{-\beta H}}{Tr(e^{-\beta H})} = \sum_i p_i \rho_A^i \otimes \rho_B^i \ .$$

This means that the density matrix is separable.

The theorems about the uniqueness of the Gibbs state at $\beta < \beta_c$ have important assumptions:

**1**

The uniform distribution $\mathbb{I}$ over the space of states $\Omega$ exists

**2**

$\Omega = \times\Omega_i$, i.e. there are no constraints and the space of states is a direct product.

Both of these are basically never true in quantum field theory.
They can be also violated on the lattice.
We will therefore pose three questions

Does large entropy really mean lack of order?

Are there local lattice models which are ordered at high temperature?

Are there nontrivial CFTs which break a global symmetry at finite temperature?

Let us first consider classical stat-mech in 2d

$$\mathbb{P}(\mathbf{n}) = \frac{\mathrm{e}^{-\beta H(\mathbf{n})}}{Z(\beta)} \tag{1}$$

where $H(\mathbf{n})$ is the energy (Hamiltonian) of the microstate, and the partition function

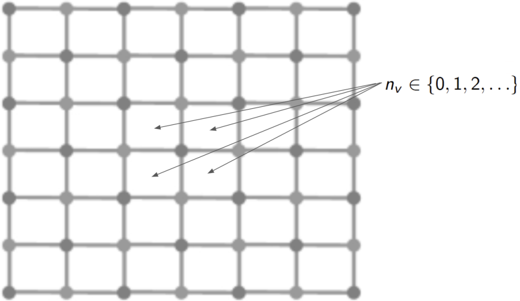
$$Z(\beta) = \sum_{\mathbf{n}} \mathrm{e}^{-\beta H(\mathbf{n})} = \mathrm{e}^{-\beta F(\beta)} \tag{2}$$

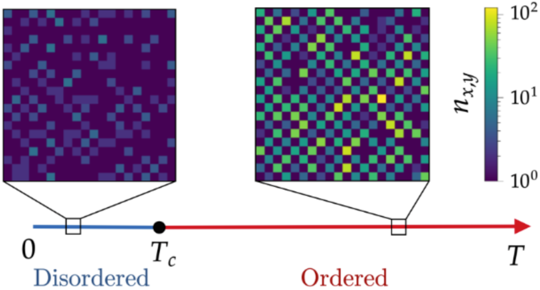
both normalizes the probability distribution, and defines the free energy $F$.

On the $L \times L$ square lattice, at each plaquette we can have $n_v \in \{0, 1, 2, \ldots\}$ bosons. We choose

$$H = U \sum_{u \sim v} n_u^2 n_v^2 + \sum_v n_v . \quad (3)$$

This means that bosons repel on adjacent plaquettes and there is some chemical potential.






Result of Monte-Carlo. We see solidification at high temperature!

Mean Field Theory:

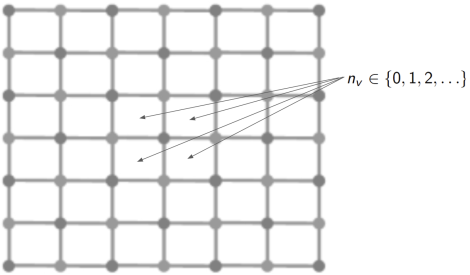
If one site is occupied but none of its neighbors are (solid phase), the typical number of particles $\bar{n}_1$ obeys $\beta \bar{n}_1 \sim 1$, or $\bar{n}_1 \sim T$. In contrast, if two adjacent sites both have $\bar{n}_2$ particles, $\beta \bar{n}_2^4 \sim 1$ or $\bar{n}_2 \sim T^{1/4}$.
If we consider the checkerboard arrangement from before, we can occupy half of the sites leading to partition function $\bar{n}_1^{L^2/2} \sim T^{L^2/2}$, which is much larger than $\bar{n}_2^{L^2} \sim T^{L^2/4}$ if we consider the disordered state.

This suggests that the dominant contribution to $Z(\beta)$ comes from checkerboard-like states for sufficiently small $\beta$. Therefore the high-temperature phase is a solid phase that spontaneously breaks the lattice translational symmetry.

The mechanism is therefore that order in one degree of freedom can enable other degrees of freedom to strongly fluctuate, leading to "entropic order", whereby typical high energy states are ordered!

A more natural and deceptively simple model is

$$H = U \sum_{u \sim v} n_u n_v + \sum_v n_v$$



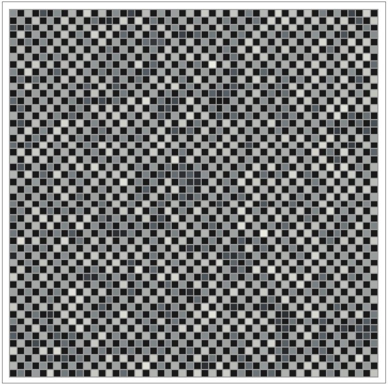
Here guessing the answer is less straightforward.
Both for the solid and gas we obtain to leading order at high temperature

$$\mathcal{F}_{gas,solid} = -\frac{L^2}{2} T \log T + \cdots \ .$$

Mean field theory suggests that

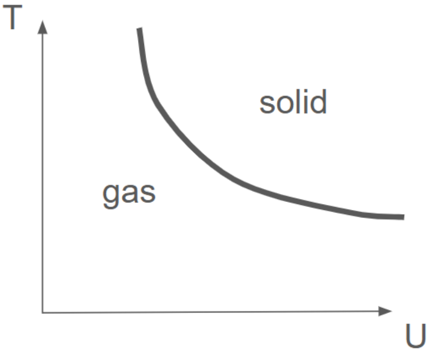
$$\mathcal{F}_{gas} - \mathcal{F}_{solid} = \frac{T}{2} \left( \log(U) + const. \right) + \cdots \ .$$

Therefore for $U > U_c \sim 1$ we again find symmetry breaking, while for $U < U_c \sim 1$ the gas is the high temperature phase.



[figure from Tin Sulejmanpasic]

$$H = U \sum_{u \sim v} n_u n_v + \sum_v n_v$$




(We do not know yet if there are additional phases.)

Now we will show how the same ideas can be used to engineer systems where the typical high energy states are ferromagnetic and superconducting.

As a continuous analog of the $n \in 0, 1, ..., \infty$ variable we will adopt the $O(N)$ order parameter at criticality

$$S_{O(N)} = \int d^2x dt \left( \frac{1}{2} (\partial \vec{\phi})^2 + (\vec{\phi}^2)^2 \right) .$$

From the large $N$ solution we know

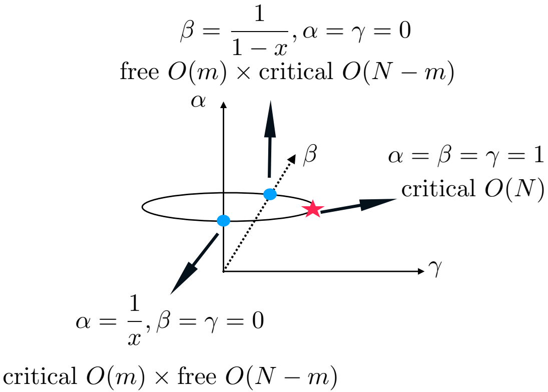
$$\Delta(\vec{\phi}^2) = 2 - \frac{32}{3\pi^2 N} + \cdots .$$

Next we take a real quantum field $\Psi$ and couple the two by

$$S_{O(N)} + \int d^2x dt \, \frac{1}{2} (\partial \Psi)^2 - \underbrace{\lambda \int d^2x dt \vec{\phi}^2 \Psi^2} .$$

$$S_{O(N)} + \int d^2xdt\ \frac{1}{2}(\partial\Psi)^2 - \underbrace{\lambda \int d^2xdt\vec{\phi}^2\Psi^2}\ .$$

$\lambda$ is a relevant coupling constant. The dynamics can be analyzed in the $\epsilon$ expansion in $2.99+1$ space-time dimensions or in the large $N$ expansion directly in 2+1 d.



In the large $N$ expansion in 3d

$$\beta_\lambda = \frac{32}{3\pi^2 N}(\lambda - \lambda^3) + \cdots$$

The most interesting fixed point for us is that with

$$\lambda = 1 + ...$$

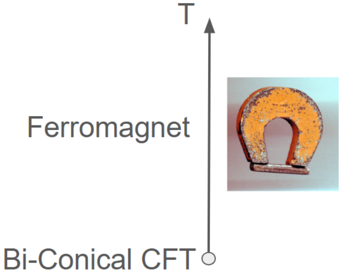
This is a bi-conical fixed point with symmetry

$$O(N) \times \mathbb{Z}_2$$

We can determine the finite temperature behavior by solving the gap equations (Schwinger-Dyson) equations. We find

$$\langle \Psi \rangle \sim 0.8\, T^{1/2} \ .$$

The theory is gapless critical at $T = 0$ and ferromagnetic at any temperature $T \neq 0$.



What are the values of $N$ for which this remarkable behavior occurs?

Indications in [Bilal Hawashin, Junchen Rong, and Michael M. Scherer] that it is valid for $N > 10$.

Some calculations we did suggest $N > 4$.

Can something like this occur in holographic models? See the work of [Buchel].

Finally, let us do something similar for superconductivity. We consider the BCS order parameter

$$\Delta = \frac{1}{N}\sum_{a=1}^{N} \psi_{\uparrow}^{a}\psi_{\downarrow}^{a} \ .$$

Usually we have superconductivity with $\Delta \neq 0$ for

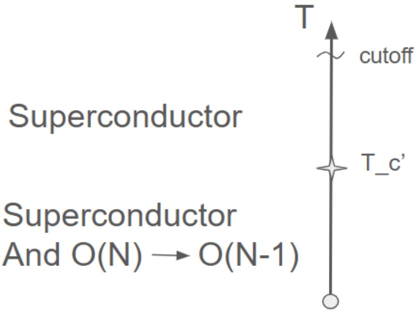
$$T < T_c = \omega_* e^{-\frac{1}{g\nu}}$$

($g$ the attractive interaction and $\nu$ the density of states).

Now assume that the order parameter talks to some internal $O(N)$ degrees of freedom through

$$-\lambda \int d^3x dt \vec{\phi}^2 |\Delta|^2 \ .$$

This model has an interesting phase diagram:



We have provided a general mechanism that allows for typical excited states to be ordered. We have discussed spatial order (solid), magnetic order (ferromagnet), and cooper pairs (superconductor). Some obvious open questions

- Can any of this be realized in real life?
- What is the full phase diagram of the model $n + nn$?
- What about holographic theories? UV complete gauge theories in 3+1 dimensions?
- What is so special about the Bi-Conical model? is this behavior actually generic?

Thank You!