Challenges for Conformal Bootstrap

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Conformal Field Theories

- experimentally important (phase transitions)
- more tractable than other QFTs

Two points of view on CFTs:

- 1) IR fixed points of microscopic theories described by a Lagrangian or a lattice model

conformal symmetry is emergent

- **2) Defined algebro-analytically as systems of correlators of local operators satisfying the operator product expansion**

= conformal bootstrap

conformal symmetry is built in

Polyakov 1974
Belavin, Polyakov, Zamolodchikov 1984

Conformal Bootstrap Algorithm (2008-present)

A physical system (Lagrangian, lattice model…)
giving rise to a unitary CFT in D or (D-1)+1 dimensions

- Symmetry
- A guess about operator spectrum (e.g. relevant scalars)

Split Operators = **Low** + **High**

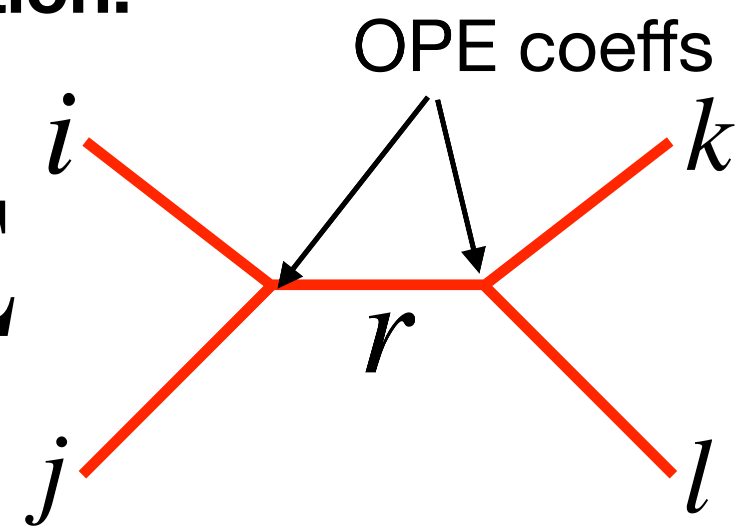
E.g. **Low**= relevant scalars, symmetry currents, stress tensor,…

High = Everyone else

Stop and ask yourself…

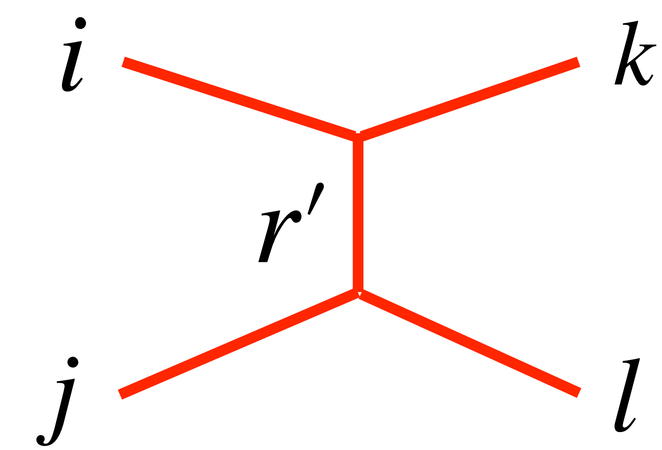
Crossing equation:

$$\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \mathcal{O}_l \rangle = \sum_r$$



"s-channel"

$$= \sum_{r'}$$



"t-channel"

$i, j, k, l \in$ **Low** Exchanged r, r' = **Low+High**

- Equation for functions of $x_1, x_2, x_3, x_4 \in \mathbb{R}^D$

$\Longrightarrow$ reduced by conformal symmetry to $(z, \bar{z}) \in \mathbb{C}$

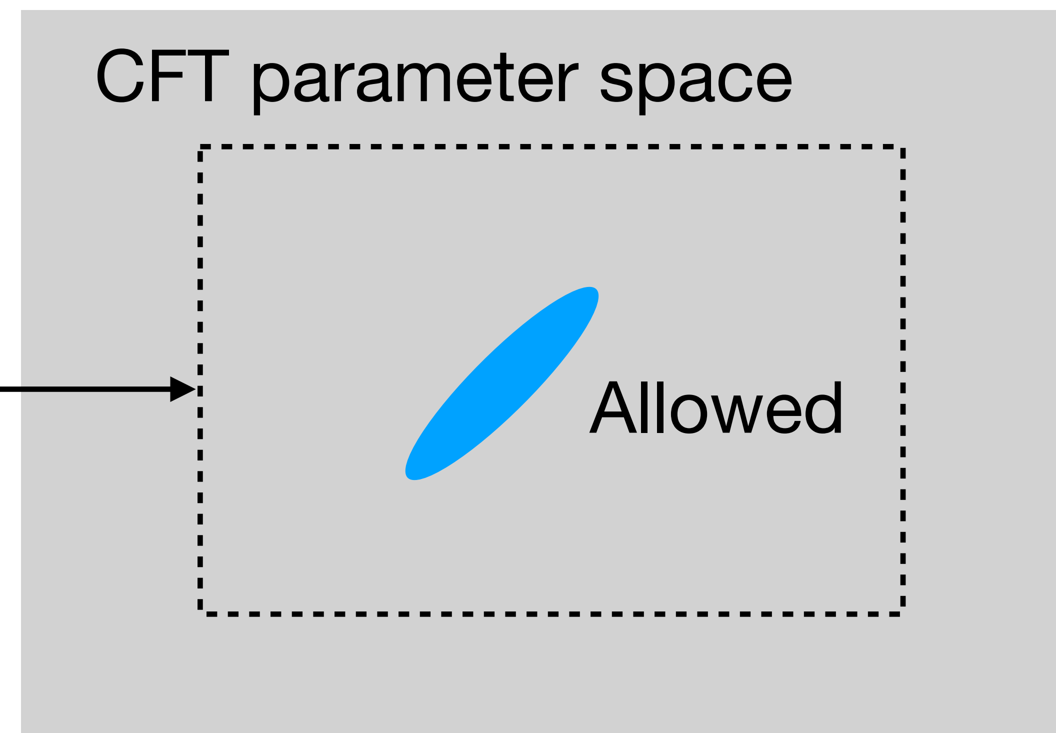
- Expand to order Λ around "half-way" point $z = \bar{z} = 1/2$ where
  - both sides converge exponentially fast
  - terms have good positivity properties

Rattazzi, **SR**, Tonni, Vichi
JHEP **0812** (2008) 031

Expansion order $\Lambda \lesssim 1000$ depending on your computer

“Scan” over the **Low** operators and their OPE coefficients
to find **allowed regions**
(defined as regions where *some* **High** operators exist
so that crossing holds, i.e. marginalizing over **High**)

“Guess” from microscopics



Smart ways of scanning

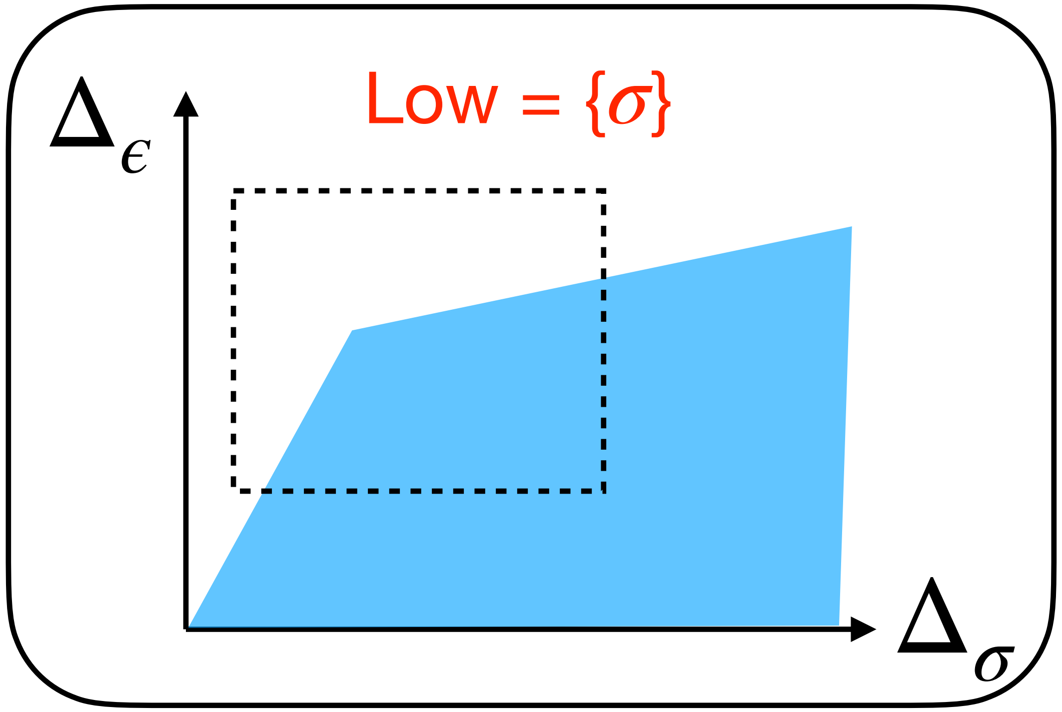
(**Ning Su**)

- cutting surface algorithm
- navigator function, …

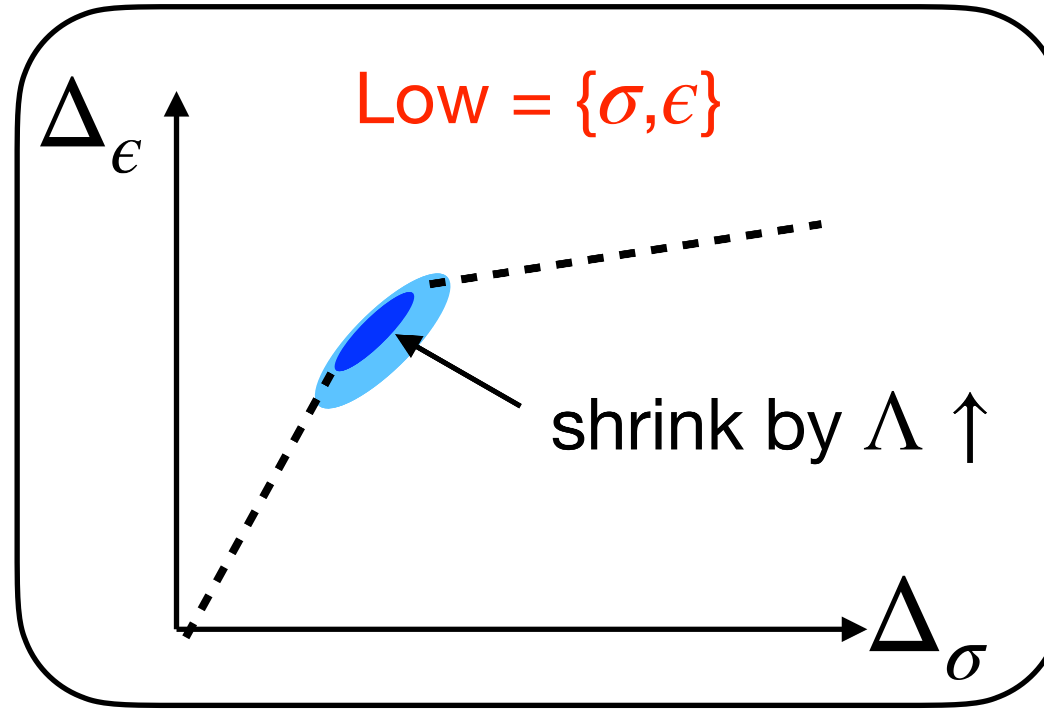
- Analysis is rigorous because of positivity (LP, SDP)
- Allowed regions always shrink imposing more constraints (higher Λ, more **Low** operators)

Highlight 1: 3D Ising CFT

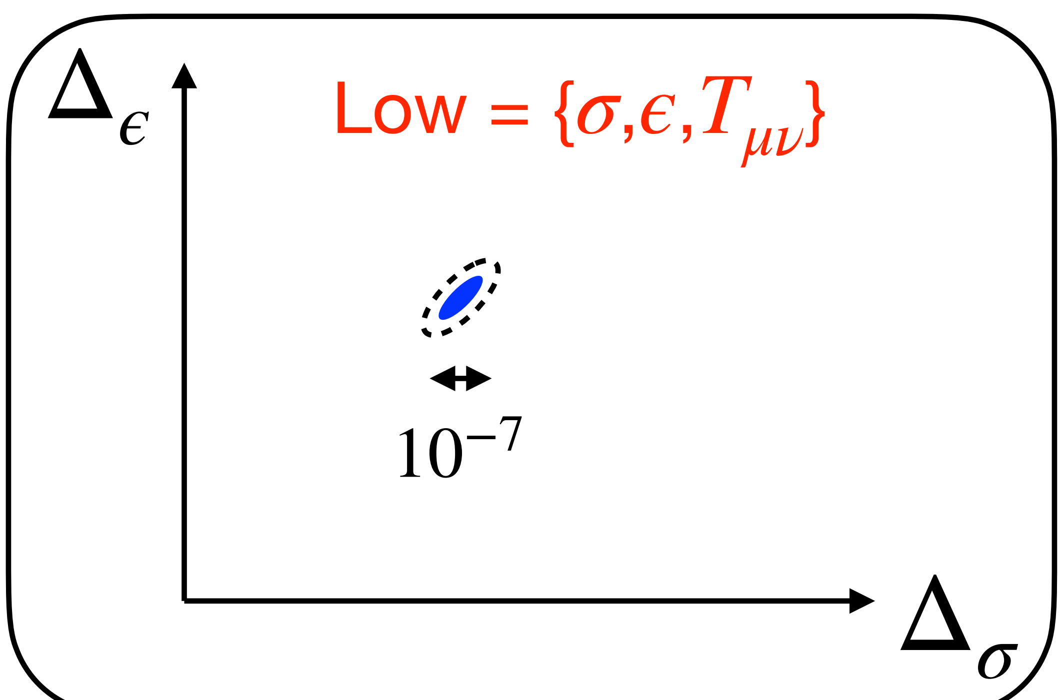
$\mathbb{Z}_2$ symmetry. Two relevant scalars: σ ($\mathbb{Z}_2$ odd), ϵ ($\mathbb{Z}_2$ even)



El-Showk, Paulos, Poland, **SR**, Simmons-Duffin, Vichi 2012, 2014



Kos, Poland, Simmons-Duffin 2014, 2016

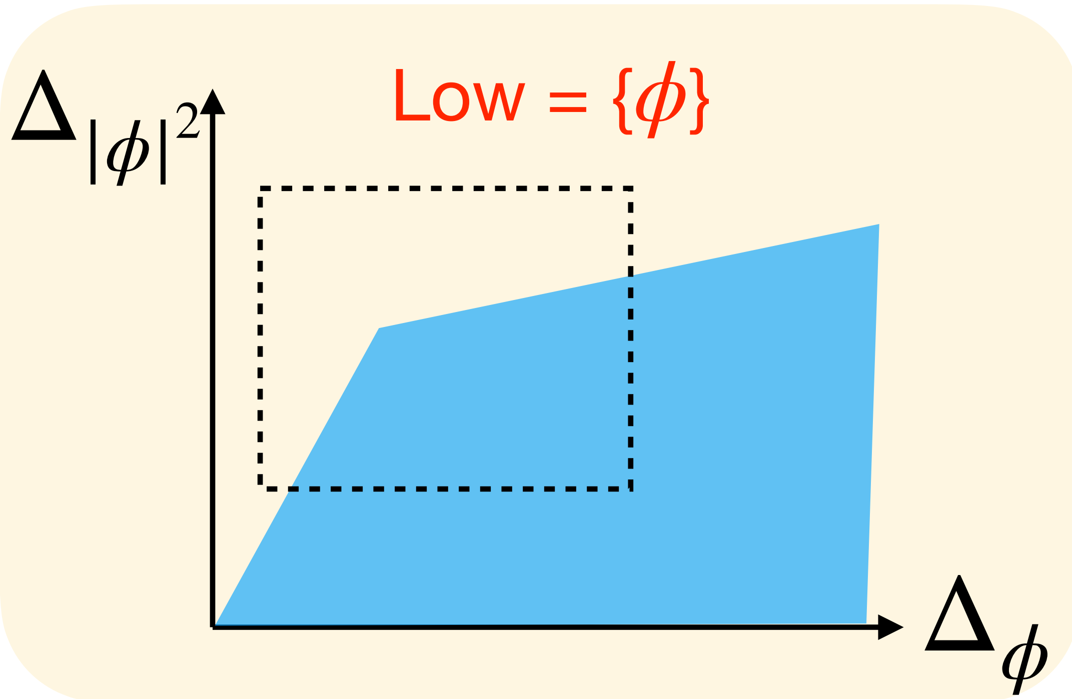


$$\Delta_\sigma = 0.518148806(24)$$
$$\Delta_\epsilon = 1.41262528(29)$$

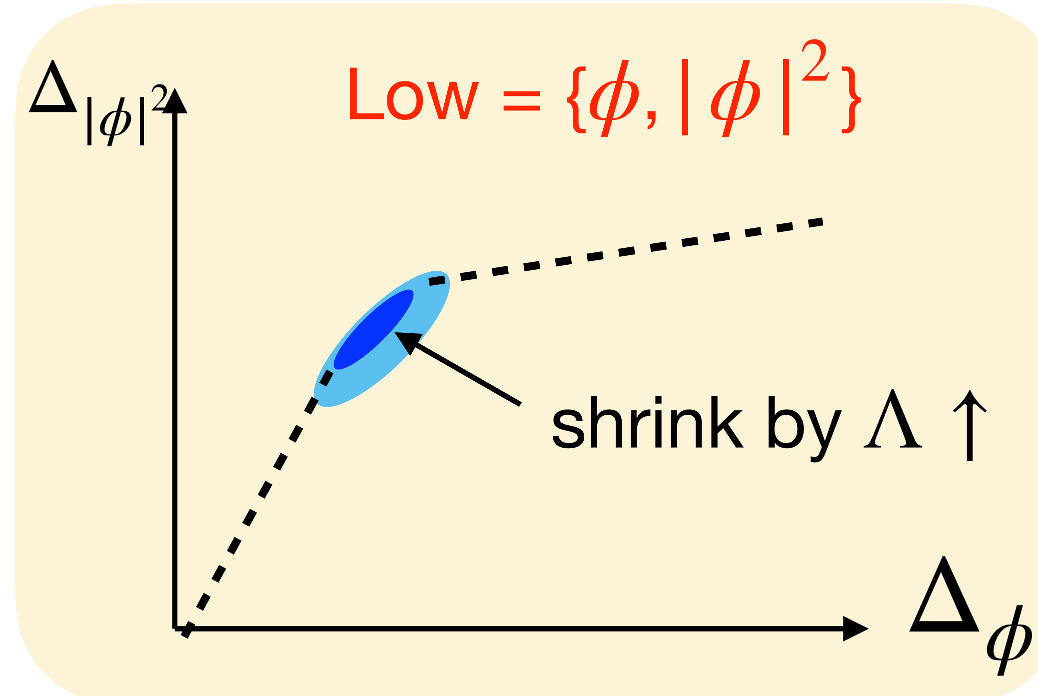
Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin 2024

Highlight 2: 3D XY model CFT

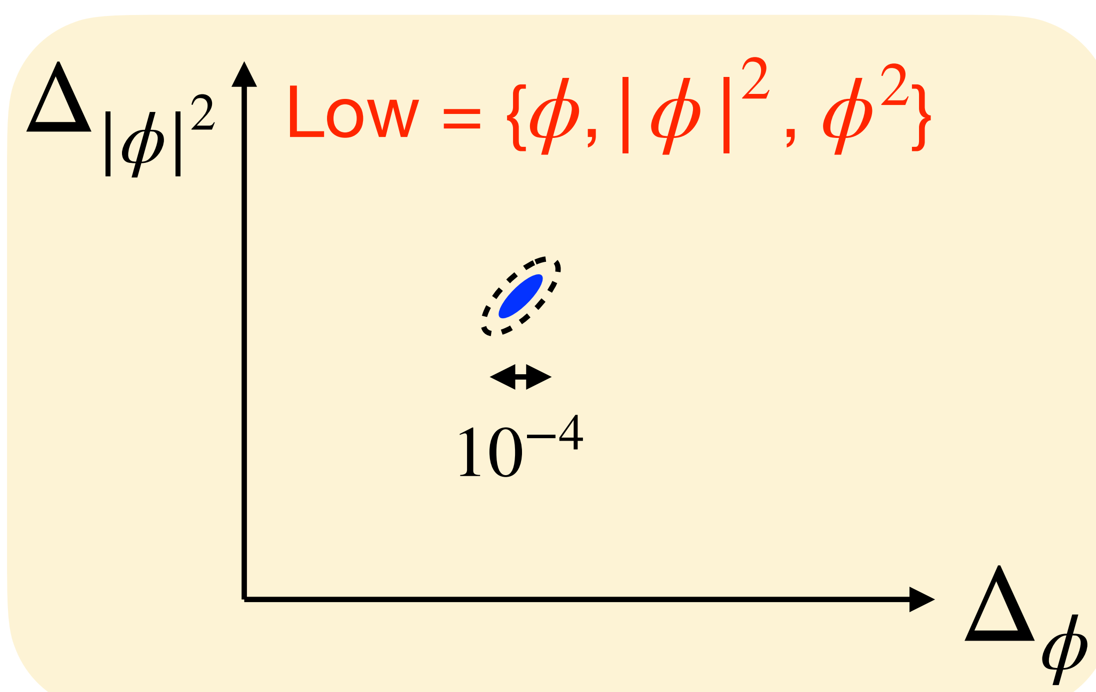
O(2) symmetry. 4 relevant scalars: $\phi, |\phi|^2, \phi^2, \phi^3$



Kos, Poland, Simmons-Duffin 2013



Kos, Poland, Simmons-Duffin, Vichi 2015

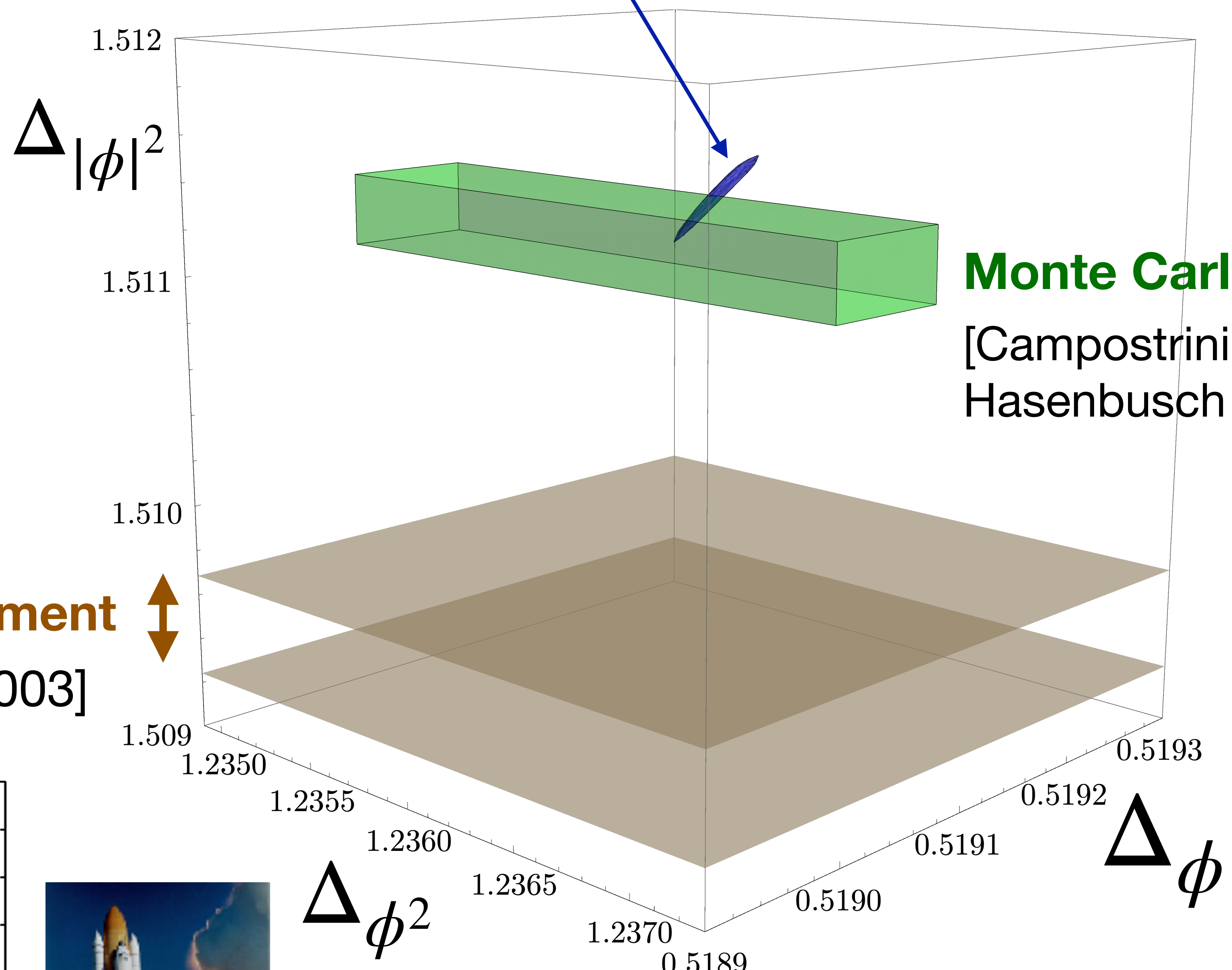


$$\Delta_\phi = 0.519088(22)$$

$$\Delta_{|\phi|^2} = 1.51136(22)$$

Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi 2020

Bootstrap [Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi, 2019]

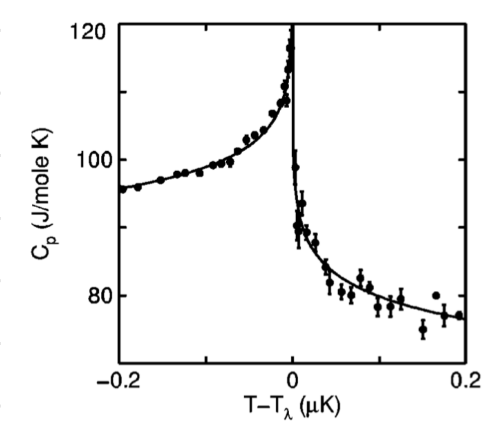


Monte Carlo

[Campostrini et al 2006, Hasenbusch 2019]

He-4 experiment

[Lipa et al, 2003]



Similar results for other “scalar” or “fermionic” CFTs, such as:

- O(N) models, $N \geq 3$

  Kos, Poland, Simmons-Duffin, Vichi 2015
  Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi 2021

- Gross-Neveu-Yukawa models

  Erramilli, Iliesiu, Kravchuk, Liu, Poland, Simmons-Duffin, Su, Vichi 2023

$$\mathcal{L}_{\rm GNY} = -\frac{1}{2}(\partial\phi)^2 - i\frac{1}{2}\psi_i \partial\!\!\!/ \psi_i - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 - i\frac{g}{2}\phi\psi_i\psi_i.$$

- $\mathcal{N} = 1$ SUSY-Ising model

  Rong, Su 2018
  Atanasov, Hillman, Poland, Rong, Su 2022

All these models were isolated into small closed regions of CFT parameter space

With more resources we can keep shrinking those regions

Selected open problems

1) Uniqueness/Non-existence problems

2) Bootstrapping gauge theories

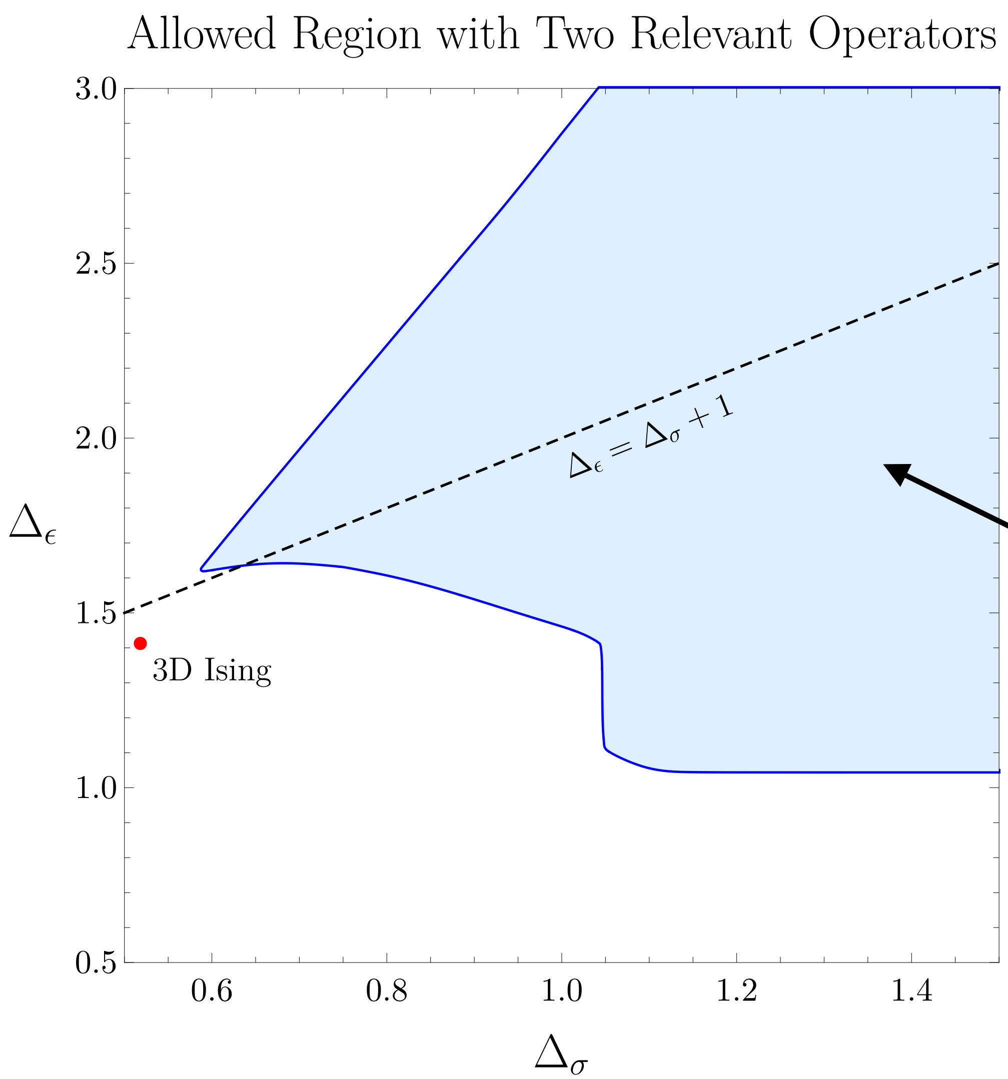
3) ’Large Δ problem’ => “analytic functional bootstrap”?

1A) Uniqueness problem

Is 3D Ising CFT unique?

Experiments suggests **yes**
(if not we'd see Ising magnets/liquid-vapor critical points with other exponents)

Can we show this rigorously via bootstrap?

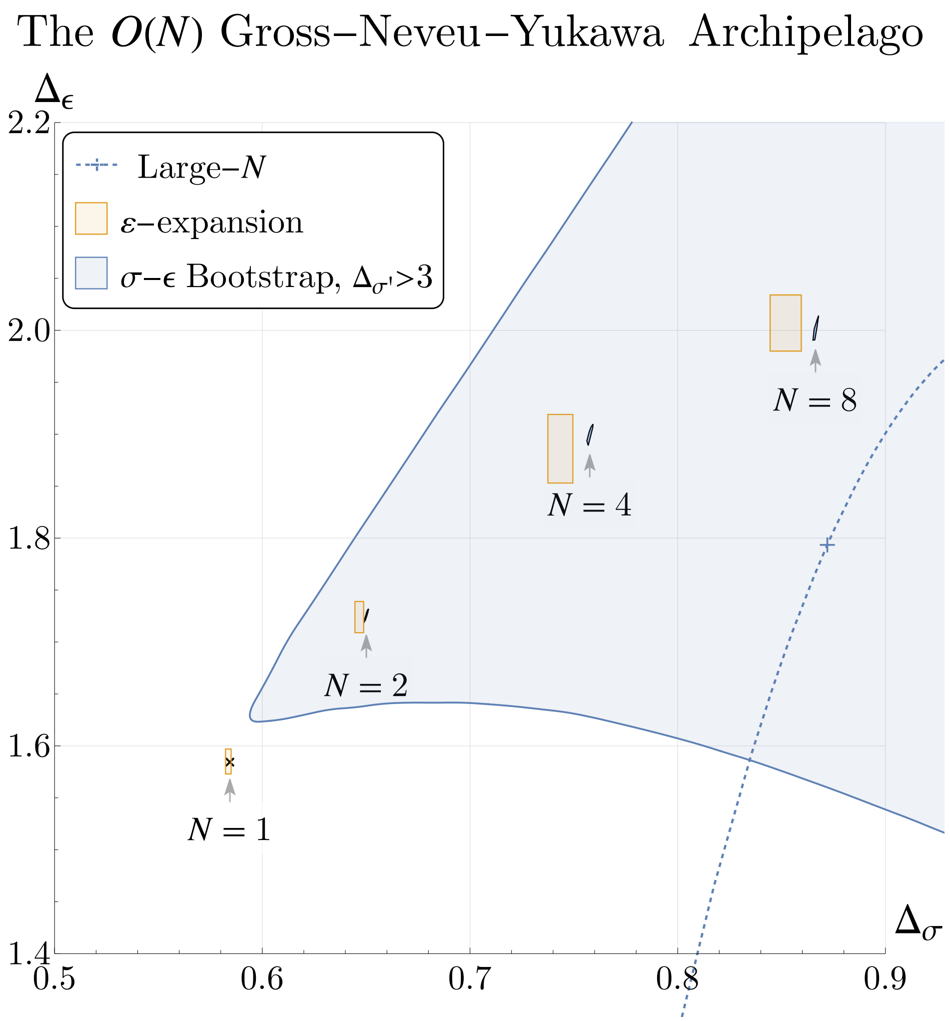


Atanasov, Hillman, Poland 2022

What's the meaning of this region?

Gross-Neveu-Yukawa model masquerading as Ising

$$\mathcal{L}_{\mathrm{GNY}} = -\frac{1}{2}(\partial\phi)^2 - i\frac{1}{2}\psi_i \partial\!\!\!/ \psi_i - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 - i\frac{g}{2}\phi\psi_i\psi_i$$



Under spatial parity

$$P: \psi\psi \to -\psi\psi, \quad \phi \to -\phi$$

Erramilli, Iliesiu, Kravchuk, Liu, Poland, Simmons-Duffin, Su, Vichi 2023

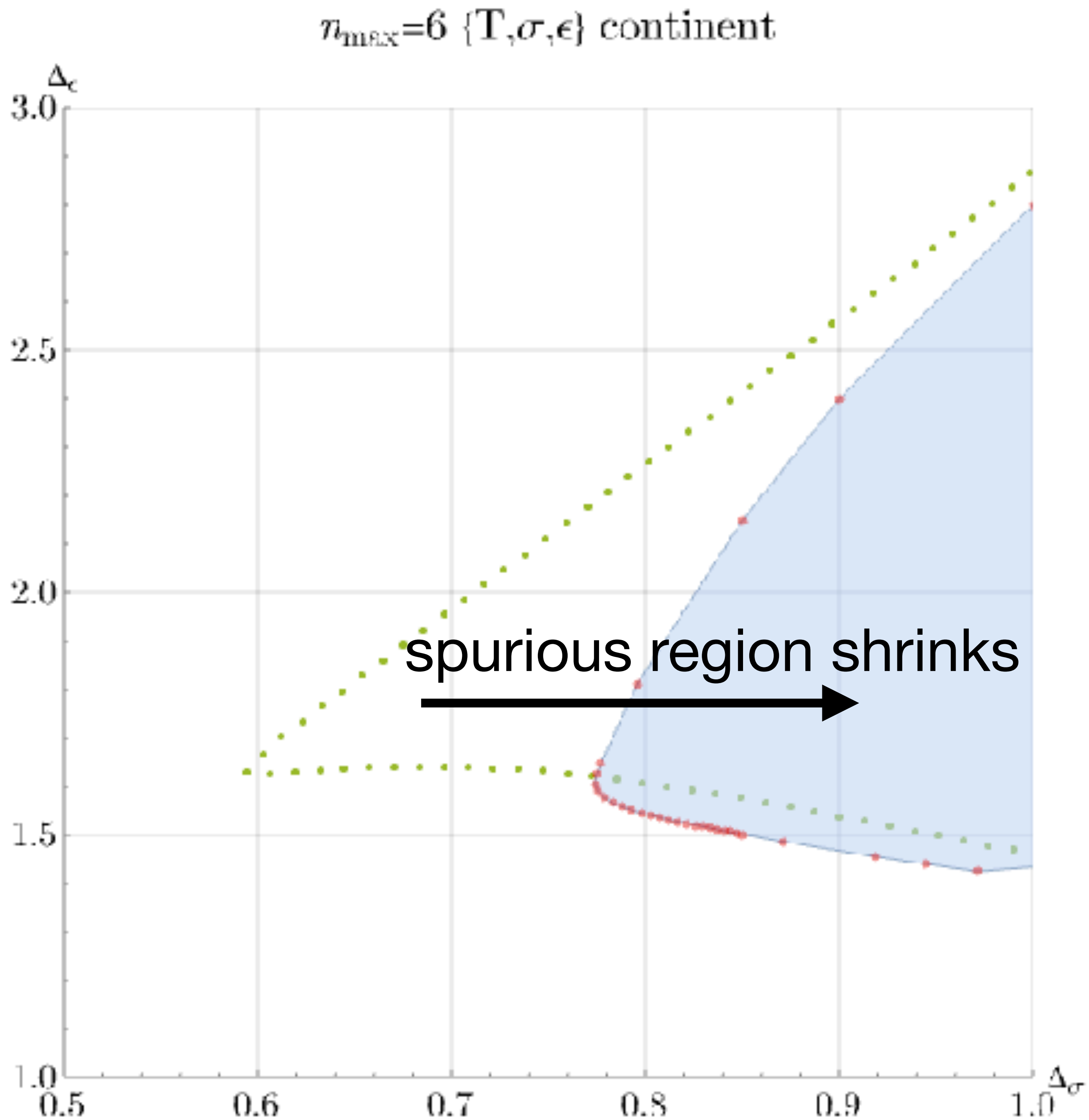
- For Low = $\{\phi, \phi^2\}$, *P* is indistinguishable from Ising $\mathbb{Z}_2$

- May be distinguished for Low = $\{\phi, \phi^2, T_{\mu\nu}\}$

  $T \times T \supset \phi$ in GNY but not in Ising

Ising and GNY may be distinguished for Low = $\{\phi, \phi^2, T_{\mu\nu}\}$

$T \times T \supset \phi$ in GNY but not in Ising



deltaGaps

spin=0, parity even, Z2Even	3.0
spin=0, parity even, Z2Odd	3.0
spin=0, parity odd, Z2Even	3.0
spin=2, parity even, Z2Even	4.0

Erramilli, Iliesiu, Kravchuk, Liu, Poland, Simmons-Duffin, Su, Vichi, unpublished

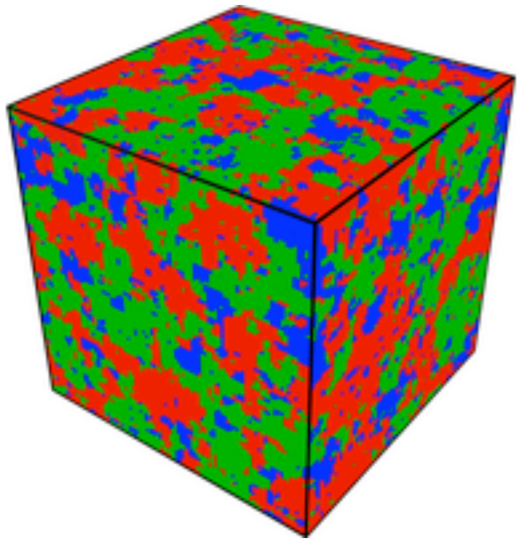
Open problem: Can the spurious region be eliminated altogether?

1B) Non-existence problem

For some models experiments and Monte Carlo suggest 1st order transition

But one can never quite exclude 2nd order in a slightly modified model.

A **proof** can be obtained by showing that there is no CFT with requisite symmetry.



Simplest case: 3-state Potts model in D=3

Lattice Monte Carlo: correlation length $\xi \sim 10$

[Janke, Villanova 1997]

Bootstrap open problem:
Show that there is no unitary 3D CFT
- with S_3 global symmetry
- a single relevant singlet scalar
- one (or more) scalars in the fundamental of S_3

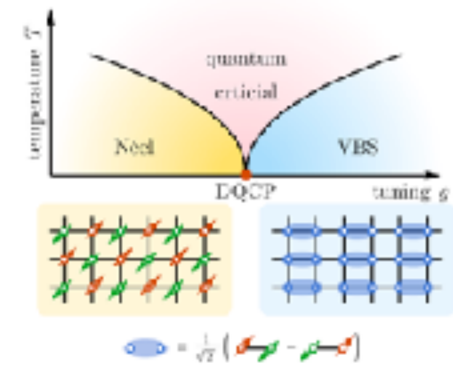
https://sites.google.com/site/slavarychkov/open-problems-in-conformal-bootstrap

2) Bootstrapping 3D conformal gauge theories

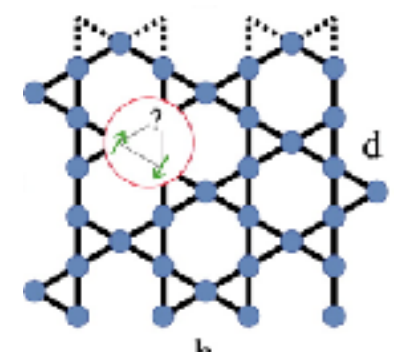
QED3 (bosonic/fermionic) = 3D U(1) Maxwell field + N_f bosons/fermions

Global symmetry: $G \simeq SU(N_f) \times U(1)_{top}$

Bosonic QED3 $N_f = 2$ ⟷ Deconfined Quantum Critical Point



Fermionic QED3 $N_f = 4$ ⟷ Dirac Spin Liquid



Herbertsmithite

Symmetry breaking CFT

$N_f^* = \,?$ N_f

Can we bootstrap these CFTs and determine N_f^* ?

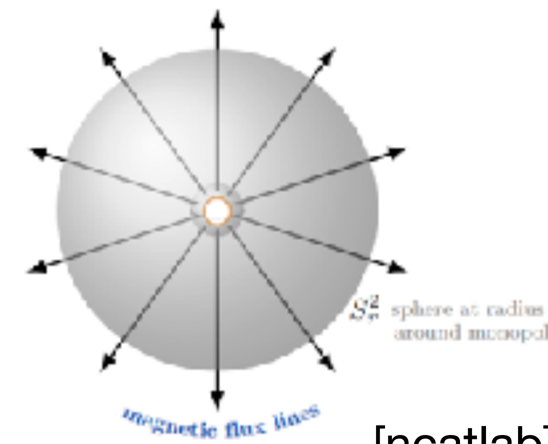
Features of QED3

- Physical CFT operators are gauge invariant combinations of elementary fields

$$\bar\psi_i \psi_j, \quad \phi_i^* \phi_j, \ldots \qquad \cancel{\psi, \quad \phi}$$

- There are also "monopole operators" charged under $U(1)_{top}$

$$\Delta_q \propto N_f$$



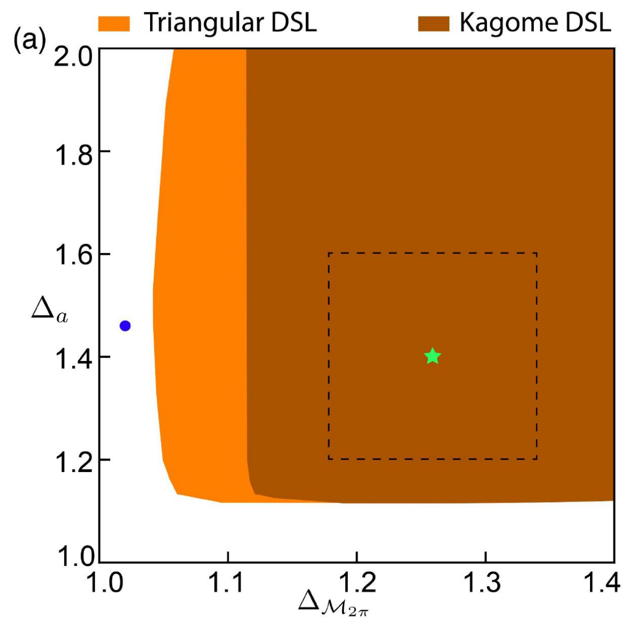
[ncatlab]

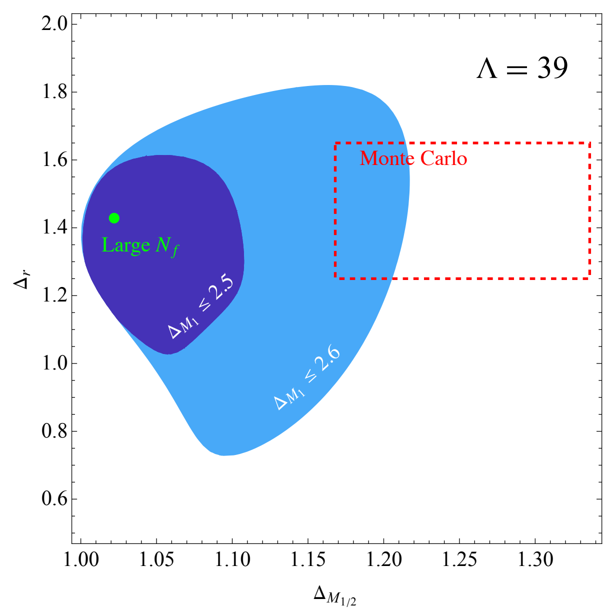
Difficulties:

- These operators are heavier than the lightest scalars in scalar/fermionic CFTs
  - expect slower convergence as $\Lambda \uparrow$ (cf "large Δ problem")

- How to distinguish from "QCD3" theories where the gauge group is $U(N_c)$?

Various bootstrap bounds on QED3 were derived but these CFTs were not yet isolated into small closed regions





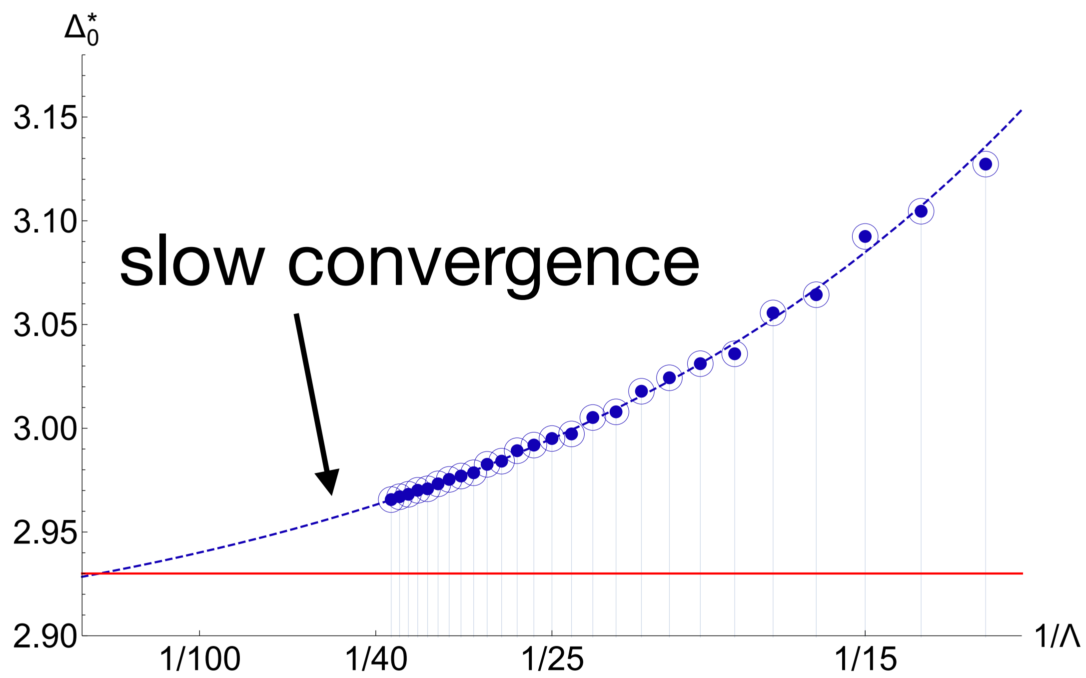
see [**SR**, Su RMP 2024] for a discussion

[Reehorst, Refinetti, Vichi 2020, He, Rong, Su 2021] identified gaps in the operator spectrum (“decoupling operators”) which could help distinguish QED3 from QCD3

(color indices allow for more antisymmetrization)

3) Large Δ problem

= Slow $\Lambda\uparrow$ convergence for bounds on correlators of large-Δ operators



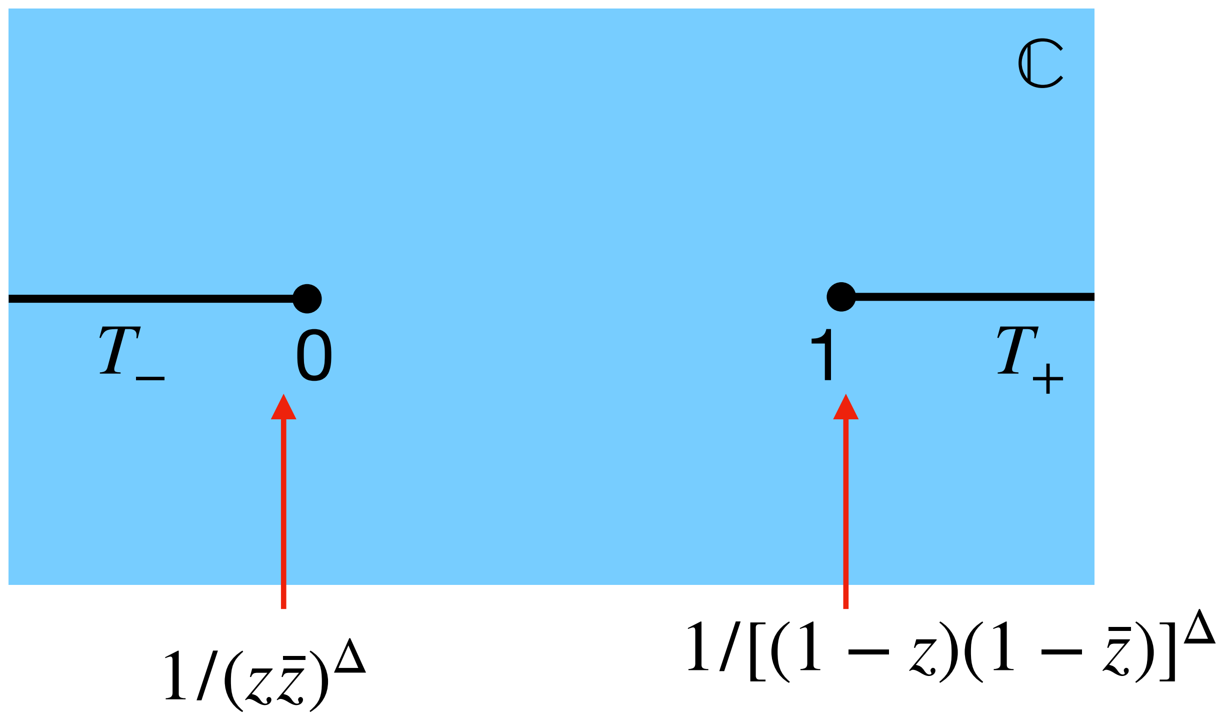
Upper bound on 1st unprotected scalar for $\mathcal{N}=4$ SCFT with c=3/4 (Konishi in SYM with SU(2) gauge group)

Beem, Rastelli, van Rees 2016

From 4pt function of protected scalar in $\mathbf{20'}$ of “large” dimension $\Delta=2$

Origin of large Δ problem

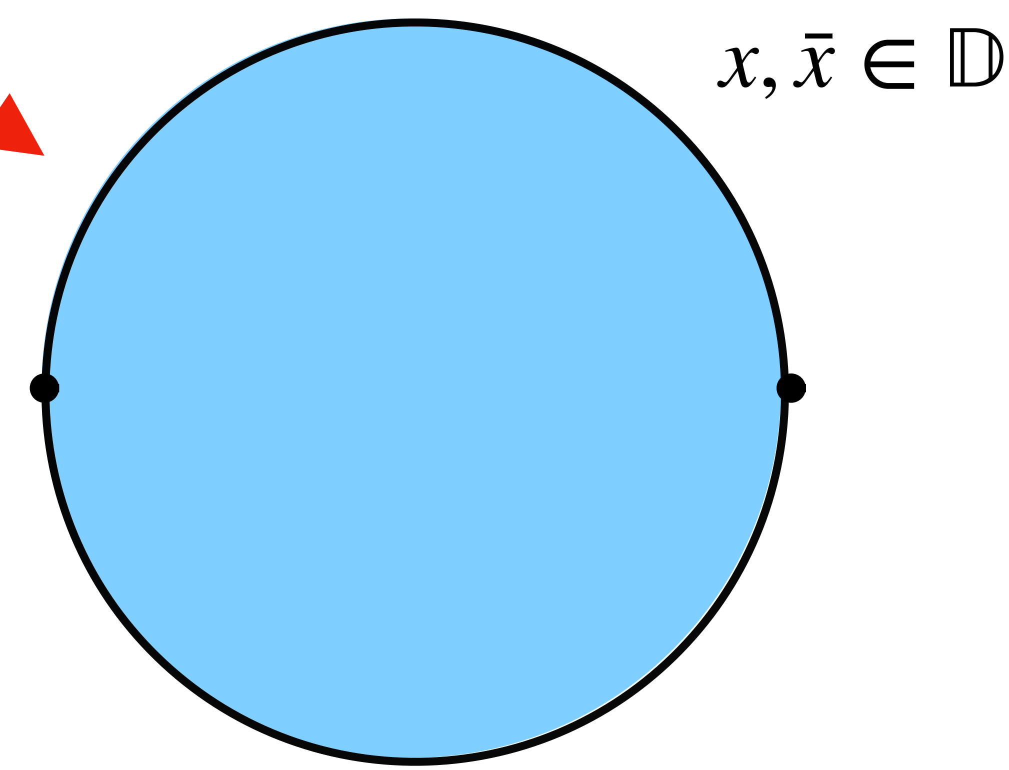
CFT 4pt function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle$ is analytic for $z, \bar{z} \in \mathbb{C} \backslash (T_+ \cup T_-)$

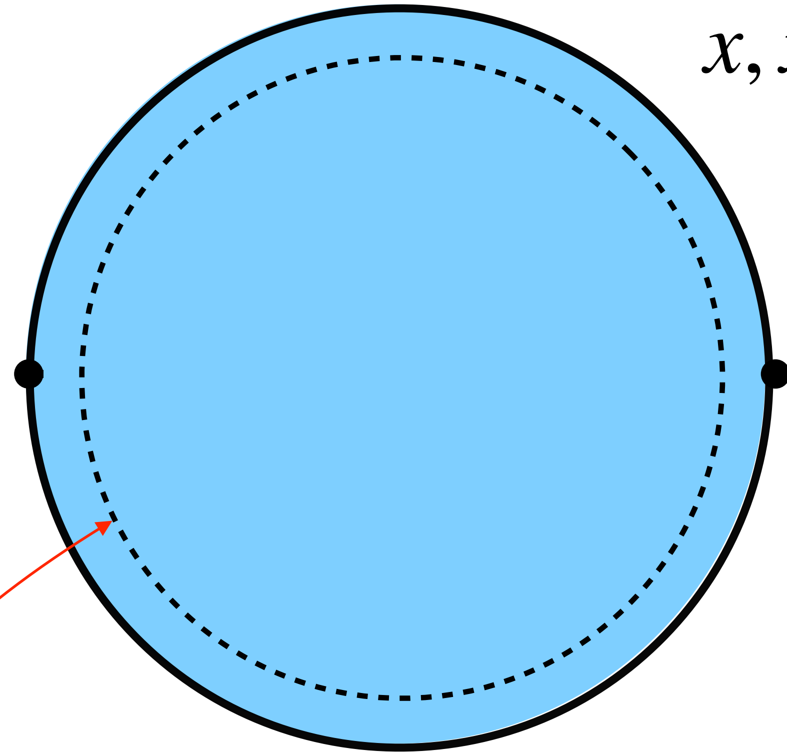


s-channel analytic in $\mathbb{C} \backslash T_+$

t-channel analytic in $\mathbb{C} \backslash T_-$

conformal map





$x, \bar{x} \in \mathbb{D}$

Expand crossing equation around $x = \bar{x} = 0$
= act on it with “derivative functionals”

$$\partial_x^n \partial_{\bar{x}}^m \big|_{x=\bar{x}=0} \quad n + m \leq \Lambda$$

(that’s standard way since our 2008 work)

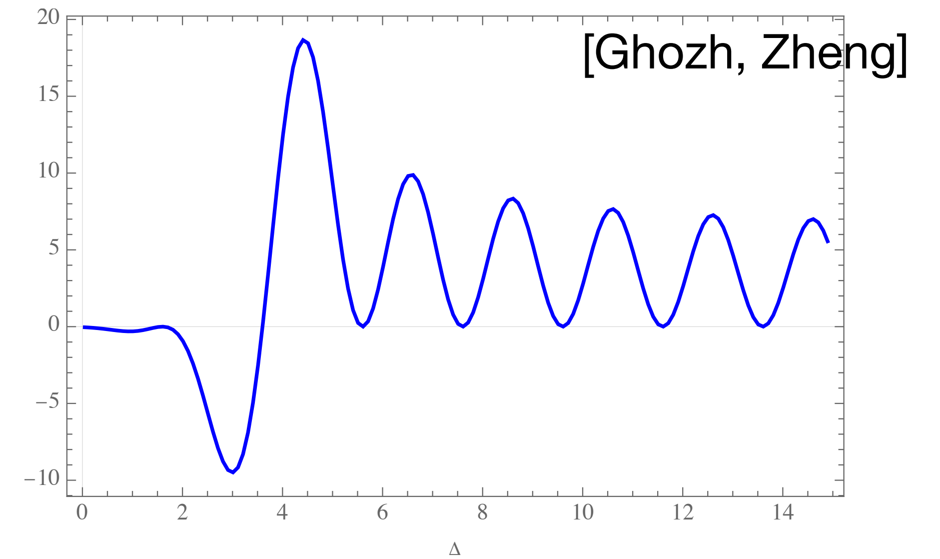
The largest class of functionals given the analyticity domain
can be obtained as contour integrals pushed to the boundary.

“analytic functionals” Mazac 2016

Mazac, Paulos 2018

Such functionals can be expanded in “derivative functionals”
but convergence becomes slow for large Δ because of s,t-channel sing’s

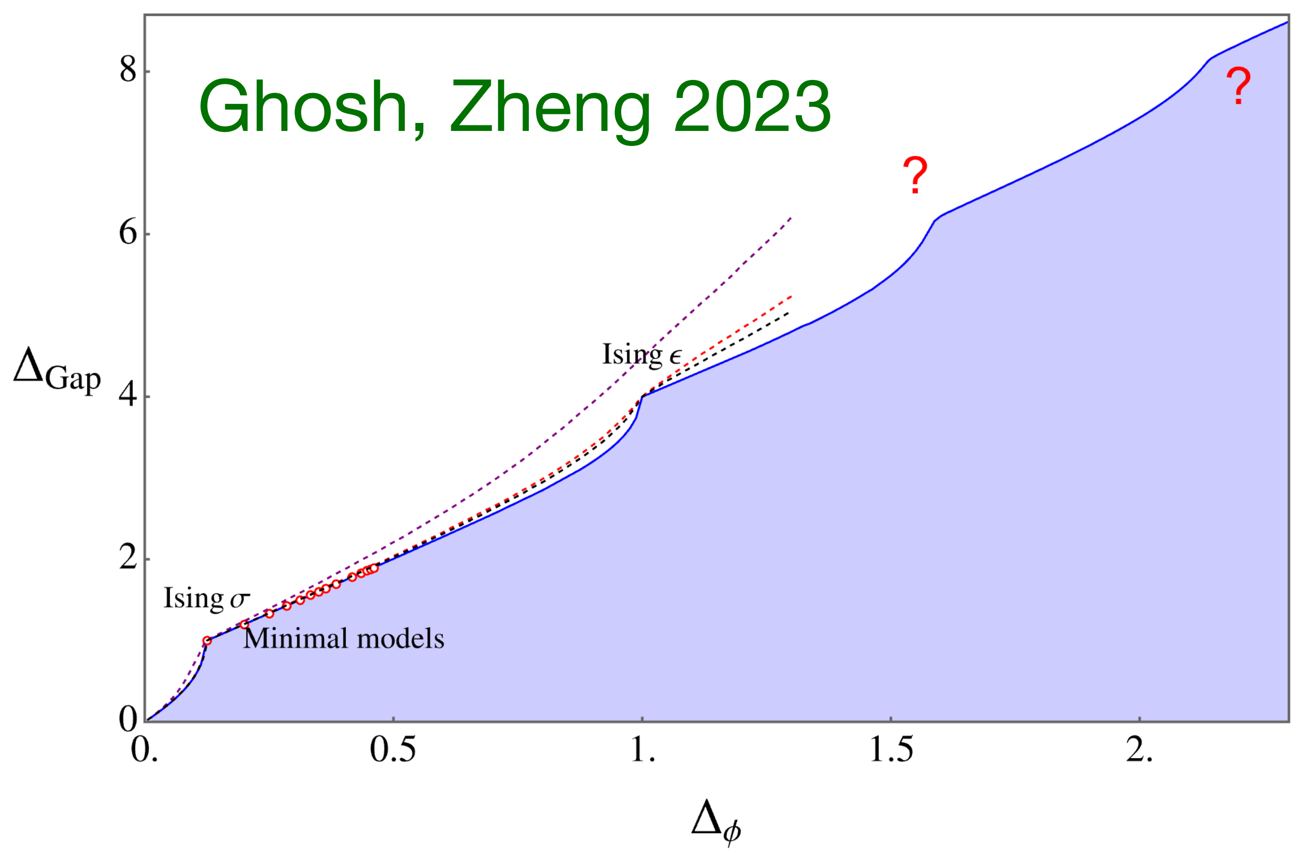
Analytic functionals



Have various magic properties,
give exact solutions to some max gap problems

More generally, *a faster-convergent basis for numerical bootstrap calculations*
Paulos, Zan 2019; Ghosh, Zheng 2023

D=2:



D=3 - works (Ghosh, Zheng 2023) but need more efficient implementation

A future of numerical bootstrap?

Selected open problems

1) Uniqueness/Non-existence problems

2) Bootstrapping gauge theories

3) 'Large Δ problem' => "analytic functional bootstrap"?

Thank you!