Cavity Geometry & Figures of Merit

Tom Krokotsch, 4.12.2024

Idea

- The coupling coefficients as we use them are not 'scale' independent
  - $\Rightarrow$ Suggests false statements like $P_{sig} \propto U_0$
- The GW → Vibration → EM interaction only happens in/near the wall and should not depend on the fields in the entire volume
- Find 'figures of merit' that quantify the effect of the geometry on GW sensitivity
- Use figures of merit to find best possible sensitivities that a MAGO-type cavity can reach taking all dependencies (more carefully) into account

Note: the following is just a reformulation, not a correction. But I think it helps to identify how a better detector design should look like. So far, quantities like U_0 were not well separated from the coupling coefficients

Equations of Motion

Vibration-EM Coupling

$$\sqrt{U_1}\left(\ddot{b}_1 + \frac{\omega_1}{Q_1}\dot{b}_1 + \omega_1^2 b_1\right) = \frac{1}{2}\omega_1^2 \frac{q_m b_0}{\mu_0\sqrt{U_1}} \int_A d\boldsymbol{A}\cdot\boldsymbol{\xi}_m \left(\boldsymbol{B}_0\cdot\boldsymbol{B}_1 - \boldsymbol{E}_0\cdot\boldsymbol{E}_1\right),$$

$$\ddot{q}_m + \frac{\omega_m}{Q_m}\dot{q}_m + \omega_m^2 q_m = -\frac{1}{2}\ddot{h}_{ij}\int_{V_{\rm wall}} \frac{dV}{V_{\rm wall}}\xi^i x^j,$$

GW-Vibration Coupling

$$\Longrightarrow P_{sig} \propto \omega_1\, Q_{cpl}\, V_{cav}\, B_0^2\, (C\Gamma)^2\, h^2$$

Previously we defined:

$$\Gamma_m^{ij} = \frac{1}{V_{\rm cav}^{1/3} M}\int_{V_{\rm cav}} d^3x\,\rho(\vec{x})\,x^i\cdot\xi_m^j(\vec{x})$$

$$C_{01}^m = \frac{V_{\rm cav}^{1/3}}{2\sqrt{U_1 U_0}}\int_{\partial V_{\rm cav}} d\vec{S}\cdot\vec{\xi}_m(\vec{x})\left(\frac{1}{\mu_0}\vec{B}_1^*\cdot\vec{B}_0 - \epsilon_0\vec{E}_1^*\cdot\vec{E}_0\right)$$

GW-Vibration Coupling

old:

$$\Gamma_m^{ij} = \frac{1}{V_{\text{cav}}^{1/3} M} \int_{V_{\text{cav}}} d^3x \rho(\vec{x}) x^i \cdot \xi_m^j(\vec{x})$$

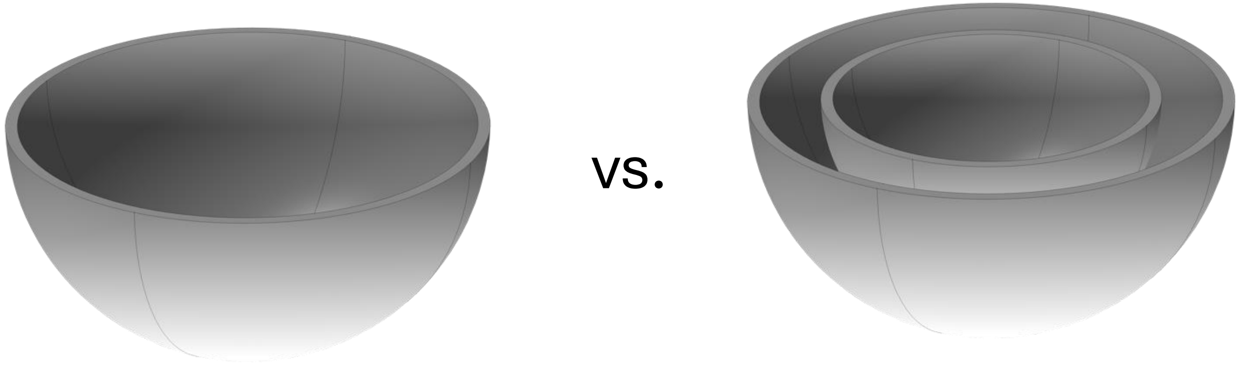
new:

$$\Gamma_m^{ij} = \frac{\int_{V_{\text{wall}}} dV \rho \, \xi^i x^j}{\sqrt{\int_{V_{\text{wall}}} dV \rho \, |\boldsymbol{x}|^2}}$$

Effective Radius of the Cavity R_{eff}

$\Rightarrow$

E.g. concentric geometries -> GW coupling should be similar



Old Definition: $\propto V^{-\frac{1}{3}}$ (can even diverge for $R_i \to R_o$)

New Definition: **Approx. same coupling**

- $V^{-\frac{1}{3}}$ cancels would cancel in the C_{01} definition, but the sensitivity often scales with Γ alone
- In new definition $q_{vib} \propto R_{eff} \Gamma^{ij} h_{ij}$

Vibration-EM Coupling

old:

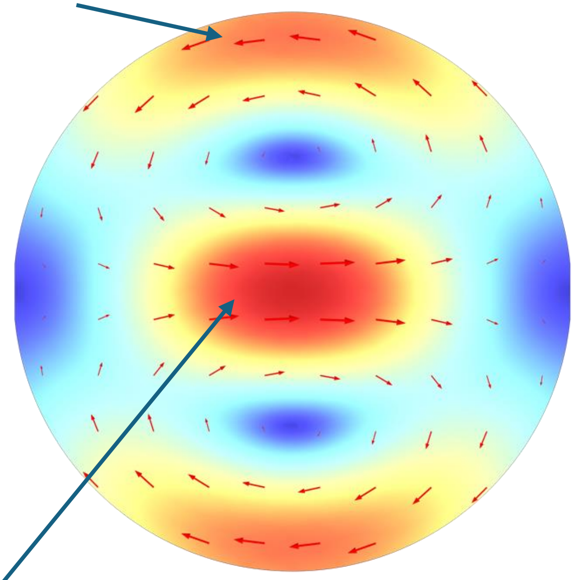
$$C_{01}^{m} = \frac{V_{\text{cav}}^{1/3}}{2\sqrt{U_1 U_0}} \int_{\partial V_{\text{cav}}} d\vec{S} \cdot \vec{\xi}_m(\vec{x}) \left(\frac{1}{\mu_0} \vec{B}_1^* \cdot \vec{B}_0 - \epsilon_0 \vec{E}_1^* \cdot \vec{E}_0 \right)$$

new:

$$C_{01}^{m} := \frac{\frac{1}{A} \int_A d\boldsymbol{A} \cdot \boldsymbol{\xi}_m \left(\boldsymbol{B}_0 \cdot \boldsymbol{B}_1 - \boldsymbol{E}_0 \cdot \boldsymbol{E}_1 \right)/2}{\underbrace{\|\boldsymbol{B}_0\|_S \, \|\boldsymbol{B}_1\|_S}}$$

Surface averaged fields

Increases Coupling



Does nothing to the coupling

$\Rightarrow$ $P_{sig} \propto U_0$ is false.

$\Rightarrow$ Actually: $P_{sig} \propto \|\boldsymbol{B}_0\|_S^2$

$\Rightarrow$ Again, concentric geometries do better

Rewritten Signal Power

With the new coefficients, the signal PSD becomes:

Critical field of Niobium

$$S_{\text{sig}}(\omega) = \frac{1}{2}\frac{\beta}{(1+\beta)^2} Q_{\text{int}}\omega_1 \frac{(R_{\text{eff}}A_S)^2}{V}\frac{B_c^2}{\mu_0}\frac{\|\boldsymbol{B}_0\|_S^2}{\max\limits_{\boldsymbol{x}\in S}|\boldsymbol{B}_0(\boldsymbol{x})|^2}\frac{\|\boldsymbol{B}_1\|_S^2}{\|\boldsymbol{B}_1\|_V^2} S_{C_{01}^m\Gamma_m^{ij}h_{ij}}(\omega-\omega_0)$$

Large extend of cavity & surface good. Large volume bad → All needs to be excited

The $\frac{\|\boldsymbol{B}_1\|_S^2}{\|\boldsymbol{B}_1\|_V^2}$ term suggests that high fields away from wall are bad for sensitivity.

However, $Q_{\text{int}} = \frac{\omega_1}{R_S}\frac{V}{A_S}\frac{\|\boldsymbol{B}_1\|_V^2}{\|\boldsymbol{B}_1\|_S^2}$ i.e. what's good for Vib-EM coupling is bad for quality factor.

On the other hand, Q_{int} only matters if we don't overcouple

Figures of Merit of Cavity Geometry

Optimal signal power:

Critical coupling: $\mathcal{F}^{ij}_{\mathrm{crit}} := \omega_1 R_{\mathrm{eff}} \sqrt{A_S} \frac{\|\boldsymbol{B}_0\|_S}{\max_{\boldsymbol{x}\in S} |\boldsymbol{B}_0(\boldsymbol{x})|} |C^m_{01}| \Gamma^{ij}_m ,$

Overcoupling: $\mathcal{F}^{ij}_{\mathrm{over}} := \sqrt{\omega_1} \frac{R_{\mathrm{eff}} A_S}{\sqrt{V}} \frac{\|\boldsymbol{B}_0\|_S}{\max_{\boldsymbol{x}\in S} |\boldsymbol{B}_0(\boldsymbol{x})|} \frac{\|\boldsymbol{B}_1\|_S}{\|\boldsymbol{B}_1\|_V} |C^m_{01}| \Gamma^{ij}_m ;$

Usually $\omega_1 \propto 1/\sqrt{A_S}$

Optimal sensitivity:

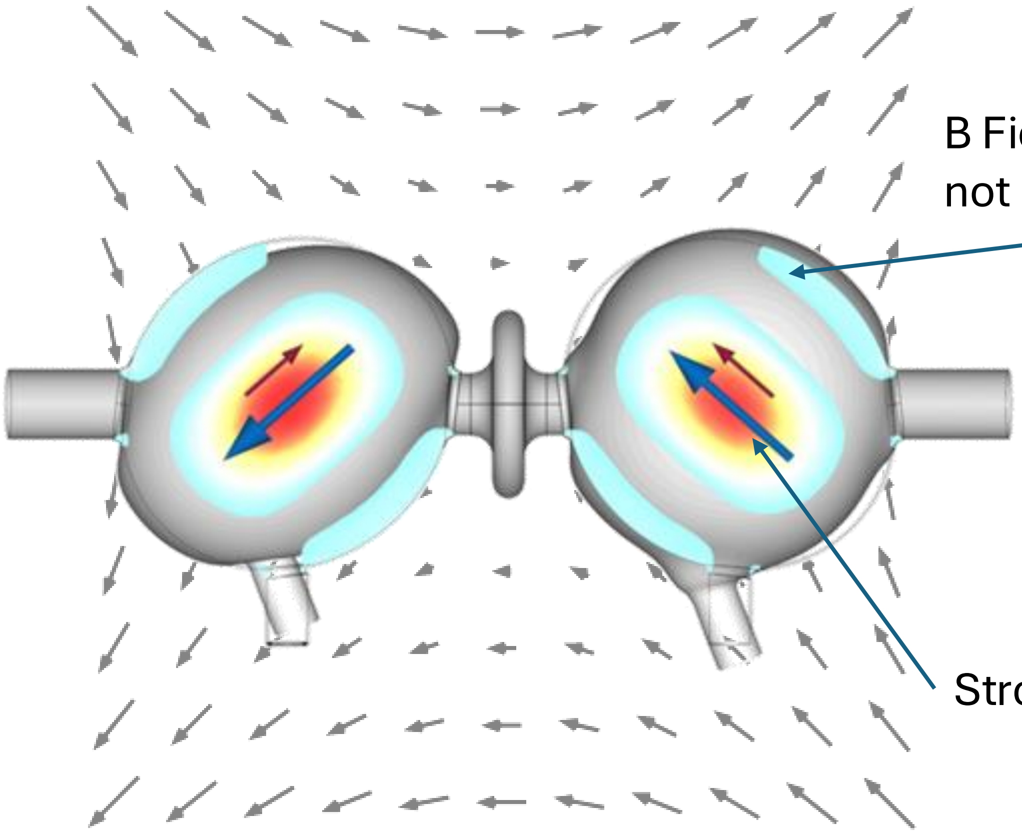
Thermal noise in cavity: $\mathcal{F}^{ij}_{\mathrm{crit}}$

SQL readout noise: $\mathcal{F}^{ij}_{\mathrm{r}} = \mathcal{F}^{ij} / \sqrt{\omega_1}.$

Thermal or mechanical vibrations: $\mathcal{F}^{ij}_{\mathrm{th.}\,m} = R_{\mathrm{eff}} \Gamma^{ij}_m .$

The remaining parameters **surface resistance** R_S, **mass** M, **critical field** B_c, **temperature** T don't need to be part of an optimization process and don't (directly) depend on the geometry.

Example for a geometry



B Fields on the wall not homogeneous

Strongest fields in center

$R_{eff} = 0.17$ m

$A_S = 0.26\ \mathrm{m}^2$

$\Gamma_{\max} = 0.3$

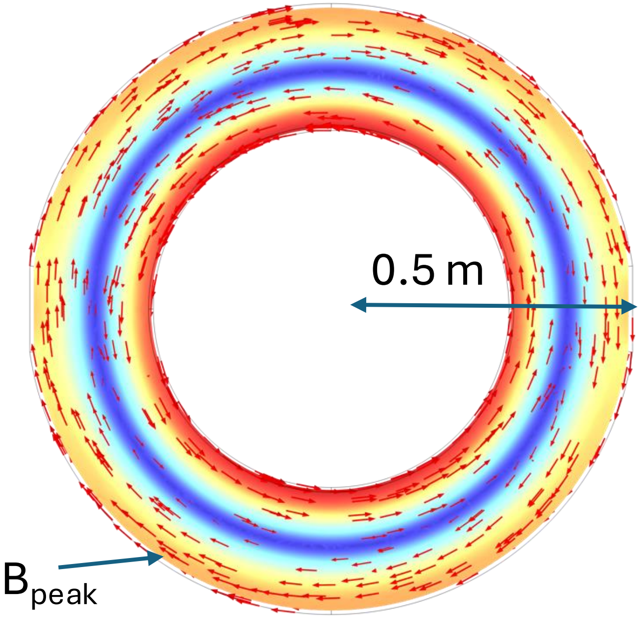
$C_{01} = 0.15$

$$\frac{\|\boldsymbol{B}_0\|_S}{\max_{\boldsymbol{x}\in S} |\boldsymbol{B}_0(\boldsymbol{x})|} = 0.6$$

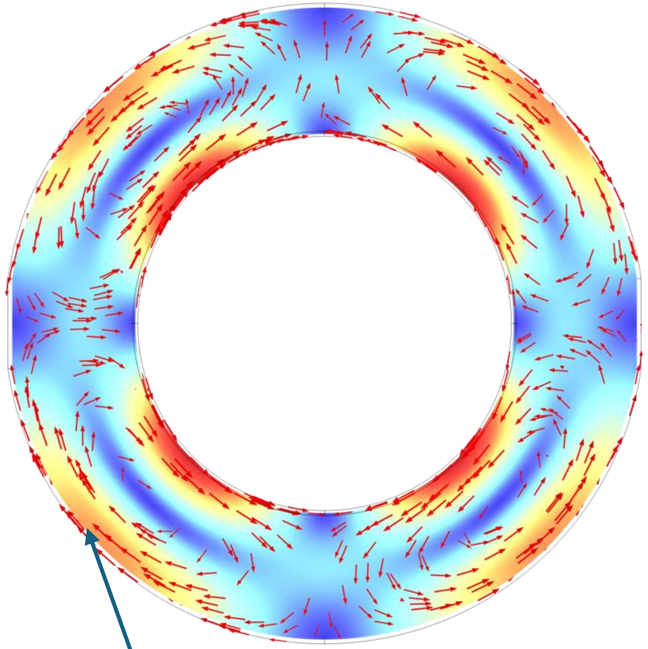
$$\frac{\|\boldsymbol{B}_1\|_S}{\|\boldsymbol{B}_1\|_V} \ll 1$$

Γ coupling to GW ok, but not perfect
C_{01} is comparatively high!

Concentric Cylinder



$B_0 = B_{peak}$



Surface Average greater than volume average

ω_0, ω_1 dont depend on R_{eff}

$R_{eff} = 0.4$ m

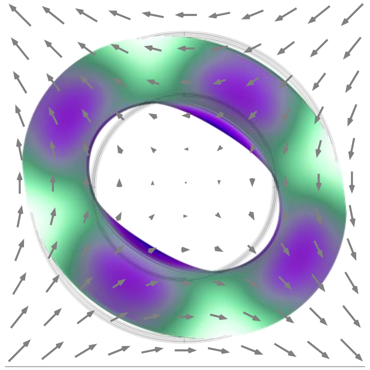
$A_S = 2\ \mathrm{m}^2$

$\Gamma_{\max} = 0.74$

$C_{01} = 0.1$

$$\frac{\|\boldsymbol{B}_0\|_S}{\max_{\boldsymbol{x}\in S} |\boldsymbol{B}_0(\boldsymbol{x})|} = 0.73$$

$$\frac{\|\boldsymbol{B}_1\|_S}{\|\boldsymbol{B}_1\|_V} = 1.12$$

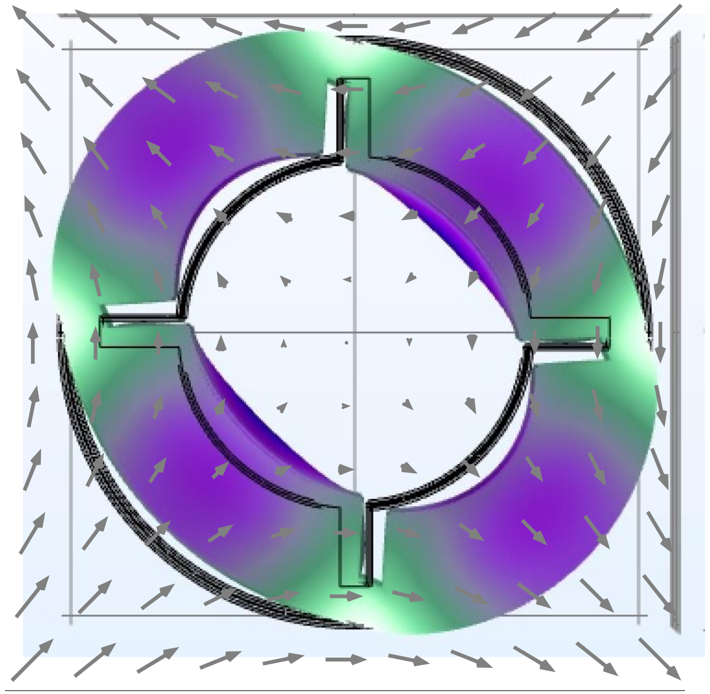


Γ coupling to GW as good as it gets

Disadvantage: Modes only for >> MHz GWs possible

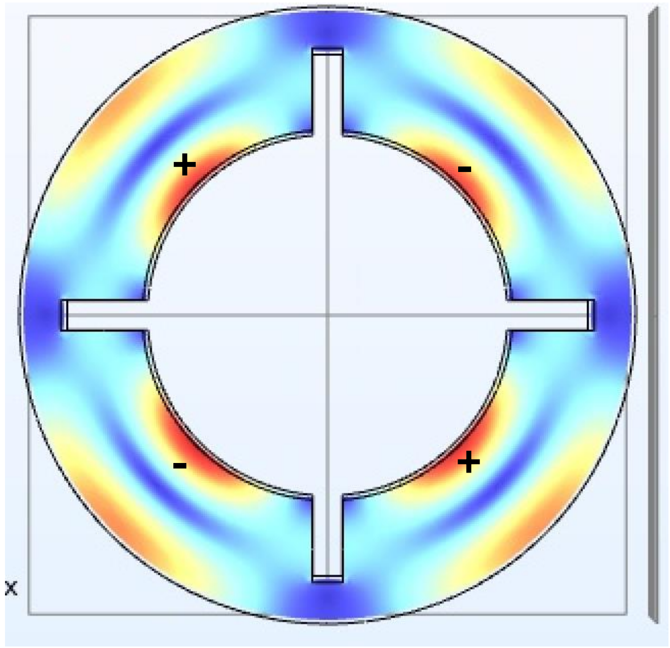
Sectioned Concentric Cylinder

Mechanical Displacement



Γ coupling to GW as good as it gets

Signal B Field



$TM_{110\,(sym)} \rightarrow TM_{110\,(anti\text{-}sym)}$

Similar performance as previous but frequency spacing arbitrarily small

$$R_{eff} = 0.4 \text{ m}$$

$$A_S = 2 \text{ m}^2$$

$$\Gamma_{\max} = 0.8$$

$$C_{01} = 0.11$$

$$\frac{\|\boldsymbol{B}_0\|_S}{\max_{\boldsymbol{x}\in S} |\boldsymbol{B}_0(\boldsymbol{x})|} = 0.5$$

$$\frac{\|\boldsymbol{B}_1\|_S}{\|\boldsymbol{B}_1\|_V} = 1.1$$

Could also have more than 4 sections. Not all need to be coupled

→ Increased sensitivity due to more cavities & improved directional sensitivity

→ In most noise limited cases no loss of sensitivity due to smaller surface area

Conclusion

- Rewriting the signal power helps identifying how an ideal cavity should look like
- Examples of concentric geometries support this interpretation
- Figures of merit help setting upper bounds on how well the detection concept could possibly function → in progress
- A similar analysis can be done for heterodyne Gertsenshtein interactions → in progress with Lars