Numerical evaluation of two-loop QCD helicity amplitudes for $gg \to t\bar{t}g$ at leading colour

Towards NNLO QCD corrections

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Based on: arXiv:2412.13876

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(Thanks to Gaia Fontana for the wonderful drawings)

Theory-Experimental Top Quark Mass Workshop

DESY, Hamburg
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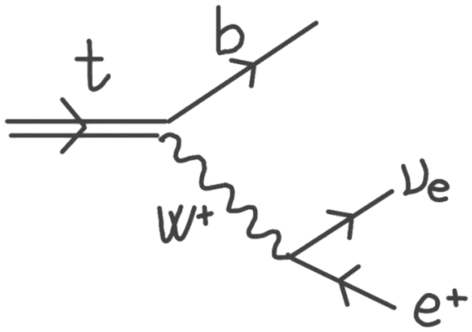
INTRODUCTION

Motivations for $t\bar{t}j$ production
& 2→3 amplitudes frontier

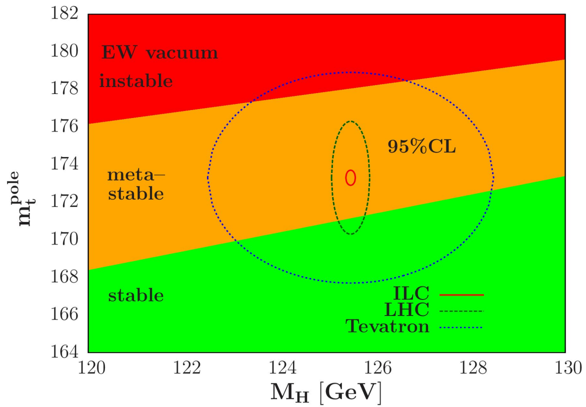
The top quark

- To-date **heaviest** fundamental particle

- **Decays** before hadronising
- Spin correlation



- Affects the **EW vacuum stability**

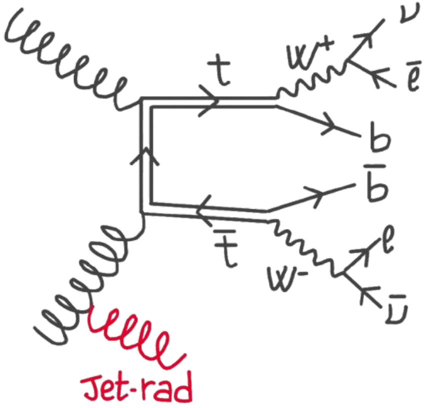


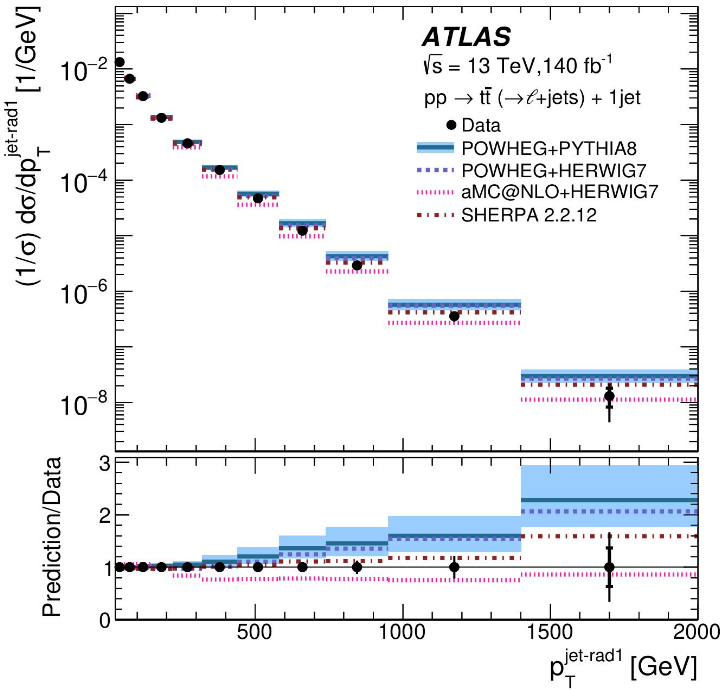
[Alekhin, Djouadi, Moch ’12]

$t\bar{t}$+jet production at the LHC

50% of $t\bar{t}$ events produced at LHC are associated with a jet

$t\bar{t}$+jet is crucial for **precision tests** of the Standard Model and for extracting the **top quark mass**



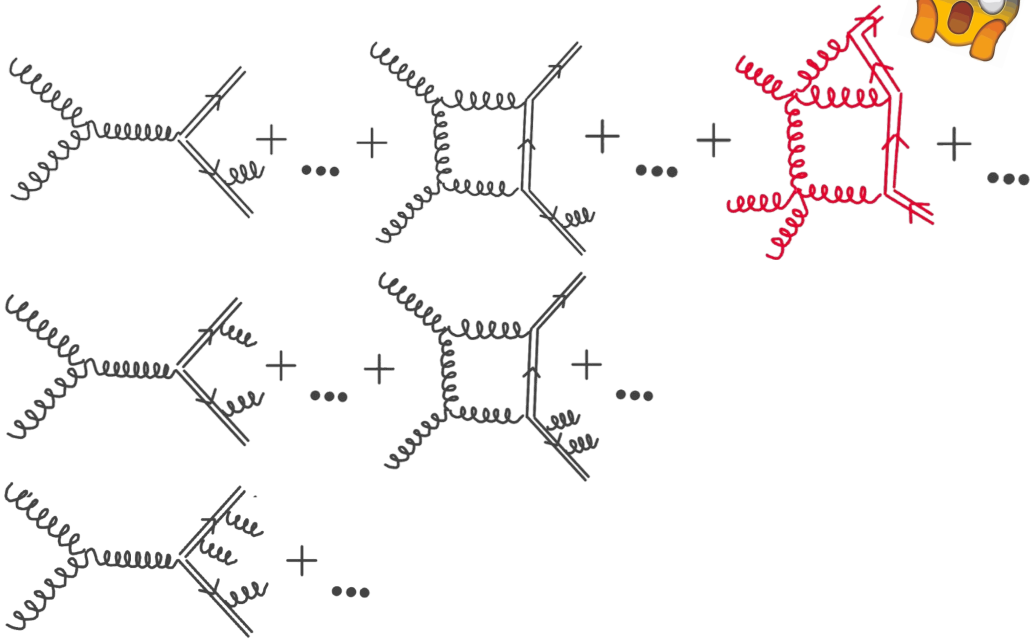


[ATLAS collaboration, '24]

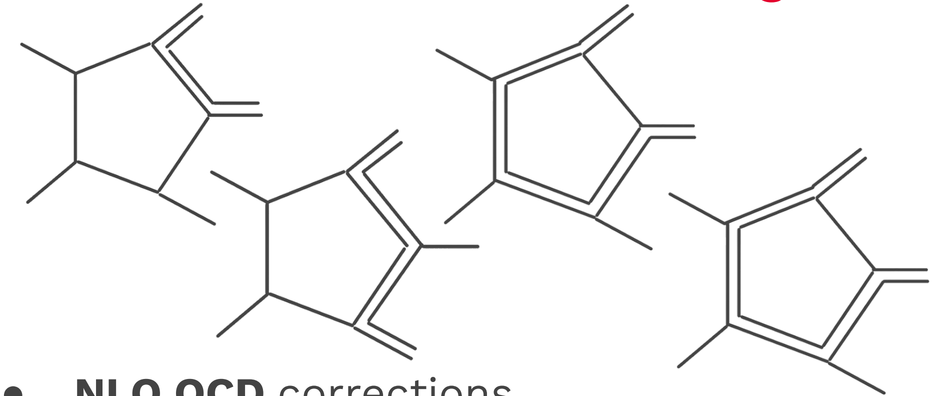
Towards NNLO QCD corrections tt̄+jet

Cross section = Amplitude + Phase space integration

$$\sigma_n^{NNLO} = \int_n d\sigma^B + d\sigma^V + d\sigma^{VV} + \int_{n+1} d\sigma^R + d\sigma^{RV} + \int_{n+2} d\sigma^{RR}$$

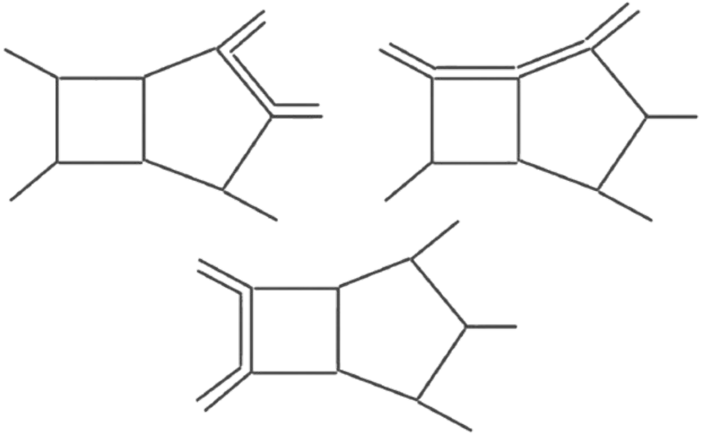


Current status of $t\bar{t}$+jet corrections



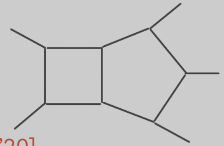
- **NLO QCD** corrections [Dittmaier, Uwer, Weinzierl, ’07]
- Full off-shell decays and interfaces with parton shower [Melnikov and Schulze ’10] [Alioli, Moch, Uwer ’12] [Bevilacqua, Czakon, Hartanto, Kraus, Worek ’15-’16]
- Mixed QCD and EW corrections [Gütschow, Lindert, Schönherr ’18]

- **NNLO QCD** corrections needed ➡ initial steps towards this challenge
  [Badger, Becchetti, Chaubey, Marzucca, Sarandrea ’22]
  [Badger, Becchetti, Chaubey, Marzucca ’23]
  [Badger, Becchetti, Giraudo, Zoia ’24]



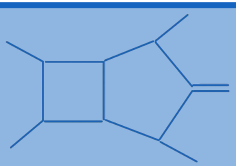
Current frontier of 2 → 3 scattering amplitudes

Massless external particles:



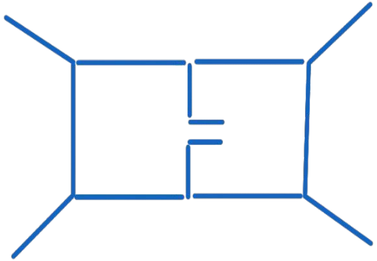
- $pp \to \gamma\gamma\gamma$ [Abreu, Page, Pascual, Sotnikov '20] [Chawdhry, Czakon, Mitov, Poncelet '21] [Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]
- $pp \to \gamma\gamma j$ [Agarwal, Buccioni, von Manteuffel, Tancredi '21] [Chawdhry, Czakon, Mitov, Poncelet '21] [Badger, Brönnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21] [Buccioni, Chen, Feng, Gehrmann, Huss, Marcoli, '25]
- $pp \to \gamma jj$ [Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia '23]
- $pp \to jjj$ [Abreu, Febres Cordero, Ita, Page, Sotnikov '21] [De Laurentis, Ita, Klinkert, Sotnikov '23] [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi '23] [De Laurentis, Ita, Sotnikov '23]

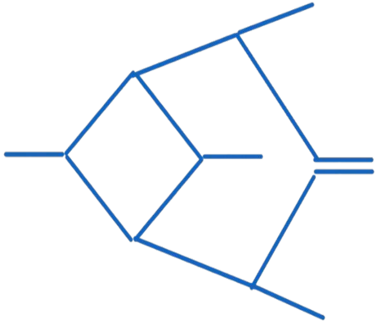
One massive external particle leading colour:



- $pp \to Wb\bar{b}$ [Badger, Hartanto, Zoia '21] [Hartanto, Poncelet, Popescu, Zoia '22]
- $pp \to Wjj$ [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '22]
- $pp \to Hb\bar{b}$ [Badger, Hartanto, Krys, Zoia '21]
- $pp \to W\gamma j$ [Badger, Hartanto, Krys, Zoia '22]
- $pp \to W/Zb\bar{b}$ [Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini '22] [Mazzitelli, Sotnikov, Wiesemann '24]

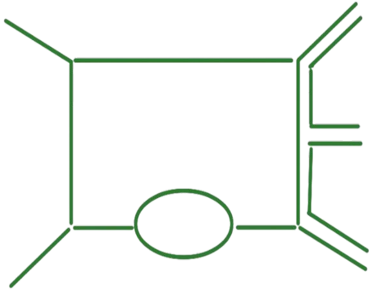
Current frontier of 2 → 3 scattering amplitudes

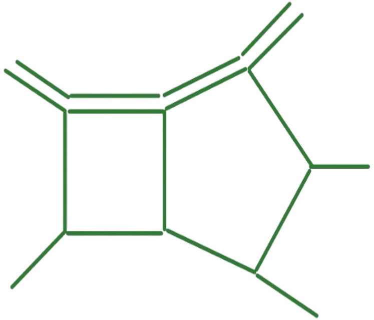




One massive external particle full colour:

- pp → $W\gamma\gamma$ [Badger, Hartanto, Wu, Zhang, Zoia ’24]
- pp → $Hb\bar{b}$ [Badger, Hartanto, Poncelet, Wu, Zhang, Zoia ’24]





More masses:

- pp → $t\bar{t}H$ (nf part) [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson, ’24]
- pp → $t\bar{t}j$ (leading colour) [Badger, Becchetti, Brancaccio, Hartanto, Zoia, ‘24]


INTERNAL MASSES

SCATTERING AMPLITUDES

Introduction to methodology

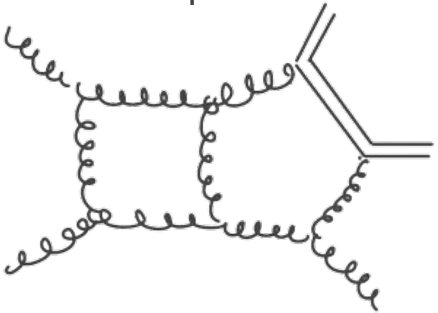
Colour decomposition

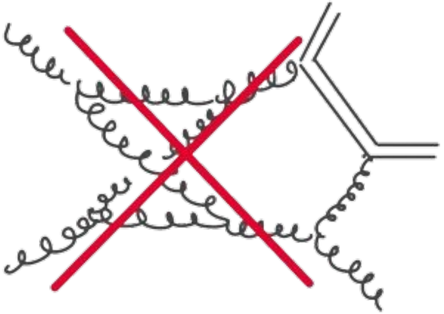
- Consider all diagrams contributing to the process

$$A^{(L)}(\vec{x},\epsilon) = \sum \Big(\quad \Big)$$

- **Colour expansion** ➡ take the leading colour limit
  ➡ reduce the complexity of the loop integrals

At two loops:





diagrams	tot	LC
LO	16	6
NLO	384	77
NNLO	11370	1357

Leading colour ∝ Nc^2 ➡ only planar diagrams

Helicity amplitudes

$$A^{(L)} = \sum_{i=1}^{N} \mathcal{T}_i \, \mathcal{F}_i^{(L)}$$

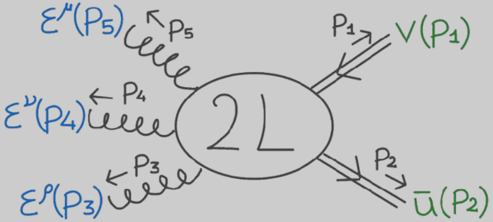
Tensor structure

Ex. gg -> ttg

$$\mathcal{T}_1 = m_t^2 \; \bar{u}(p_2) v(p_1) \varepsilon(p_3, q_3) \cdot p_1 \; \varepsilon(p_4, q_4) \cdot p_1 \; \varepsilon(p_5, q_5) \cdot p_1$$

$\vdots$

$\mathcal{T}_{32} = \ldots$



Form factors

$$\mathcal{F}_i^{(L)} = \sum_{j=1}^{N} \left(\Theta^{-1}\right)_{ij} \underbrace{\sum_{\text{pol.}} \mathcal{T}_j^{\dagger} A^{(L)}}_{A^{(L),\text{proj}}}$$

with $\Theta_{ij} = \mathcal{T}_i^{\dagger} \mathcal{T}_j$

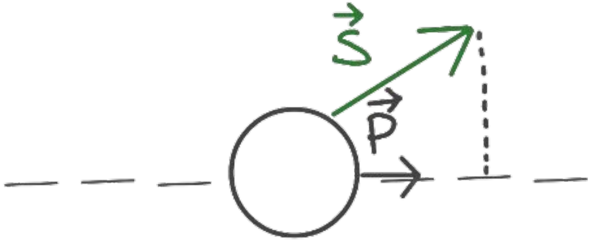
Specify helicities: $$A^{(L),h_1h_2h_3h_4h_5} = \sum_{i=1}^{N} \mathcal{T}_i^{h_1h_2h_3h_4h_5} \; \mathcal{F}_i^{(L)}$$

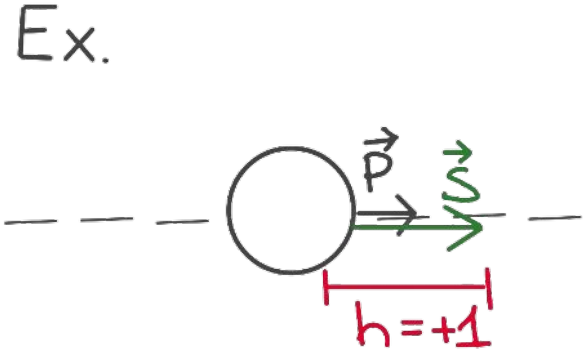
Massive spinors

Helicity amplitudes encode spin correlation information
➔ inclusion of top-quark decay in narrow-width approximation

Helicity: projection of the spin along the direction of momentum

Massless case:





Massive spinors

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→ inclusion of top-quark decay in narrow-width approximation

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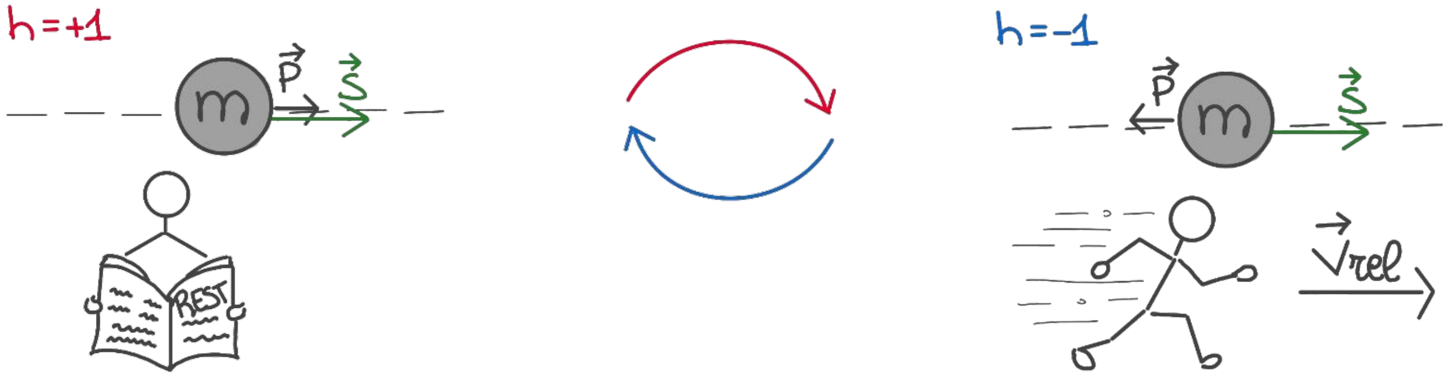


Massive spinors

Helicity amplitudes encode spin correlation information
→ inclusion of top-quark decay in narrow-width approximation

Helicity: projection of the spin along the direction of momentum

Massive case:

$$u_+(p,m) = \frac{(\not{p}+m)|n\rangle}{\langle p^\flat\, n\rangle} \qquad u_-(p,m) = \frac{(\not{p}+m)|n]}{[p^\flat\, n]}$$

where *n* an arbitrary light-like momentum and $p^{\flat,\mu} = p^\mu - \dfrac{m^2}{2p\cdot n} n^\mu$

see [Kleiss, Stirling ’85] [Arkani-Hamed, Huang, Huang ’17] [Badger, Chaubey, Hartanto, Marzucca ’21] [Badger, Becchetti, Chaubey, Marzucca, Sarandrea ’22]

Reduction to MIs

- The amplitude is a linear combination of Feynman integrals:

$$\boxed{A^{(L),\mathrm{proj}}(\vec{x},\epsilon) = \sum_i c_i(\vec{x},\epsilon) I_i(\vec{x},\epsilon)}$$

i.e. $I(\vec{x},\epsilon) = \int \frac{d^{\mathrm{D}}k_1 d^{\mathrm{D}}k_2}{k_1^2(k_1+p_1)^2(k_1+p_1+p_2)^2\ldots}$ with $\mathrm{D} = 4-2\epsilon$

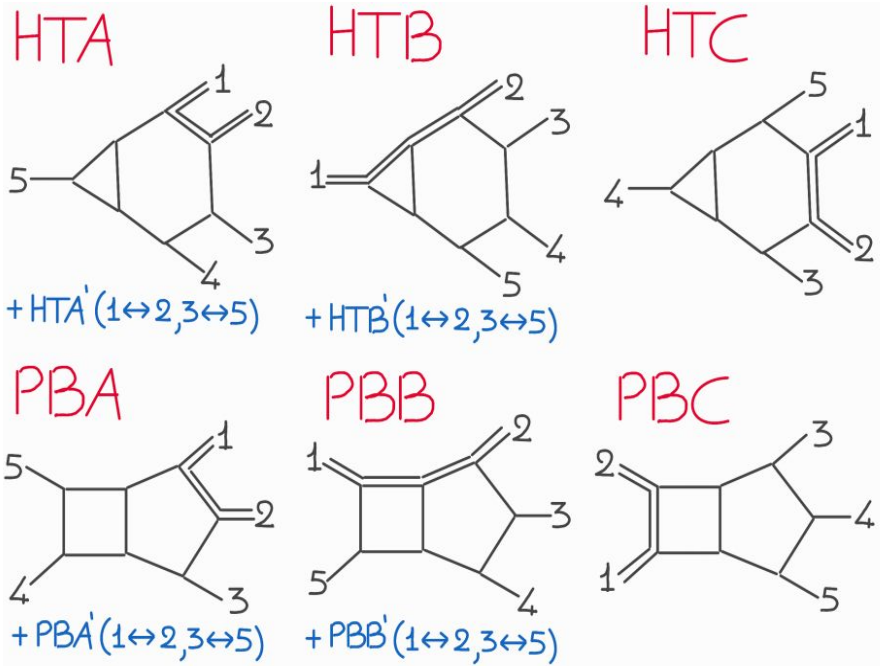
- Feynman integrals $\longrightarrow$ linear combination of Master integrals (MIs) using:

Integration by Parts Identities (IBPs) [Chetyrkin, Kataev, Tkachov, '80]

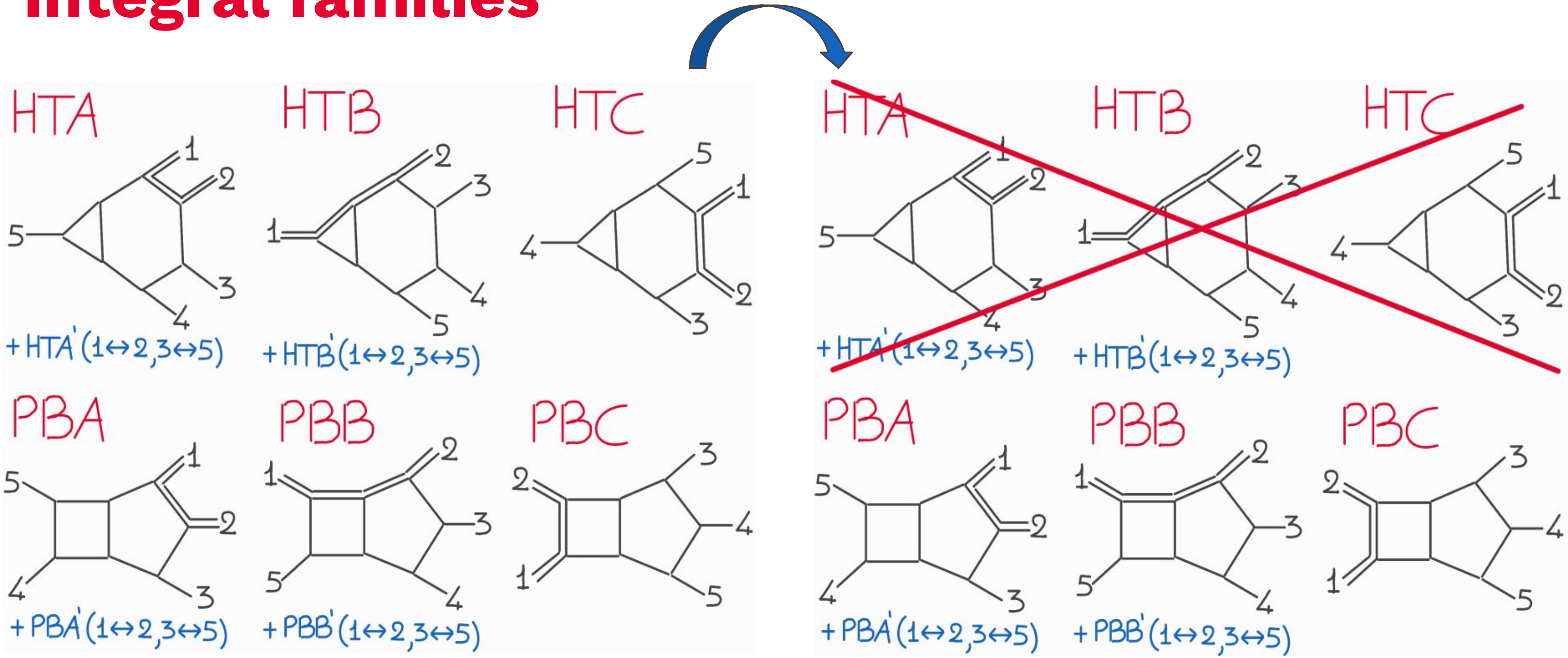
$$\int d^{\mathrm{D}}k_1 d^{\mathrm{D}}k_2 \frac{\partial}{\partial k_1^{\mu}} \left(p_1^{\mu} \frac{1}{k_1^2(k_1+p_1)^2(k_1+p_1+p_2)^2\ldots} \right) = 0$$

- IBPs generated with *NeatIBP* [Wu, Boehm, Ma, Xu, Zhang '23]

Integral families



Integral families



Algebraic complexity

- Amplitude in terms of MIs

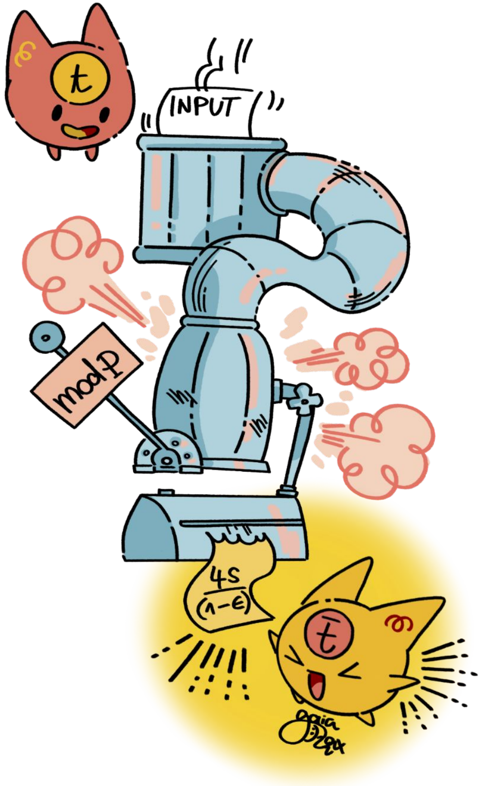
$$A^{(2),\mathrm{proj}}(\vec{x},\epsilon) = \sum_i d_i(\vec{x},\epsilon)\,\mathrm{MI}_i(\vec{x},\epsilon)$$

Algebraic complexity
→ 6 variables

Analytic complexity
→ internal massive propagators

- Replace symbolic operations with numerical evaluations in a **finite field** (integers mod prime P) [von Manteuffel, Schabinger ’14] [Peraro ’16]

- Numerical framework: *FiniteFlow* [Peraro ’19]



Differential equations for MIs

MIs satisfy the following differential equation:

$$\mathrm{d}\vec{f}(\vec{x},\epsilon) = \mathrm{d}A(\vec{x},\epsilon)\ \vec{f}(\vec{x},\epsilon)$$

where $\vec{x}$ are the kinematic invariants (6 variables)

For **PBA** and **PBC**:

[Badger, Becchetti, Chaubey, Marzucca ’23]
[Badger, Becchetti, Giraudo, Zoia ’24]

$$\mathrm{d}A(\vec{x},\epsilon) = \epsilon \sum_j c_j\ \mathrm{dlog}(\alpha_j(\vec{x}))$$

ε factorised — dlog-form

For **PBB**:

[Badger, Becchetti, Giraudo, Zoia ’24]

$$\mathrm{d}A(\vec{x},\epsilon) = \epsilon^k \sum_j c_{kj}\ \omega_j(\vec{x})$$

DEs quadratic in ε — one-form

solution in terms of elliptic functions

Laurent expansion of the MIs

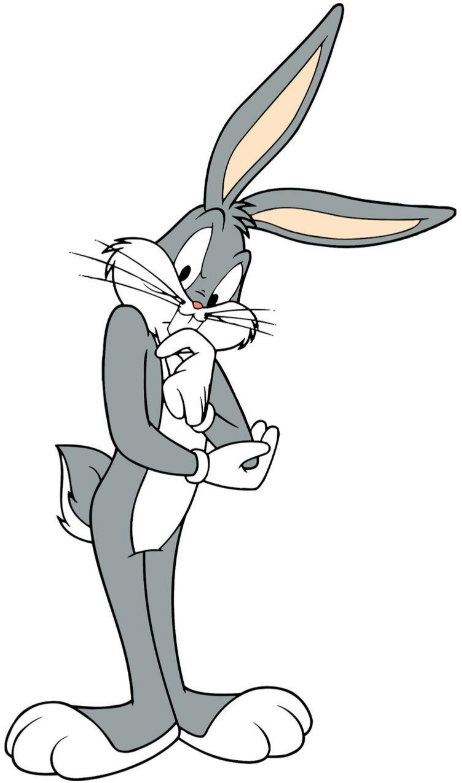
- **Expand** the MIs around **ε = 0**:

$$f(\vec{x},\epsilon) = \sum_{k=0}^{4} \epsilon^k f^{(k)}(\vec{x})$$

- Advantages for algebraically independent MIs components (pentagon functions):
    1. **Analytic UV/IR pole subtraction**
    2. **Simplification of finite remainders**
    3. **Improved numerical evaluation**

- Pentagon functions method applies to DEs:

$$\mathrm{d}\vec{f}(\vec{x},\epsilon) = \epsilon \sum_j c_j \, \mathrm{dlog}(\alpha_j(\vec{x})) \; \vec{f}(\vec{x},\epsilon)$$



see [Gehrmann, Henn, Lo Presti '18] [Chicherin, Sotnikov '20] [Chicherin, Sotnikov, Zoia '22] [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia '24]

A basis of special functions for tt̄j

Pentagon functions method applies straightforwardly to PBA and PBC, while for **PBB non-canonical DEs**

⬇

MIs of non-canonical sectors are:
non-zero only starting from $O(\varepsilon^4)$!

⬇

Potentially **over-complete basis** with non-polylogarithmic functions only in the finite reminder!

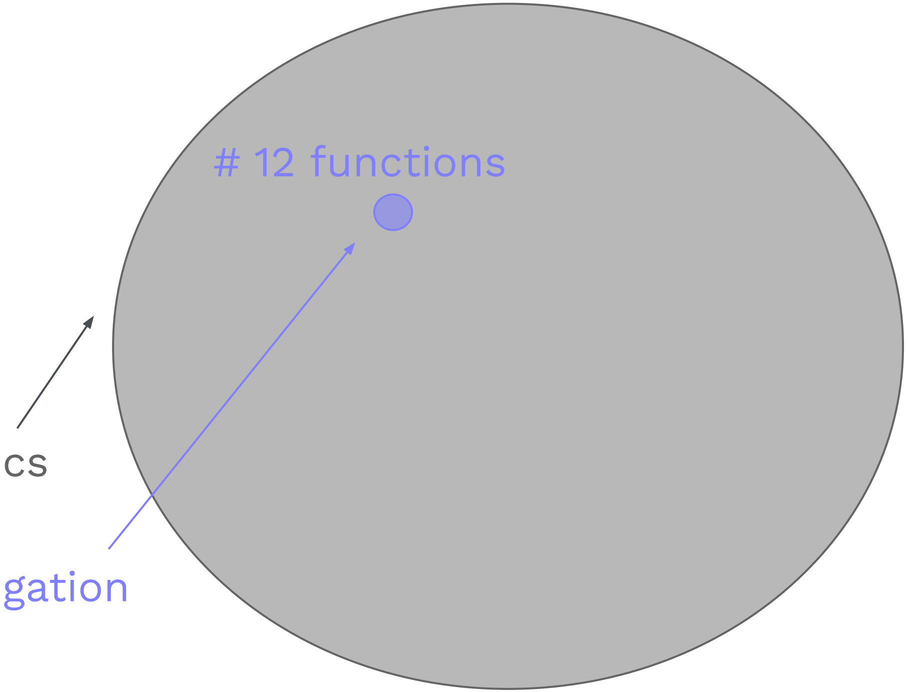
⬇

Finally, the amplitude takes the form

$$A^{(2L),\mathrm{proj}}(\epsilon,\vec{x}) = \sum_i \sum_{k=-4}^{0} \epsilon^k r_{ki}(\vec{x}) \mathrm{F}_i(\vec{x})$$

Polylogarithmic Vs Non-polylogarithmic

- All special functions
- Non-polylogarithmic functions



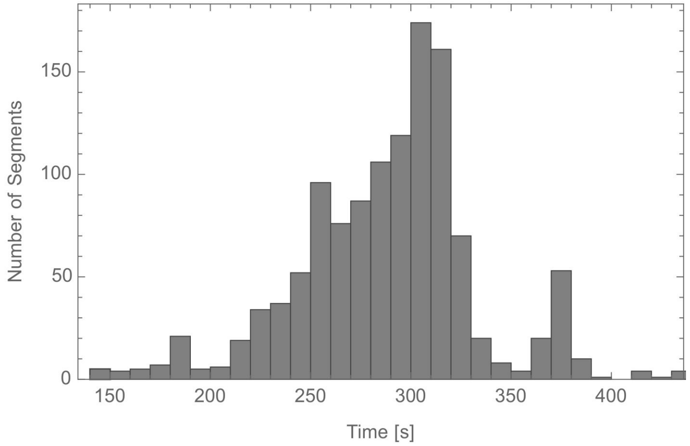
fast+stable numerics

need further investigation

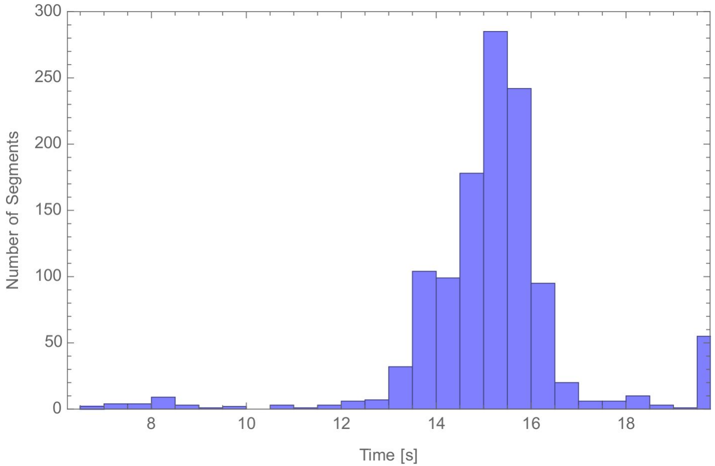
Numerical evaluation of special functions

Special functions evaluated using **generalised power series** method [Moriello '20] as implemented in *DiffExp* [Hidding '21]

Evaluation time per segment - Special functions



Evaluation time per segment - Non-polylog. functions

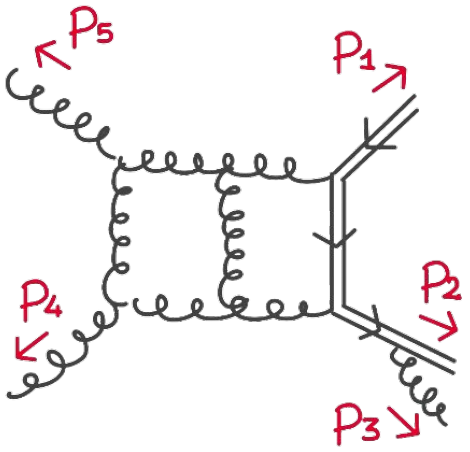


	PB_A	PB_B	PB_C	all MIs	special func. (all)	special func. (non-polylog.)
$\langle T \rangle$	$43\,s$	$77\,s$	$66\,s$	$309\,s$	$297\,s$	$16\,s$
σ	$7\,s$	$17\,s$	$14\,s$	$27\,s$	$65\,s$	$3\,s$

RESULTS

Numerical evaluation of two-loop $gg \to t\bar{t}g$ helicity amplitudes

Notation and kinematics



Process:

$$g(-p_4) + g(-p_5) \to \bar{t}(p_1) + t(p_2) + g(p_3)$$

Kinematics:

$$\vec{d} = (d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2)$$

Spin structure basis for helicity states:

$$A^{(L)}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5}; n_t, n_{\bar{t}}) = m_t \Phi^{h_3 h_4 h_5} \times \sum_{i=1}^{4} \Theta_i(n_t, n_{\bar{t}}) \, A^{(L),[i]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$$

Finite remainder reconstruction

- Mass-renormalised amplitudes are gauge invariant
  ➡ **Gauge invariance check**
- UV/IR poles identified analytically and finite remainder computed directly
  ➡ **Pole check**
- Results cross-checked against independent computation of helicity amplitudes in terms of momentum-twistor variables
  ➡ **Simplification of the amplitude**

helicity	max degrees MIs recon.	max degrees SF recon.
$+++++$	294	131
$+++-+$	384	269
$++++-$	395	264

Numerical evaluation

Numerical evaluation of finite remainders up to two loops in the leading color limit for gg → t$\bar{t}$g production

Helicity	$R^{(0),[1]}$	$R^{(0),[2]}$	$R^{(0),[3]}$	$R^{(0),[4]}$
$+++$	$0.26326 - 0.0097514\,i$	0	0	0
$+-+$	$5.9619 - 0.16047\,i$	0	0	$-0.31659 - 0.097935\,i$
$++-$	$-5.9575 + 0.0089231\,i$	$-12.606 - 0.067440\,i$	$4.6564 + 0.024911\,i$	$-1.9692 - 0.010535\,i$
Helicity	$R^{(1),[1]}/R^{(0),[1]}$	$R^{(1),[2]}/R^{(0),[1]}$	$R^{(1),[3]}/R^{(0),[1]}$	$R^{(1),[4]}/R^{(0),[1]}$
$+++$	$38.396 - 5.8002\,i$	$71.982 - 4.0653\,i$	$-14.289 + 0.70866\,i$	$17.909 - 0.39528\,i$
$+-+$	$19.221 - 8.4151\,i$	$-4.8506 + 4.8015\,i$	$0.67096 - 0.09959\,i$	$-1.2201 + 2.1594\,i$
$++-$	$20.369 - 19.991\,i$	$41.522 - 41.969\,i$	$-15.990 + 15.739\,i$	$6.2964 - 6.4584\,i$
Helicity	$R^{(2),[1]}/R^{(0),[1]}$	$R^{(2),[2]}/R^{(0),[1]}$	$R^{(2),[3]}/R^{(0),[1]}$	$R^{(2),[4]}/R^{(0),[1]}$
$+++$	$882.48 - 91.619\,i$	$2489.7 - 266.72\,i$	$-492.28 + 8.1003\,i$	$593.35 - 87.569\,i$
$+-+$	$414.16 - 206.87\,i$	$-171.78 + 189.69\,i$	$25.226 - 1.5639\,i$	$-54.820 + 95.716\,i$
$++-$	$332.97 - 646.02\,i$	$623.01 - 1325.1\,i$	$-259.14 + 512.33\,i$	$89.185 - 198.65\,i$

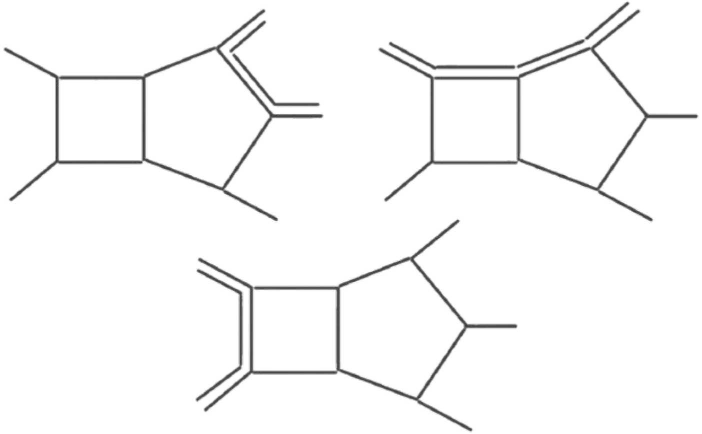
Check out our paper arXiv:2412.13876 😊

CONCLUSIONS

Conclusions & outlook

Conclusions

- **Numerical evaluation** of two-loop helicity finite remainders for $gg \to t\bar{t}g$
- Developed a **new strategy for master integrals** bypassing canonical form challenges
- Efficiently isolated **polylogarithmic and non-polylogarithmic components** for numerical evaluation



Outlook

- Pursue **analytic reconstruction**
- Optimize the numerical evaluation of special functions
- Integration over phase space to deliver **cross-section** results (using available subtraction schemes)

Conclusions

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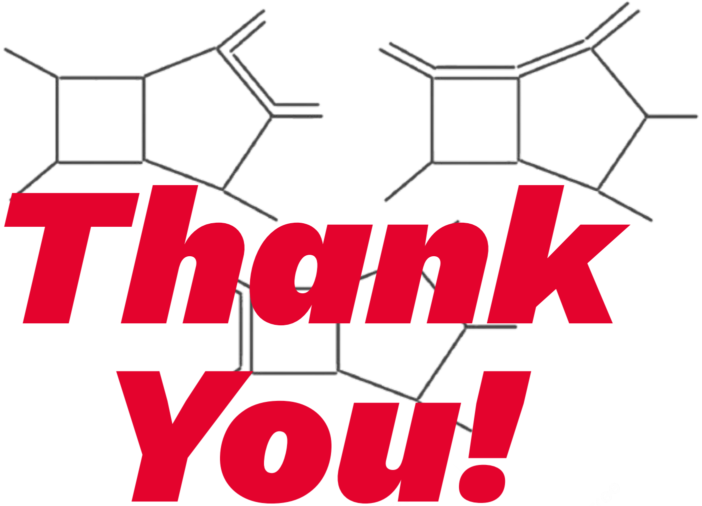

Outlook

- Pursue **analytic reconstruction**
- Optimize the numerical evaluation of special functions
- Integration over phase space to deliver **cross-section** results (using available subtraction schemes)

BACKUP

Physical point

The point chosen in the **physical region** for the sub-amplitudes evaluation is

$$d_{12} = \frac{1617782845110651539}{1506833389797120000}\,, \quad d_{23} = \frac{335}{1232}\,, \quad d_{34} = -\frac{5}{32}\,,$$
$$d_{45} = \frac{3665}{7328}\,, \quad d_{15} = -\frac{45}{1408}\,, \quad m_t^2 = \frac{376940175237098461}{1506833389797120000}\,,$$

$$\text{with} \quad \mathrm{tr}_5 = \mathrm{i}\,\frac{\sqrt{582950030096630501}}{426229309440}\,.$$

In terms of **momentum twistor variables**

$$s_{34} = -\frac{5}{16}\,, \quad t_{45} = -\frac{733}{229}\,, \quad t_{12} = -\frac{61}{72}\,,$$
$$t_{23} = -\frac{134}{77}\,, \quad t_{51} = \frac{9}{44}\,, \quad x_{5123} = \frac{11}{51} + \frac{1}{125}\mathrm{i}\,.$$