Non-Locality induces Isometry and Factorisation in Holography

DIP online Seminar

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Outline

- Motivation
- Black hole microstates
- Counting the microstates
- The factorisation puzzle from an algebraic perspective
- Conclusion

Motivation

- The **holographic principle** states that in quantum gravity the information stored in a $(d+1)$ dimensional volume is encoded on its d dimensional boundary ['t Hooft '93, Susskind '95]
- Motivated from **Bekenstein-Hawking entropy** of black holes

$$S_{\text{BH}} = \frac{A}{4G_N},$$

  which indicates that the number of degrees of freedom in QG scales with the area and not the volume [Bekenstein '73, Hawking '75]
- Strong evidence for statistical interpretation of black hole entropy, from explicit microstate counting [Strominger, Vafa '96, Balasubramanian et al. '22]

Motivation

- The most successful realisation of the holographic principle is the **AdS/CFT correspondence**

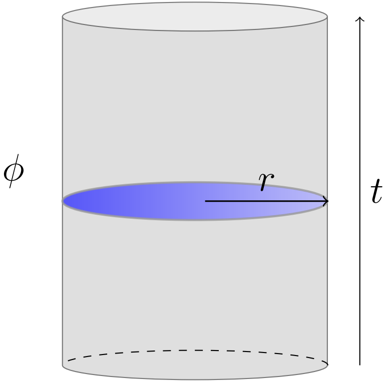
  [Maldacena '97]

- The correspondence provides a map between bulk an boundary Hilbert spaces

$$V : \mathcal{H}_{\text{Bulk}} \to \mathcal{H}_{\text{Bdry}}$$

- The holographic map V has several inconsistencies in the semiclassical $G_N \to 0$ limit

AdS$_{2+1}$



Motivation

Problem 1

- The bulk Hilbert space is much larger than the boundary Hilbert space [Engelhardt et al. '22, Faulkner, Li '22]
  - $\Rightarrow$ The holographic map V becomes non-isometric
  - $\Rightarrow$ $\mathcal{H}_{\text{Bulk}}$ contains large set of null states, which get annihilated by V
  - $\Rightarrow$ This can not be detected by a local low energy observer, having only access to simple measurements [Engelhardt et al. '22]
- The size of the bulk Hilbert space violates central dogma of black hole physics

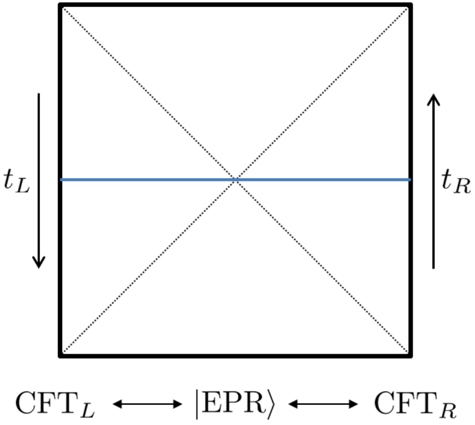
Motivation

Problem 2: The factorisation puzzle

- The eternal black hole is dual to two copies of the CFT in an entangled state [Maldacena '01]
- In this setup

$$\mathcal{H}_{\text{Bdry}} = \mathcal{H}_L \otimes \mathcal{H}_R \,,$$

  while $\mathcal{H}_{\text{bulk}}$ does not factorize
- **Algebraic perspective:** The boundary algebra of observables is type I while the bulk operator algebra is type III_1



Black Hole Microstates

Consider infinite family of **generalized TFD states** [Papadodimas, Raju '15, Papadodimas, Raju '16]

$$|\mathrm{TFD}\rangle_\alpha = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{i\alpha_n} e^{-\frac{\beta}{2}E_n} |E_n\rangle_L \otimes |E_n\rangle_R$$

with phases α_n. The TFD corresponds to $\alpha_n = 0$

- Same one-sided correlation functions as TFD
- Entanglement equivalent of TFD

$$S_{\mathrm{EE}}(\rho_R) = S_{\mathrm{EE}}(\rho_R^\alpha) \quad \text{with} \quad \rho_R^{(\alpha)} = \mathrm{Tr}_L\, |\mathrm{TFD}\rangle_{(\alpha)}\langle\mathrm{TFD}|_{(\alpha)}$$

- According to ER=EPR [Maldacena, Susskind '13, Verlinde H. '20]

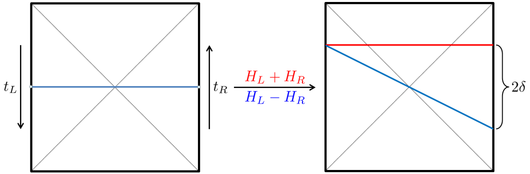
$$S_{\mathrm{EE}}(\rho_R) = S_{\mathrm{BH}}$$

Black Hole Microstates

Each $|\text{TFD}\rangle_\alpha$ corresponds to time-evolved TFD

$$|\text{TFD}\rangle_\alpha = e^{i\frac{t}{2}(H_L+H_R)}|\text{TFD}\rangle \quad \text{with} \quad \alpha_n = E_n\, t$$

Holographic dual to Eternal black hole in AdS, with different gluing to boundary time



- Time-shift δ is equivalent to length of Lorentzian wormhole, and can only be detected by **non-local** measurement
- Existence of these states corresponds to freedom of choosing an origin of time independently for $\text{CFT}_{L/R}$

Black Hole Microstates

In semiclassical, or large N limit the energy spectrum is highly random (approx. continuous)

[Verlinde H. '20]

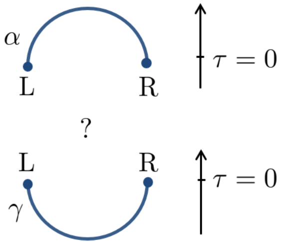
$$\Rightarrow \quad \langle \mathrm{TFD}_\alpha | \mathrm{TFD} \rangle_\gamma = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} e^{i(\gamma_n - \alpha_n)} = \delta_{\alpha\gamma} + \mathcal{O}(e^{-S_{\mathbf{BH}}/2})$$

This may also be calculated from the **gravitational path integral**

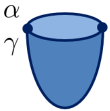
$\Rightarrow$ includes a **sum over geometries** consistent with given boundary condition

$\langle \mathrm{TFD}|_\alpha \ \widehat{=}$

$|\mathrm{TFD}\rangle_\gamma \ \widehat{=}$



$\langle \mathrm{TFD}_\alpha | \mathrm{TFD} \rangle_\gamma \simeq$



$\Rightarrow$ Due to orthogonality states form infinite basis of bulk Hilbert space $\Rightarrow$ **Problem 1**

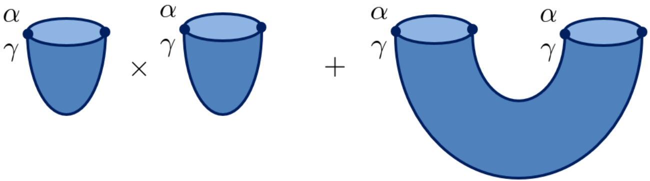
Black Hole Microstates

Problem is solved by calculating overlaps in microstates from higher moments

$$\frac{1}{\mathcal{N}}\sum_{\gamma}|\langle \mathsf{TFD}_\alpha|\mathsf{TFD}\rangle_\gamma|^2=\frac{1}{\mathcal{N}}+\frac{1}{Z^2(\beta)}\sum_n e^{-2\beta E_n}=\frac{1}{\mathcal{N}}+\frac{Z(2\beta)}{Z^2(\beta)}$$

Correction term understood from appearance of **replica wormholes** in gravitational path integral [Verlinde H. '20, Verlinde H. '21]

$$|\langle \mathrm{TFD}_\alpha|\mathrm{TFD}\rangle_\gamma|^2\simeq$$



⇒ **non-perturbative** and **non-local** correction

Black Hole Microstates

Appearance of non-trivial overlaps in higher moments understood from an averaging procedure

$$\langle \mathsf{TFD}_\alpha | \mathsf{TFD} \rangle_\gamma = \overline{\mathcal{M}_{\alpha\gamma}} \quad \text{with} \quad \mathcal{M}_{\alpha\gamma} = \delta_{\alpha\gamma} + e^{-S_{\mathbf{BH}}/2} \mathcal{R}_{\alpha\gamma}$$

and $\mathcal{R}_{\alpha\gamma}$ is random matrix with mean zero $\Rightarrow$ Higher moments contain variance of $\mathcal{R}$

- Degrees of freedom of fundamental theory encoded in $\mathcal{R}$
- Path integral averages over fundamental DoF [Penington, Shenker, Stanford, Yang '19]
- Here this is a state average over Hilbert space of phase-shifted states

The Euclidean replica wormholes arise from an average over states corresponding to Lorentzian wormholes of different length.
This connects two seemingly different notions of non-locality in QG

Counting the Microstates

Counting only linearly independent microstates: **Strategy** [Balasubramanian et al. '22, Emparan et al. '24]

- Consider Hilbert space $\mathcal{H}_\Omega$ spanned by Ω microstates
- Calculate $d_\Omega = \text{dim}(\mathcal{H}_\Omega) \Rightarrow$ Dimension of full HS given by limit of d_Ω
- d_Ω given by rank of Gram matrix $G_{ij} := \langle \text{TFD}_{\alpha_i} | \text{TFD} \rangle_{\alpha_j} \quad i, j = 1, ..., \Omega$

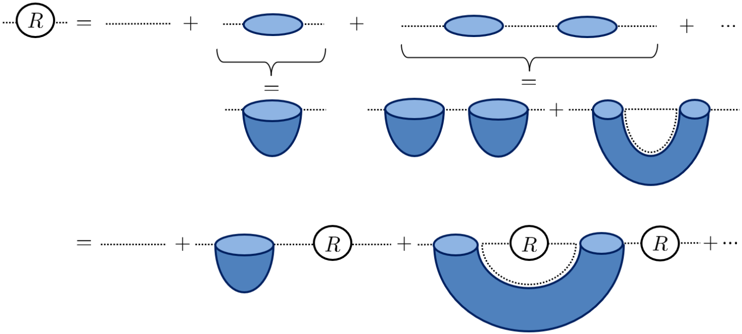
$$d_\Omega = \lim_{\epsilon \to 0^+} \int_\epsilon^\infty d\lambda\, D(\lambda) = \Omega - \text{Ker}(G) \quad \text{with} \quad D(\lambda) := \text{Tr}\left(\delta(\lambda \mathbb{1} - G)\right) = \sum_{i=1}^{\Omega} \delta(\lambda - \lambda_i)$$

- Higher powers G^n encode overlaps $\Rightarrow$ **resolvent method** [Penington, Shenker, Stanford, Yang '19]
- Eigenvalue density $D(\lambda)$ given in terms of trace of resolvent

$$D(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left(R(\lambda - i\epsilon) - R(\lambda + i\epsilon)\right)$$

Counting the Microstates

Resolvent is defined as $R_{ij}(\lambda) = \left(\frac{1}{\lambda\mathbb{1} - G}\right)_{ij} = \frac{\delta_{ij}}{\lambda} + \sum_{n=1}^{\infty} \frac{(G^n)_{ij}}{\lambda^{n+1}}$ or pictorially:



$$\Longrightarrow R_{ij}(\lambda) = \frac{\delta_{ij}}{\lambda} + \frac{1}{\lambda}\sum_{n=1}^{\infty} \frac{Z_n}{Z_1^n} R^{n-1}(\lambda)\, R_{ij}(\lambda) \quad \text{with} \quad R(\lambda) = \sum_i R_{ii}(\lambda)$$

Counting the Microstates

Properly normalized n-boundary wormhole partition function given by [Verlinde H. '21]

$$\frac{Z_n}{Z_1^n} = \frac{Z(n\beta)}{Z^n(\beta)} \quad \text{with} \quad Z(\beta) = e^{-\beta\, M + S_{\mathsf{BH}}} = \int_0^\infty dE\, e^{-\beta E}\, z(E)$$

with microcanonical partition function $z(E)$
⇒ Project into a fixed energy window + introduce microcanonical wormhole partition function [Balasubramanian et al. '22]

$$\mathbf{Z}_n = e^{-n\beta\, E} z(E)\Delta E\,, \quad e^{S_M} = z(E)\Delta E = e^{S_{\mathsf{BH}}}$$

Inserting this into the previous result for $R(\lambda)$ gives

$$R(\lambda) = \frac{\Omega}{\lambda} + \frac{e^{S_M}}{\lambda} \sum_{n=1}^{\infty} \left(\frac{R(\lambda)}{e^{S_M}}\right)^n = \frac{\Omega}{\lambda} + \frac{1}{\lambda} \frac{e^{S_M}\, R(\lambda)}{e^{S_M} - R(\lambda)}$$

Counting the Microstates

$R(\lambda)$ is solution to quadratic equation and we find the EV density of G

$$D(\lambda) = \frac{e^{S_{\mathsf{BH}}}}{2\pi\lambda}\sqrt{\left[\lambda - \left(1 - (\Omega\, e^{-S_{\mathsf{BH}}})^{1/2}\right)^2\right]\left[\left(1 + (\Omega\, e^{-S_{\mathsf{BH}}})^{1/2}\right)^2 - \lambda\right]}$$
$$+\,\delta(\lambda)\left(\Omega - e^{S_{\mathsf{BH}}}\right)\Theta\left(\Omega - e^{S_{\mathsf{BH}}}\right)$$

From this we find

$$d_\Omega = \min\left(\Omega, e^{S_{\mathsf{BH}}}\right)$$

Counting the Microstates

- Microstates span HS of dimension $e^{S_{\mathsf{BH}}}$ $\Rightarrow$ span full $\mathcal{H}_{\mathrm{BH}}$
- Embedding of (infinite) $\mathcal{H}_{\mathrm{eff}}$ into $\mathcal{H}_{\mathrm{BH}}$ explicitly realises non-isometric map
  [Engelhardt et al. '22, Faulkner, Li '22]
- Null states correspond to kernel of G
- Existence of these states can only be noticed once $\Omega = e^{S_{\mathsf{BH}}}$ states are included in $\mathcal{H}_{\Omega}$
  $\Rightarrow$ In particular requires **non-local** measurement
- For our calculation we don't have to add any external DoF, but only rely on the **fundamental non-locality of quantum gravity**

Algebraic Perspective on Factorisation

- In AQFT a physical system is defined in terms of its **algebra of observables AoO** $\mathcal{A}$
- Given a faithful algebraic state $\omega : \mathcal{A} \to \mathbb{C}$ we construct the GNS representation $(\mathcal{H}_{\mathsf{GNS}}, \pi)$

$$\mathcal{H}_{\mathsf{GNS}} := \{\, |a\rangle\,, a \in \mathcal{A} \,|\, \langle a|b\rangle := \omega(a^* b)\,\}\,, \quad \pi(a)|b\rangle = |ab\rangle\,,$$

$\Rightarrow \pi(\mathcal{A})$ is a **von Neumann algebra**

- **Theorem:** ω is pure if and only if the GNS representation is irreducible

Algebraic Perspective on Factorisation

Type	pure states/ irreps	Trace/ density matrices	Entropies
I	$\exists \Rightarrow$ irrep.	$\exists$	$< \infty$
II	no irrep. $\Rightarrow \nexists$	$\exists$ (renormalized)	∞
III	no irrep. $\Rightarrow \nexists$	$\nexists$	∞

- The algebra of observables associated to a local subregion U of a QFT is type III_1
  $\Rightarrow$ This is one manifestation of the information paradox
- Solving this paradox requires a description of the AoO exterior to a black hole as an algebra of type I, which is the operator algebra of ordinary quantum mechanical systems [Witten '21]
- **Conjecture:** The type I description requires **non-perturbative** corrections [Witten '21]

Algebraic Perspective on Factorisation

- **Boundary perspective:** AoO of the left/right CFT is of **type I** $\mathcal{B}(\mathcal{H}_{L/R})$, and the TFD is an element of $\mathcal{H}_L \otimes \mathcal{H}_R \Rightarrow$ relies on **discrete energy spectrum**
- **Bulk perspective:** For $G_N \to 0$ the bulk algebra is **type III**$_1$
  $\iff$ At large N **continuous energy** spectrum [Lashkari et al. '23]
- TFD only exists as element, associated to $\mathbb{1}$ of GNS Hilbert space $\mathcal{H}_{\text{GNS}}$, constructed from **algebra of single-trace operators** $\mathcal{A}_{L/R}$, with

$$\omega(a) = \langle a \rangle_\beta = \lim_{N \to \infty} \langle \text{TFD}|a|\text{TFD} \rangle$$

  and $\pi(\mathcal{A}_{L/R})$ is type III$_1$ [Leutheusser, Liu '21]
- Continuous energy spectrum $\Rightarrow \langle \text{TFD}_\alpha|\text{TFD}\rangle_\gamma = \delta_{\alpha\gamma}$

Algebraic Perspective on Factorisation

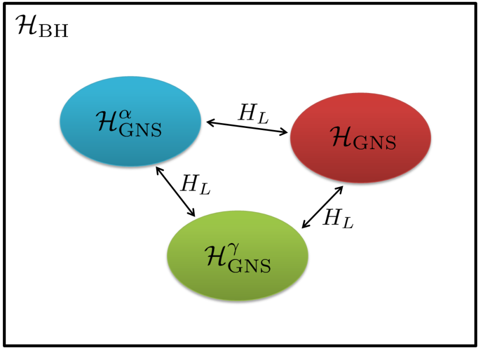
- H_L is not a single trace-operator
  $\Rightarrow |\text{TFD}\rangle_\alpha \notin \mathcal{H}_{\text{GNS}}$
  $\Rightarrow$ Each microstate contained in separate $\mathcal{H}^\alpha_{\text{GNS}}$
- Include H_L into AoO through crossed product construction [Witten '21, Chandrasekaran, Penington, Witten '22]

$$\hat{\mathcal{A}}_L = \mathcal{A}_L \rtimes \mathcal{A}_{H_L}$$

which acts on

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_{\text{GNS}} \otimes L^2(\mathbb{R})$$

and is of type II_∞



Algebraic Perspective on Factorisation

- In the type II description all microstates are orthogonal, and the entropy is infinite
- We showed explicitly that the inclusion of **non-perturbative** wormhole corrections imply:
  $\Rightarrow$ microstates span finite dimensional Hilbert space $\Rightarrow$ Type I
- **Claim:** The non-perturbative corrections yield a discrete energy spectrum ($\Rightarrow$ type I)
  proof: Let us assume that the spectrum is continuous
  $\Rightarrow$ Microstates are orthogonal $\Rightarrow$ Hilbert space is infinite dimensional
  $\lightning$ wormholes imply non-trivial overlaps $\Rightarrow$ Hilbert space finite dimensional
  $\Rightarrow$ Spectrum is discrete
- Consequently all microstates can be written as an element of a **factorized Hilbert space**

$$\mathcal{H}_l \otimes \mathcal{H}_r = \text{span}(\,|E_n\rangle_l \otimes |E_m\rangle_r\,)$$

Conclusion

The holographic map seems to be inconsistent in the semiclassical limit

Problem 1: Bulk Hilbert space has way too many states

- Infinite family of generalized TFD states are orthogonal for $G_N \to 0$
- Including **non-local** effects in the gravitational path integral leads to non-trivial overlaps
  $\Rightarrow$ linearly independent states span a HS consistent with holographic entropy bounds

Problem 2: Factorisation puzzle (type I vs type III/II)

- Reduction in size corresponds to a transition from type III/II to type I
  $\Rightarrow$ Confirms Witten's conjecture regarding non-perturbative corrections
- Transition is understood in terms of a discrete energy spectrum
  $\Rightarrow$ Energy eigenstates span **factorized Hilbert space**

A consistent formulation of a quantum theory of gravity has to be non-local