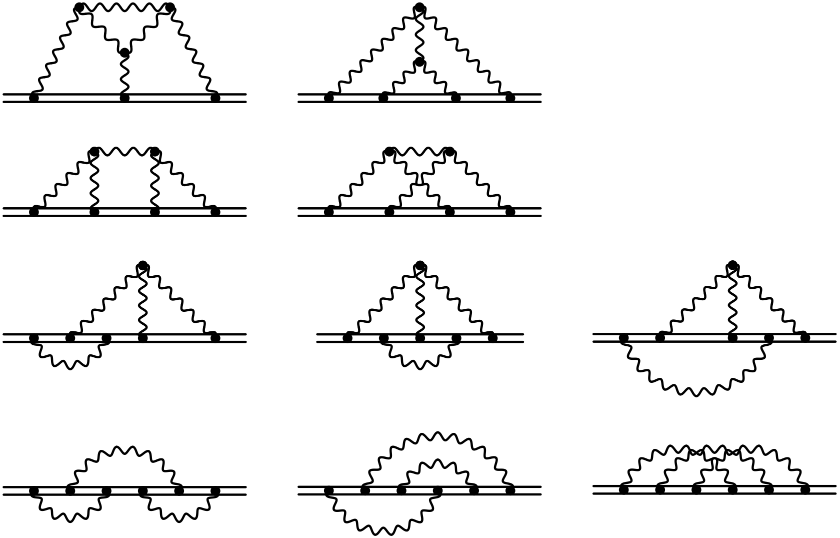
# Three-loop results in HQET

Andrey Grozin
grozin@particle.uni-karlsruhe.de

Institut für Theoretische Teilchenphysik
Universität Karlsruhe

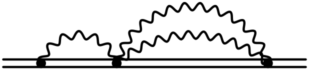
# Propagator diagrams



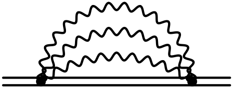
Grinder (REDUCE), AG (2000)

# Master integrals

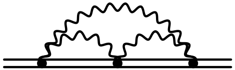
$$= I_1^3$$



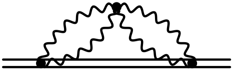
$$= I_1 I_2$$



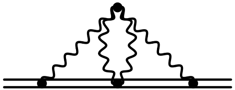
$$= I_3$$



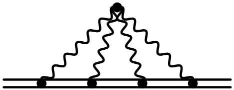
$$\sim I_3 \frac{I_1^2}{I_2}$$

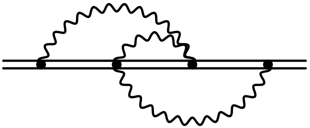


$$\sim I_3 \frac{G_1^2}{G_2}$$

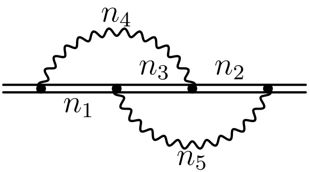


$$= G_1 I(1,1,1,1,\varepsilon)$$



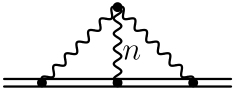


$$= I_1 J(1,1,-1+2\varepsilon,1,1)$$




$$J(n_1,n_2,n_3,n_4,n_5) = \Gamma(n_1+n_2+n_3+2(n_4+n_5-d))$$
$$\times \frac{\Gamma(n_1+n_3+2n_4-d)\Gamma(d/2-n_4)\Gamma(d/2-n_5)}{\Gamma(n_1+n_2+n_3+2n_4-d)\Gamma(n_4)\Gamma(n_5)\Gamma(n_1+n_3)}$$
$$\times {}_3F_2\left( \begin{array}{c} n_1, d-2n_5, n_1+n_3+2n_4-d \\ n_1+n_3, n_1+n_2+n_3+2n_4-d \end{array} \middle| \, 1 \right)$$

AG (2000)




$$I(1,1,1,1,n) = \frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-n-1\right)}{\Gamma(d-2)}$$

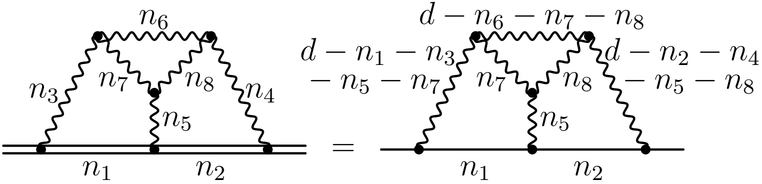
$$\times \Bigg[ 2\frac{\Gamma(2n-d+3)\Gamma(2n-2d+6)}{(n-d+3)\Gamma(3n-2d+6)}$$

$$\times {}_3F_2\left( \begin{array}{c} n-d+3, n-d+3, 2n-2d+6 \\ n-d+4, 3n-2d+6 \end{array} \,\middle|\, 1 \right)$$

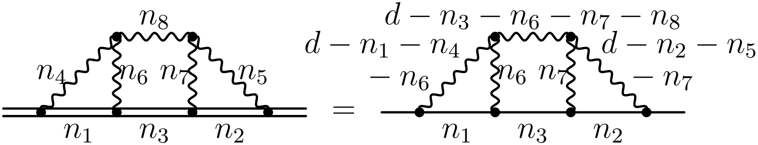
$$- \Gamma(d-n-2)\Gamma^2(n-d+3) \Bigg]$$

M. Beneke, V. Braun (1994)

# Inversion

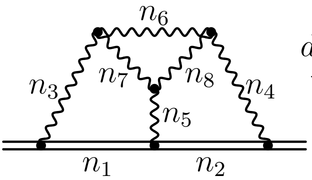


$n_6$
$n_3$
$n_7$
$n_8$
$n_4$
$n_5$
$n_1$
$n_2$
=
$d - n_6 - n_7 - n_8$
$d - n_1 - n_3$
$- n_5 - n_7$
$d - n_2 - n_4$
$- n_5 - n_8$
$n_7$
$n_8$
$n_5$
$n_1$
$n_2$

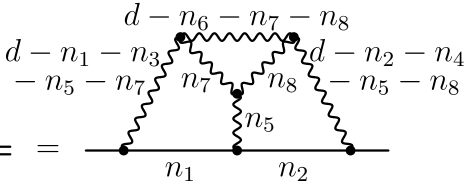



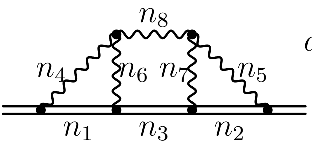
$n_8$
$n_4$
$n_6$
$n_7$
$n_5$
$n_1$
$n_3$
$n_2$
=
$d - n_3 - n_6 - n_7 - n_8$
$d - n_1 - n_4$
$- n_6$
$d - n_2 - n_5$
$- n_7$
$n_6$
$n_7$
$n_1$
$n_3$
$n_2$

# Inversion

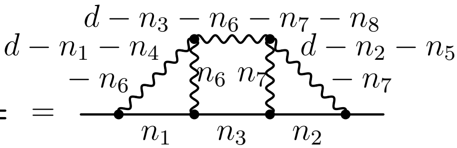


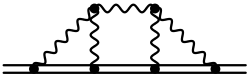

$=$



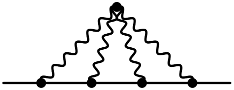




$=$



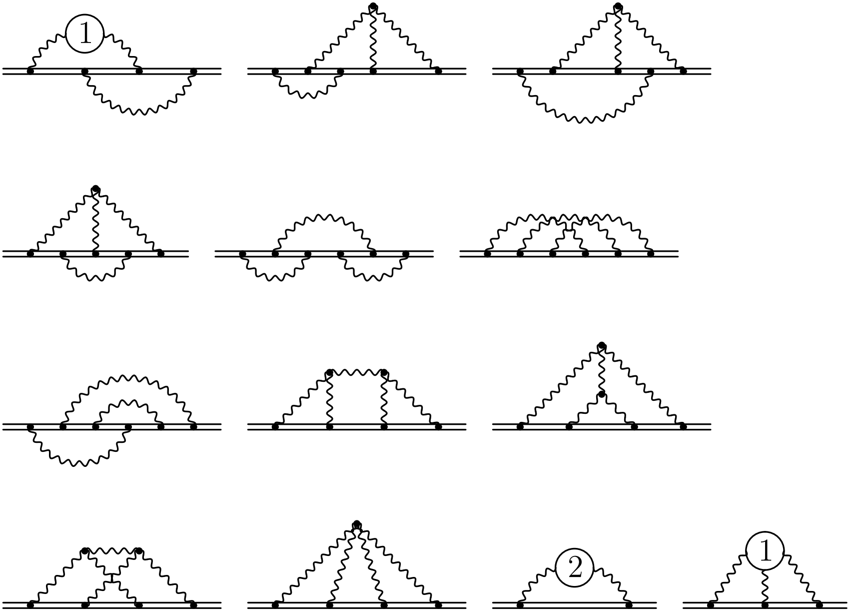
$=$



$= -5\zeta_5 + 12\zeta_2\zeta_3$

A. Czarnecki, K. Melnikov (2002)

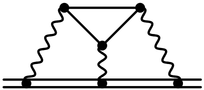
# Heavy–quark propagator



# Heavy–light current (K. Chetyrkin, AG 2003)

$$
\begin{aligned}
\tilde{\gamma}_j = {} & -3C_F \frac{\alpha_s}{4\pi} \\
& + C_F \left[ C_F \left( -16\zeta_2 + \frac{5}{2} \right) + C_A \left( 4\zeta_2 - \frac{49}{6} \right) + \frac{10}{3} T_F n_l \right] \left( \frac{\alpha_s}{4\pi} \right)^2 \\
& + C_F \left[ C_F^2 \left( -80\zeta_4 - 36\zeta_3 + 64\zeta_2 - \frac{37}{2} \right) \right. \\
& \quad + C_F C_A \left( -16\zeta_4 + \frac{142}{3}\zeta_3 - \frac{1184}{9}\zeta_2 - \frac{655}{36} \right) \\
& \quad + C_A^2 \left( -24\zeta_4 - \frac{22}{3}\zeta_3 + \frac{260}{9}\zeta_2 + \frac{1451}{108} \right) \\
& \quad + C_F T_F n_l \left( -\frac{176}{3}\zeta_3 + \frac{448}{9}\zeta_2 + \frac{470}{9} \right) \\
& \quad \left. + C_A T_F n_l \left( \frac{152}{3}\zeta_3 - \frac{112}{9}\zeta_2 - \frac{512}{27} \right) + \frac{140}{27} (T_F n_l)^2 \right] \left( \frac{\alpha_s}{4\pi} \right)^3 + \cdots
\end{aligned}
$$

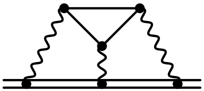
# On-shell diagrams with mass 1



Gröbner bases

AG, A. Smirnov, V. Smirnov (2006)

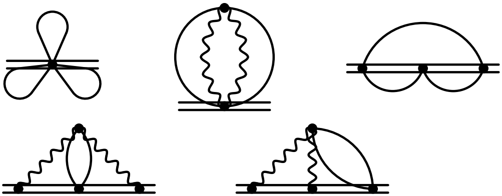
# On-shell diagrams with mass 1



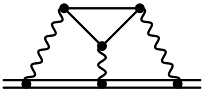
Gröbner bases

AG, A. Smirnov, V. Smirnov (2006)

Apparently even



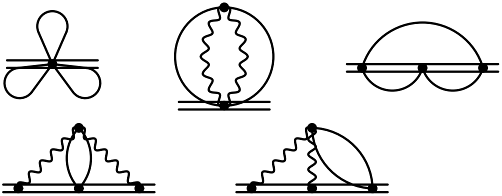
# On-shell diagrams with mass 1



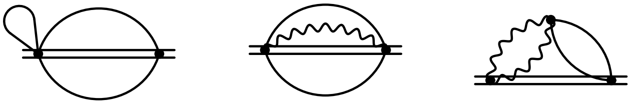
Gröbner bases

AG, A. Smirnov, V. Smirnov (2006)

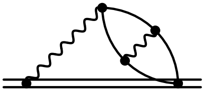
Apparently even



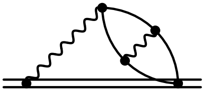
Apparently odd



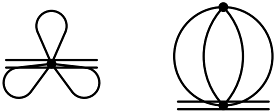
# On-shell diagrams with mass 2



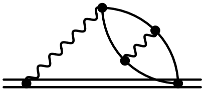
# On-shell diagrams with mass 2



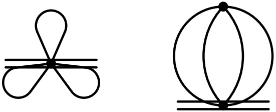
Apparently even



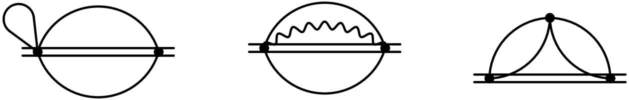
# On-shell diagrams with mass 2

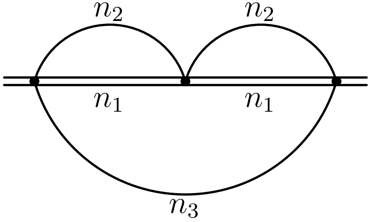


Apparently even



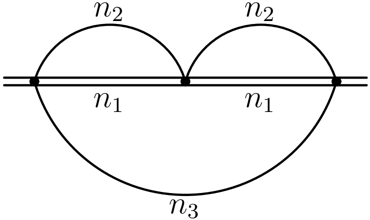
Apparently odd





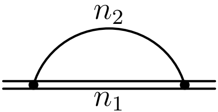

AG, T. Huber, D. Maître (2007)

$$I_{n_1 n_2 n_3} = \frac{\Gamma(n_3 - 3/2 + \varepsilon)}{\pi^{1/2}\Gamma(n_3)} \int_{-\infty}^{+\infty} I^2_{n_1 n_2}(ip_{E0})(1+p_{E0}^2)^{3/2-n_3-\varepsilon} dp_{E0}$$



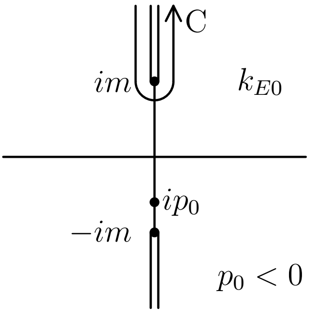

AG, T. Huber, D. Maître (2007)

$$I_{n_1 n_2 n_3} = \frac{\Gamma(n_3 - 3/2 + \varepsilon)}{\pi^{1/2}\Gamma(n_3)} \int_{-\infty}^{+\infty} I^2_{n_1 n_2}(ip_{E0})(1+p_{E0}^2)^{3/2-n_3-\varepsilon} dp_{E0}$$




$$I_{n_1 n_2}(p_0) = \frac{1}{i\pi^{d/2}} \int \frac{dk_0\, d^{d-1}\vec{k}}{[-2(k_0+p_0) - i0]^{n_1}[m^2 - k^2 - i0]^{n_2}}$$

$$= \frac{\Gamma(n_2 - (d-1)/2)}{\pi^{1/2}\Gamma(n_2)} \int\limits_{-\infty}^{+\infty} dk_{E0} \frac{(k_{E0}^2 + m^2)^{(d-1)/2-n_2}}{(-2p_0 - 2ik_{E0})^{n_1}}$$




$$I_{n_1 n_2}(p_0) = 2\frac{\Gamma(n_2 - (d-1)/2)}{\pi^{1/2}\Gamma(n_2)}\cos\left[\pi\left(\frac{d}{2} - n_2\right)\right] \times \int_m^\infty dk \frac{(k^2 - m^2)^{(d-1)/2 - n_2}}{(2k - 2p_0)^{n_1}}$$

$$
I_{n_1 n_2}(p_0) = \frac{\Gamma(n_1+n_2-2+\varepsilon)\Gamma(n_1+2n_2-4+2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1+n_2-2+\varepsilon))}
$$

$$
\times\, {}_2F_1\left( \begin{array}{c} n_1, n_1+2n_2-4+2\varepsilon \\ n_1+n_2-\frac{3}{2}+\varepsilon \end{array} \middle| \frac{1}{2}\left(1+\frac{p_0}{m}\right)\right) m^{d-n_1-2n_2}
$$

$$
= \frac{\Gamma(n_1+n_2-2+\varepsilon)\Gamma(n_1+2n_2-4+2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1+n_2-2+\varepsilon))}
$$

$$
\times\, {}_2F_1\left( \begin{array}{c} \frac{1}{2}n_1, \frac{1}{2}n_1+n_2-2+\varepsilon \\ n_1+n_2-\frac{3}{2}+\varepsilon \end{array} \middle| 1-\frac{p_0^2}{m^2}\right) m^{d-n_1-2n_2}
$$

$$I_{n_1 n_2}(p_0) = \frac{\Gamma(n_1+n_2-2+\varepsilon)\Gamma(n_1+2n_2-4+2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1+n_2-2+\varepsilon))}$$

$$\times\, {}_2F_1\left( \begin{array}{c} n_1, n_1+2n_2-4+2\varepsilon \\ n_1+n_2-\frac{3}{2}+\varepsilon \end{array} \middle| \frac{1}{2}\left(1+\frac{p_0}{m}\right)\right) m^{d-n_1-2n_2}$$

$$= \frac{\Gamma(n_1+n_2-2+\varepsilon)\Gamma(n_1+2n_2-4+2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1+n_2-2+\varepsilon))}$$

$$\times\, {}_2F_1\left( \begin{array}{c} \frac{1}{2}n_1, \frac{1}{2}n_1+n_2-2+\varepsilon \\ n_1+n_2-\frac{3}{2}+\varepsilon \end{array} \middle| 1-\frac{p_0^2}{m^2}\right) m^{d-n_1-2n_2}$$

$m=0$:

$$I_{n_1 n_2}(p_0) = \frac{\Gamma(n_1+2n_2-d)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}(-2p_0)^{d-n_1-2n_2}$$

$$
\begin{aligned}
\frac{I_{122}}{\Gamma^3(1+\varepsilon)} = -\frac{1}{2\varepsilon^2}\Bigg[ & \frac{1}{1+2\varepsilon}\, {}_4F_3\left( \begin{array}{c} 1, \frac{1}{2}-\varepsilon, 1+\varepsilon, -2\varepsilon \\ \frac{3}{2}+\varepsilon, 1-\varepsilon, 1-2\varepsilon \end{array} \middle| 1 \right) \\
& - \frac{2}{1+4\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma^3(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+4\varepsilon)}\, {}_3F_2\left( \begin{array}{c} \frac{1}{2}, 1+2\varepsilon, -\varepsilon \\ \frac{3}{2}+2\varepsilon, 1-\varepsilon \end{array} \middle| 1 \right) \\
& + \frac{1}{1+6\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma^4(1+2\varepsilon)\Gamma(1-2\varepsilon)\Gamma^2(1+3\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+4\varepsilon)\Gamma(1-4\varepsilon)\Gamma(1+6\varepsilon)} \Bigg]
\end{aligned}
$$

$$\frac{I_{122}}{\Gamma^3(1+\varepsilon)} = -\frac{1}{2\varepsilon^2}\left[\frac{1}{1+2\varepsilon}\,{}_4F_3\left(\begin{array}{c} 1, \frac{1}{2}-\varepsilon, 1+\varepsilon, -2\varepsilon \\ \frac{3}{2}+\varepsilon, 1-\varepsilon, 1-2\varepsilon \end{array}\middle|\, 1\right)\right.$$
$$-\frac{2}{1+4\varepsilon}\frac{\Gamma^2(1-\varepsilon)\Gamma^3(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+4\varepsilon)}\,{}_3F_2\left(\begin{array}{c} \frac{1}{2}, 1+2\varepsilon, -\varepsilon \\ \frac{3}{2}+2\varepsilon, 1-\varepsilon \end{array}\middle|\, 1\right)$$
$$\left.+\frac{1}{1+6\varepsilon}\frac{\Gamma^2(1-\varepsilon)\Gamma^4(1+2\varepsilon)\Gamma(1-2\varepsilon)\Gamma^2(1+3\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+4\varepsilon)\Gamma(1-4\varepsilon)\Gamma(1+6\varepsilon)}\right]$$

Expansion up to $\varepsilon^7$ agrees with

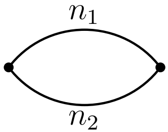
$$\frac{I_{122}}{\Gamma^3(1+\varepsilon)} = \frac{\pi^2}{3}\frac{\Gamma^3(1+2\varepsilon)\Gamma^2(1+3\varepsilon)}{\Gamma^6(1+\varepsilon)\Gamma(2+6\varepsilon)}$$

$$b(\varepsilon) = \frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)} \qquad g_n(\varepsilon) = \frac{b^n(\varepsilon)}{b(n\varepsilon)(1+2n\varepsilon)}$$

$$g_1(\varepsilon)\,{}_4F_3\left( \begin{array}{c} 1, \frac{1}{2}-\varepsilon, 1+\varepsilon, -2\varepsilon \\ \frac{3}{2}+\varepsilon, 1-\varepsilon, 1-2\varepsilon \end{array} \middle| 1 \right)$$

$$- 2g_2(\varepsilon)\,{}_3F_2\left( \begin{array}{c} \frac{1}{2}, 1+2\varepsilon, -\varepsilon \\ \frac{3}{2}+2\varepsilon, 1-\varepsilon \end{array} \middle| 1 \right) + g_3(\varepsilon) = 0$$

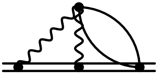
Mellin–Barnes




$$= \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{\left[m^2 - k^2 - i0\right]^{n_1} \left[m^2 - (k+p)^2 - i0\right]^{n_2}}$$

$$= \frac{m^{d-2(n_1+n_2)}}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz\, \Gamma(-z)$$

$$\times \frac{\Gamma(n_1+z)\Gamma(n_2+z)\Gamma(n_1+n_2-d/2+z)}{\Gamma(n_1+n_2+2z)} m^{-2z}$$

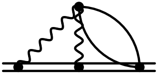


$$
= \frac{\Gamma^2(2\varepsilon)\Gamma(3\varepsilon-1)}{4\Gamma(4\varepsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz
$$

$$
\frac{\Gamma(1+z)\Gamma(1/2+\varepsilon+z)\Gamma(1+\varepsilon+z)\Gamma(-2\varepsilon-z)\Gamma(-\varepsilon-z)\Gamma(-z)}{\Gamma(3/2+\varepsilon+z)\Gamma(1-2\varepsilon-z)}
$$

$$
= -\Gamma^3(1+\varepsilon)\left[\frac{\pi^2}{9\varepsilon^2} - \frac{6\zeta_3 - 5\pi^2}{9\varepsilon} + \frac{11}{270}\pi^4 - \frac{10}{3}\zeta_3 + \frac{19}{9}\pi^2 \right.
$$

$$
\left. + \left(-\frac{8}{3}\zeta_5 + \frac{8}{9}\pi^2\zeta_3 + \frac{11}{54}\pi^4 - \frac{38}{3}\zeta_3 + \frac{65}{9}\pi^2\right)\varepsilon + \cdots\right]
$$

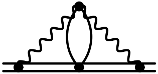


$$= \frac{\Gamma^2(2\varepsilon)\Gamma(3\varepsilon-1)}{4\Gamma(4\varepsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz$$

$$\frac{\Gamma(1+z)\Gamma(1/2+\varepsilon+z)\Gamma(1+\varepsilon+z)\Gamma(-2\varepsilon-z)\Gamma(-\varepsilon-z)\Gamma(-z)}{\Gamma(3/2+\varepsilon+z)\Gamma(1-2\varepsilon-z)}$$

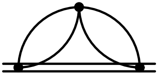
$$= -\Gamma^3(1+\varepsilon)\left[\frac{\pi^2}{9\varepsilon^2} - \frac{6\zeta_3 - 5\pi^2}{9\varepsilon} + \frac{11}{270}\pi^4 - \frac{10}{3}\zeta_3 + \frac{19}{9}\pi^2\right.$$

$$\left. + \left(-\frac{8}{3}\zeta_5 + \frac{8}{9}\pi^2\zeta_3 + \frac{11}{54}\pi^4 - \frac{38}{3}\zeta_3 + \frac{65}{9}\pi^2\right)\varepsilon + \cdots\right]$$



$$= \frac{\Gamma(1/2-\varepsilon)\Gamma(-\varepsilon)\Gamma^2(2\varepsilon)\Gamma(1+\varepsilon)\Gamma(3\varepsilon-1)}{4\Gamma(3/2-\varepsilon)\Gamma(4\varepsilon)}$$

$$\times \left[\psi(1/2-\varepsilon) + \psi(1-\varepsilon) - 2\ln 2 + 2\gamma_{\mathrm{E}}\right]$$

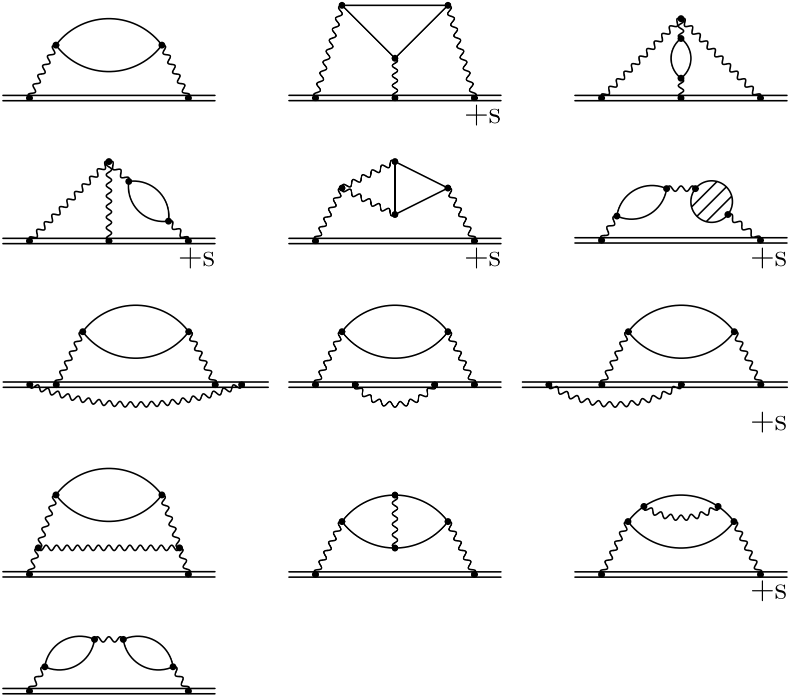


$$= \frac{\pi^{3/2}}{4^{\varepsilon}\Gamma(3/2-\varepsilon)}$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(1+z)\Gamma(3/2-\varepsilon+z)\Gamma(\varepsilon+z)}{\Gamma(3/2+z)\Gamma(\varepsilon-z)}$$

$$\times \Gamma(-1/2+\varepsilon-z)\Gamma(-3/2+2\varepsilon-z)\Gamma(-z)$$

$$= -\frac{32}{3}\pi^2 + \cdots$$


+S
+S
+S
+S
+S
+S

# Decoupling for the heavy–light current

$$\tilde{\jmath}(\mu_c) = \tilde{\jmath}'(\mu_c)\left\{1 + \frac{89}{36}C_F T_F \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right.$$

$$+ C_F T_F \left[\left(8B_4 - \frac{86}{405}\pi^4 + \frac{1427}{27}\zeta_3 + \frac{1600}{243}\pi^2 - \frac{7219}{162}\right) C_F\right.$$

$$- \left(4B_4 - \frac{43}{81}\pi^4 + \frac{1471}{54}\zeta_3 + \frac{400}{243}\pi^2 + \frac{3845}{486}\right) C_A$$

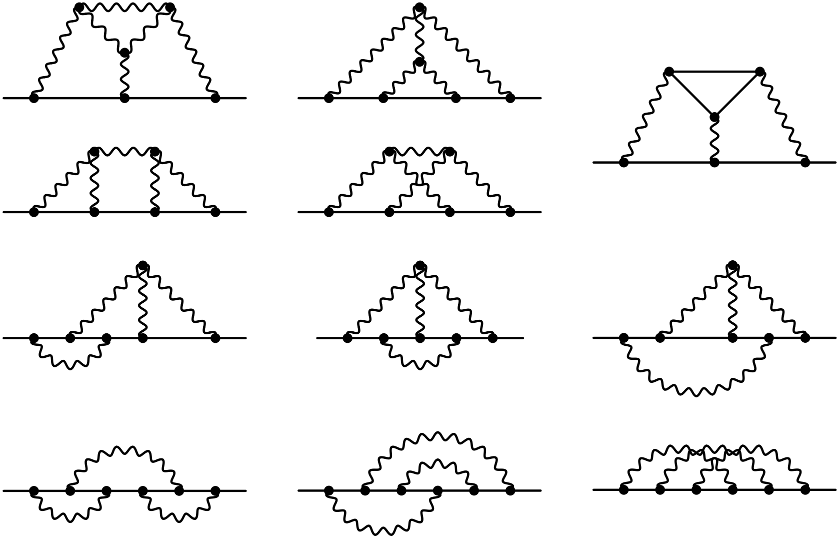
$$\left.\left. - \frac{2}{9}\left(32\zeta_3 - \frac{1327}{27}\right) T_F n_l + \frac{1}{9}\left(112\zeta_3 - \frac{1685}{27}\right) T_F\right] \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3\right\}$$

$$m_c(\mu_c) = \mu_c$$

$$B_4 = 16\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{2}{3}\ln^2 2\left(\ln^2 2 - \pi^2\right) - \frac{13}{180}\pi^4$$

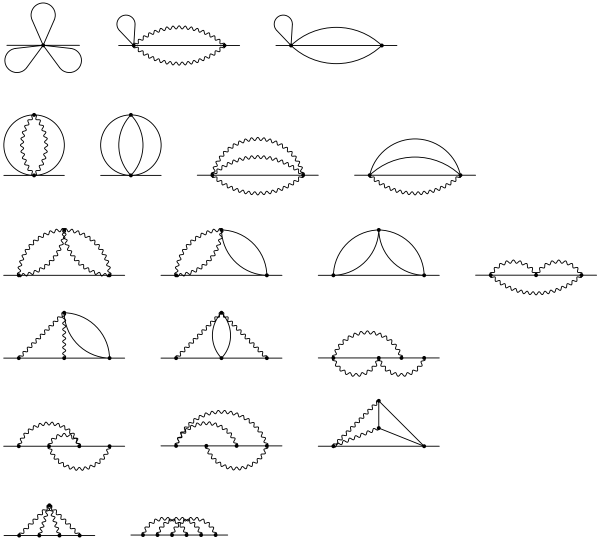
AG, A. Smirnov, V. Smirnov (2006)

# Massive on-shell diagrams



K. Melnikov, T. van Ritbergen (2000)

# Master integrals

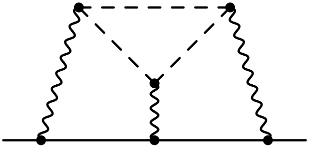


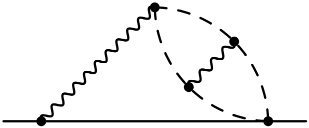
# Chromomagnetic interaction

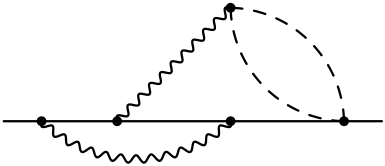
$$\gamma = \frac{\alpha_s}{\pi}\frac{1}{2}C_A + \left(\frac{\alpha_s}{\pi}\right)^2 C_A \left(\frac{17}{36}C_A - \frac{13}{36}T_F n_l\right)$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \left\{ \left(\frac{1}{8}\zeta_3 + \frac{899}{1728}\right) C_A^3 + \frac{1}{2}\pi^2 \frac{d_F^{abcd} d_A^{abcd}}{C_F N_F} \right.$$
$$- \left[ \left(\frac{1}{2}\zeta_3 + \frac{65}{216}\right) C_A^2 - \left(\frac{1}{2}\zeta_3 - \frac{49}{96}\right) C_A C_F + \frac{1}{36} C_A T_F n_l \right] T_F n_l$$
$$\left. - \frac{2}{3}\pi^2 \frac{d_F^{abcd} d_F^{abcd}}{C_F N_F} n_l \right\}$$

AG, P. Marquard, J. Piclum, M. Steinhauser (2008)

# Massive on-shell diagrams with 2 masses







# Master integrals