Higher moments of heavy quark correlators in the low energy limit at $\mathcal{O}(\alpha_s^2)$

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SFB TR9

Loops and Legs 2008

Outline

Motivation

- Comparison with moments extracted from experimental data
  $\longrightarrow$ Precise determination of charm and bottom quark masses
- Comparison with large n behaviour from threshold expansion
  $\longrightarrow$ Cross-check between different kinematical regions
- Quark masses from lattice calculations
  $\longrightarrow$ high moments can be used to study non-perturbative effects

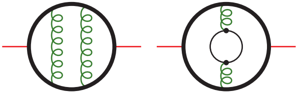
Definitions

Interested in correlators of the scalar, pseudo-scalar, vector and axial-vector currents of heavy quarks.

$$j^s = \bar{\Psi}\Psi, \quad j^p = \bar{\Psi}\gamma_5\Psi, \quad j^v_\mu = \bar{\Psi}\gamma_\mu\Psi, \quad j^a_\mu = \bar{\Psi}\gamma_\mu\gamma_5\Psi$$

$$(-q^2 g_{\mu\nu} + q_\mu q_\nu)\Pi^\delta_T(q^2) + q_\mu q_\nu \Pi^\delta_L(q^2) = i\int dx\, e^{iqx}\langle 0|T j^\delta_\mu(x) j^\delta_\nu(0)|0\rangle$$

$$q^2\Pi^\delta(q^2) = i\int dx\, e^{iqx}\langle 0|T j^\delta(x) j^\delta(0)|0\rangle$$



$$= \quad \Pi^\delta(q^2) = \frac{3}{16\pi^2}\sum\nolimits_{n>0} C^\delta_n \left(\frac{q^2}{4m^2}\right)^n$$

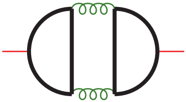
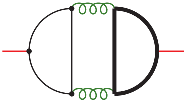
Definitions

$$C_n^{\delta} = C^{(0),\delta} + C_F \frac{\alpha_s}{\pi} C_n^{(1),\delta} + \left(\frac{\alpha_s}{\pi}\right)^2 C_n^{(2),\delta} + \cdots$$

Colour structure at three-loop

$$\begin{aligned} C_n^{(2),\delta} &= C_F^2 C_{A,n}^{(2),\delta} + C_F C_A C_{NA,n}^{(2),\delta} \\ &+ C_F T_F n_l C_{l,n}^{(2),\delta} + C_F T_F n_h C_{h,n}^{(2),\delta} + C_F C_{S,n}^{(2),\delta} \end{aligned}$$

The singlett contribution $C_{S,n}^{(2),\delta}$

 ,  ,

starting at 3-loop contains logarithms of the form $\log(\frac{q^2}{4m^2})$.

What's known?

- first 8 moments at $\mathcal{O}(\alpha_s^2)$ [Chetyrkin, Kühn, Steinhauser, '96]
- moments up to $n = 30$ at $\mathcal{O}(\alpha_s^2)$ [Czakon et. al.]
  [Maier, Maierhöfer, PM]
- first physical moment at $\mathcal{O}(\alpha_s^3)$ [Chetyrkin, Kühn, Sturm]]
  [Boughezal, Czakon, Schutzmeier]
- all orders result for n_f^2 at $\mathcal{O}(\alpha_s^3)$ [Grozin, Sturm]
- moments up to $n = 30$ for n_f^2 at $\mathcal{O}(\alpha_s^3)$ [Czakon et. al.]

Strategy I

1. Expand propagator diagrams in $\frac{q^2}{4m^2}$
2. Reduce resulting integrals to master integrals using IBP relations

- after expansion only one-scale problem (dimension)

$\longrightarrow$ Reduction using Laporta algorithm quite easy.

- each moment adds two powers of the propagators

$\longrightarrow$ Combinatorics makes reduction very time- and memory-consuming

- more than $\sim$ 10 moments not reachable

Strategy II

1. Reduce propagators to master integrals
2. Calculate 'master' propagators in expansion in $\frac{q^2}{4m^2}$

- Reduction is a two-scale problem (dimension + $\frac{q^2}{4m^2}$)

$\longrightarrow$ Reduction more complicated, but
  - only few additional powers of propagators (self-energy insertions)
  - constant effort for all moments
- calculation of masters can be completely automatized

Calculation of Master Integrals I

- master integrals satisfy scaling equation

$$(q^2 \frac{\partial}{\partial q^2} + m^2 \frac{\partial}{\partial m^2} + \hat{D}) M_i(q^2) = 0$$

[Remiddi]

  $\hat{D}$ gives the mass dimension of the master integral
- $\frac{\partial}{\partial m^2}$ raises the power of massive lines
  $\longrightarrow$ reduce to master integrals
- coupled system of differential equations for the master integrals

$$\frac{\partial}{\partial q^2} M_i(q^2) = \sum A_{i,k} M_k(q^2)$$

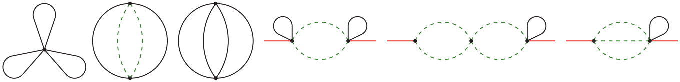
Calculation of Master Integrals II

- try to solve directly or expand in q^2

$$M_i(q^2) = \sum \left(\left(q^2\right)^k M_{i,k}^{(0)} + \left(q^2\right)^{k-\epsilon} M_{i,k}^{(1)} + \left(q^2\right)^{k-2\epsilon} M_{i,k}^{(2)} \right)$$

$\rightarrow$ system of linear equations for the coefficient $M_{i,k}^{(n)}$

- at three loops: 54 master integrals (36 non-singlet + 18 singlet)
- everything can be expressed in terms of a few tadpole master integrals.



Setup of the calculation

- diagrams generated with qgraf [Nogueira '91]
- topologies identified and mapped with the help of q2e and exp [Harlander, Seidensticker, Steinhauser '96/'97]
- calculation done with form [Vermaseren '00]
- follow Strategy II: first reduce than expand
- reduction to master integrals using IBP identities with `crusher` [PM, D. Seidel]

Results

- Example: ninth moment of the pseudo-scalar correlator

$$
\begin{aligned}
C_9^{(2),p} = &+ C_F T_R \Big(\frac{24213861812657343488}{4607166404718380025} n_l - \frac{23936615221766899739377}{13951233275688124416000} n_h \\
&+ \frac{386477922523}{289910292480} n_h \zeta_3 + \frac{365917995900928}{1504105830352125} L(n_l + n_h) + \frac{262144}{2078505} L^2 (n_l + n_h) \Big) \\
&+ C_F C_A \Big(- \frac{2774476486497511252284870305976724044391}{9695938227392717678410570137600000} + \frac{1870074094390219597606687}{7855857679623782400} \zeta_3 \\
&- \frac{151838577682432}{136736893668375} L - \frac{65536}{188955} L^2 \Big) + C_F^2 \Big(- \frac{18248707238521284112217374676684189}{1201215870545458762874880000} \\
&+ \frac{143085894538571009718455041}{11321677244163686400} \zeta_3 + \frac{372897204109312}{167122870039125} L + \frac{1048576}{230945} L^2 \Big)
\end{aligned}
$$

$$L = \log \frac{\mu^2}{4m^2}$$

Application: Mass Determination I

- By comparing the theoretical moments with the experimental ones the charm and bottom quark masses can be determined.
- Take R-ratio

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

in combination with the dispersion relation

$$\Pi(q^2) = \frac{q^2}{12\pi^2} \int ds \, \frac{R(s)}{s(s-q^2)} + \text{const}$$

- Compare Taylor moments on both sides

Application: Mass Determination II

- Each moment can be used for an independent determination of the quark masses since they are sensitive to different energy regions.
- Used at three- and four-loop accuracy.
- At four-loop only the first moment known. Estimates are used for the higher moments.
  $\longrightarrow$ Compare talk by J.H. Kühn

Threshold Expansion I

- consider production of heavy quarks near threshold
- calculate $R(s) = R(\beta)$ in expansion in the velocity β of the quarks
- moments can then be compared to the large n expansion from threshold calculations

$$C_n = \frac{4}{9}\int \frac{ds}{s} R(s) \left(\frac{4m^2}{s}\right)^n = \frac{4}{9}\int_0^1 d(\beta^2) R(\beta) (1-\beta^2)^{n-1}$$

- expansion only valid for small $\beta = \sqrt{1 - \frac{4m^2}{s}}$
  $\Rightarrow$ reliable prediction of high moments only.

Threshold Expansion II

- cross section for $e^+e^- \to Q\bar{Q}$

$$\sigma(e^+e^- \to Q\bar{Q}) = \sigma_0 \left(1 + C_F \frac{\alpha_s}{\pi} \Delta^{(1)} + C_F \left(\frac{\alpha_s}{\pi}\right)^2 \Delta^{(2)} + \cdots \right)$$

$$\sigma_0 = \frac{4}{3} \frac{\alpha^2}{s} N_c e_Q^2 \frac{\beta(3-\beta^2)}{2}$$

- leading and subleading behaviour in $\frac{\alpha_s^n}{\beta^n}$ given by Sommerfeld-Sakharov factor

$$|\Psi(0)|^2 = \frac{z}{1-\exp(-z)}, \quad z = \frac{C_F \alpha_s \pi}{\beta}$$

and a universal hard factor $(1 - 4C_F \frac{\alpha_s}{\pi})$

$$\sigma \approx \sigma_0 |\Psi(0)|^2 (1 - 4C_F \frac{\alpha_s}{\pi})$$

Threshold Expansion III

- Consider abelian part $\sigma_A^{(2)}$ at $\mathcal{O}(\alpha_s^2)$

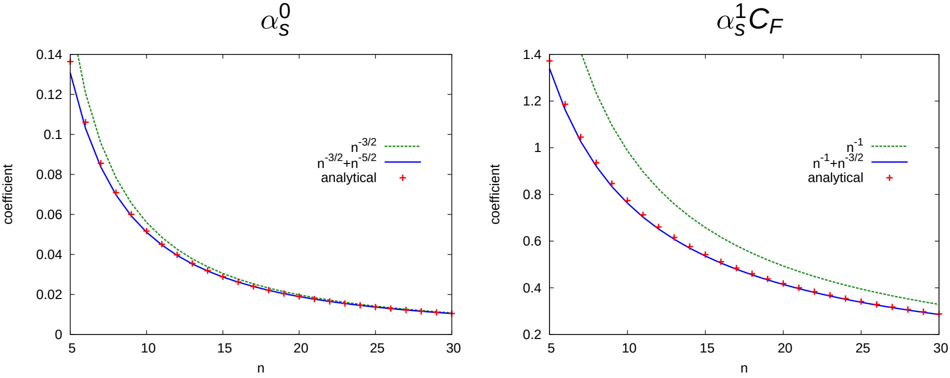
$$\sigma_A^{(2)} = \sigma_0 C_F^2 \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{\pi^4}{12\beta^2} - 2\frac{\pi^2}{\beta} + \frac{\pi^4}{6} + \pi^2\left(-\frac{35}{18} - \frac{2}{3}\log\beta + \frac{4}{3}\log 2\right) + \frac{39}{4} - \zeta_3\right]$$

[Czarnecki, Melnikov '97]

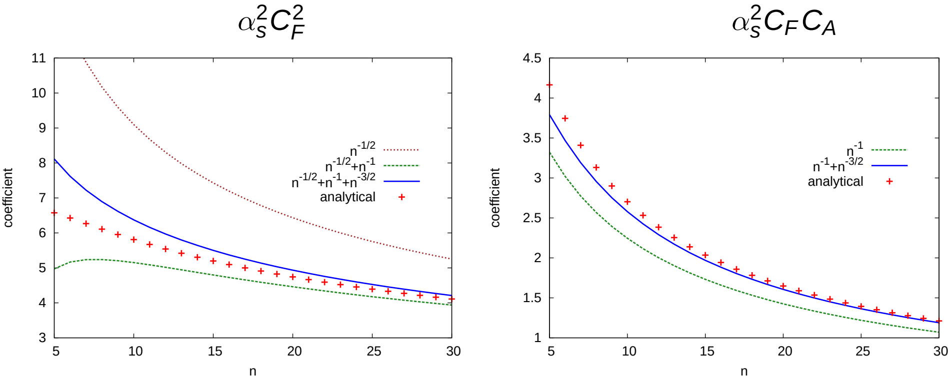
- transform to momentum space $\beta^k \to n^{-k/2-1}$ to obtain behaviour of the moments for large n

$$C_{A,n}^{(2),v} = \frac{\pi^{9/2}}{6} n^{-1/2} - 4\,\pi^2\, n^{-1} + \frac{\sqrt{\pi}}{144}\left(23\pi^4 + 8\pi^2(6H_{n+\frac{1}{2}} - 47 + 36\log(2)) + 36(39 - 4\zeta_3)\right) n^{-3/2} + \mathcal{O}(n^{-2})$$

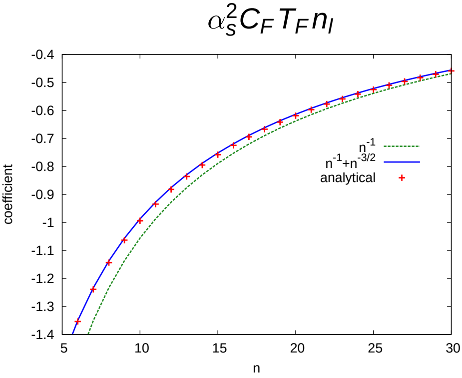
Comparison I

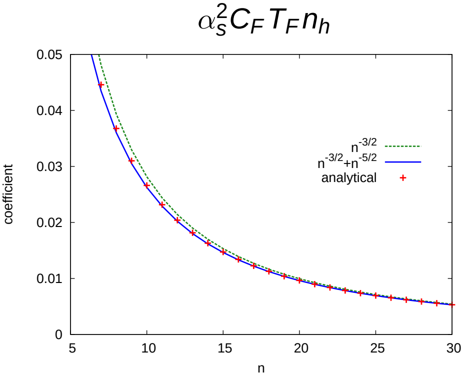


Comparison II



Comparison III





Summary and Outlook

- We calculated the first 30 moments in the low energy expansion of the correlators of scalar, pseudo-scalar, vector and axial-vector currents.
- Comparing the results with the large *n* behaviour obtained from threshold calcultations we find good agreement.
- Outlook: Extend calculation to $\mathcal{O}(\alpha_s^3)$.

Backup Slides

Second Moment of the vector correlator at $\mathcal{O}(\alpha_s^3)$

in collaboration with A. Maier, P. Maierhöfer, A. Smirnov (preliminary)

$$
\begin{aligned}
C_2 = &+\Big(-\frac{64985074258811347}{353072079360000} + \frac{1662518706713}{21016195200}\log^4 2 - \frac{362601376}{54729675}\log^5 2 + \frac{2900811008}{3648645}a5 \\
&+ \frac{1662518706713}{875674800}a4 + \frac{164928917}{270270}\zeta_5 + \frac{26401638588211}{28021593600}\zeta_4 + \frac{1684950406}{3648645}\zeta_4 \log 2 - \frac{112680551036302633}{47076277248000}\zeta_3 \\
&- \frac{1662518706713}{3502699200}\zeta_2 \log^2 2 + \frac{725202752}{10945935}\zeta_2 \log^3 2\Big) + n_l C_F C_A T_F \Big(+ \frac{22559166733}{16796160000} + \frac{520999}{4354560}\log^4 2 + \frac{520999}{181440}a4 \\
&- \frac{167529079}{5806080}\zeta_4 + \frac{309132631}{12902400}\zeta_3 - \frac{520999}{725760}\zeta_2 \log^2 2\Big) + n_l C_F^2 T_F \Big(- \frac{357543003871}{11757312000} - \frac{520999}{2177280}\log^4 2 - \frac{520999}{90720}a4 \\
&- \frac{598455689}{2903040}\zeta_4 + \frac{36896356307}{174182400}\zeta_3 + \frac{520999}{362880}\zeta_2 \log^2 2\Big) + n_l^2 C_F T_F^2 \Big(- \frac{15441973}{19136250} + \frac{32}{45}\zeta_3\Big) \\
&+ n_h C_F C_A T_F \Big(+ \frac{680718452445797}{56491532697600} + \frac{15936929}{3628800}\log^4 2 + \frac{15936929}{151200}a4 - \frac{362}{63}\zeta_5 - \frac{428540059}{3870720}\zeta_4 \\
&+ \frac{95106517892129}{1394852659200}\zeta_3 - \frac{15936929}{604800}\zeta_2 \log^2 2\Big) + n_h C_F^2 T_F \Big(+ \frac{73773357786383}{641949235200} + \frac{134095979}{2419200}\log^4 2 \\
&+ \frac{134095979}{100800}a4 - \frac{129051911}{92160}\zeta_4 + \frac{12649494611119}{15850598400}\zeta_3 - \frac{134095979}{403200}\zeta_2 \log^2 2\Big) \\
&+ n_h n_l C_F T_F^2 \Big(- \frac{95040709}{62705664} + \frac{2029}{41472}\log^4 2 + \frac{2029}{1728}a4 - \frac{99421}{55296}\zeta_4 + \frac{12159109}{4644864}\zeta_3 - \frac{2029}{6912}\zeta_2 \log^2 2\Big) \\
&+ n_h^2 C_F T_F^2 \Big(- \frac{1842464707}{646652160} + \frac{2744471}{1064448}\zeta_3\Big)
\end{aligned}
$$

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- Comparing the results with the large *n* behaviour obtained from threshold calcultations we find good agreement.
- Second moment at $\mathcal{O}(\alpha_s^3)$ calculated using Strategy I.
- Outlook: Extend calculation to $\mathcal{O}(\alpha_s^3)$ using Strategy II.