Blowing up Feynman integrals

Stefan Weinzierl

Institut für Physik, Universität Mainz

in collaboration with Ch. Bogner

- **I. Introduction**: **Sector decomposition**
- **II. Practical**: **An open-source implementation**
- **III. Formal**: **Feynman integrals and periods**

Credits

Sector decomposition is a method to compute numerically the Laurent expansion of divergent multi-loop integrals.

- K. Hepp, Commun. Math. Phys. 2, (1966), 301
- M. Roth and A. Denner, Nucl. Phys. B479, (1996), 495
- T. Binoth and G. Heinrich, Nucl. Phys. B585, (2000), 741

Multi-loop integrals

The general l-loop integral with n propagators:

$$I_G = \frac{\Gamma(\nu_1)...\Gamma(\nu_n)}{\Gamma(\nu - lD/2)} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}$$

The q_j are linear combinations of the loop momenta k_r and the external momenta.

$$I_G = \int_0^\infty \left(\prod_{j=1}^{n} dx_j x_j^{\nu_j - 1} \right) \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu - lD/2}}, \qquad \nu = \sum_{j=1}^{n} \nu_j.$$

The functions $\mathcal{U}$ and $\mathcal{F}$ are graph polynomials, homogeneous of degree l and $l+1$, respectively.

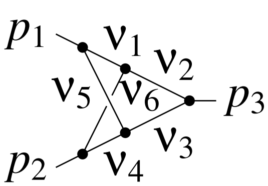
Graph polynomials

Cutting l lines of a given connected l-loop graph such that it becomes a connected tree graph defines a monomial of degree l. $\mathcal{U}$ is given as the sum over all such monomials.

Cutting one more line leads to two disconnected trees. The corresponding monomials are of degree $l+1$. The function $\mathcal{F}_0$ is the sum over all such monomials times minus the square of the sum of the momenta flowing through the cut lines and

$$\mathcal{F}(x) = \mathcal{F}_0(x) + \mathcal{U}(x) \sum_{j=1}^{n} x_j m_j^2.$$

Example: The massless two-loop non-planar vertex.



$$\begin{aligned}
\mathcal{U} &= x_{15}x_{23} + x_{15}x_{46} + x_{23}x_{46}, \\
\mathcal{F} &= \left(x_1x_3x_4 + x_5x_2x_6 + x_1x_5x_{2346}\right)\left(-p_1^2\right) \\
&\quad + \left(x_6x_3x_5 + x_4x_1x_2 + x_4x_6x_{1235}\right)\left(-p_2^2\right) \\
&\quad + \left(x_2x_4x_5 + x_3x_1x_6 + x_2x_3x_{1456}\right)\left(-p_3^2\right).
\end{aligned}$$

Laurent expansion

$$I_G = \int_0^\infty \left(\prod_{j=1}^{n} dx_j x_j^{\nu_j - 1} \right) \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu - lD/2}},$$

$\mathcal{U}$ is a homogeneous polynomial in the Feynman parameters of degree l, positive definite inside the integration region and positive semi-definite on the boundary.

$\mathcal{F}$ is a homogeneous polynomial in the Feynman parameters of degree $l+1$ and depends in addition on the masses m_i^2 and the momenta $(p_{i_1} + ... + p_{i_r})^2$.
In the euclidean region it is also positive definite inside the integration region and positive semi-definite on the boundary.

Laurent expansion in $\varepsilon = (4-D)/2$:

$$I_G = \sum_{j=-2l}^{\infty} c_j \varepsilon^j.$$

Part II: An open-source implementation of the algorithm of sector decomposition

$$I_G = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \left(\prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \right) \prod_{j=1}^{r} \left[P_j(x) \right]^{d_j + \varepsilon f_j} .$$

Algorithm:

(Binoth and Heinrich)

Step 1: Decompose into n primary sectors.
Step 2: Iterate sector decomposition until all polynomials are monomialised.
Step 3: Taylor expansion in the integration variables.
Step 4: Laurent expansion in ε.
Step 5: Numerical integration.

Important: Termination of step 2.

Annotations on the implementation

Algorithm involves symbolic and numerical part.

Implemented in C++ using the GiNaC-library.

Module to generate optimized code.

Various strategies to choose the sub-sectors.

Number of sub-sectors:

Integral	Strategy A	Strategy B	Strategy C	Strategy X
Tbubble	58	48	48	48
Planar double-box	755	586	586	293
Non-planar double-box	1138	698	698	395

Available from http://wwwthep.physik.uni-mainz.de/˜stefanw

Part III: Feynman integrals and periods

- Statement of the theorem
- Definition of a period
- Proof of the theorem: Sector decomposition
- Hironaka's polyhedra game

Object of investigation

$$J = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \left(\prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \right) \prod_{j=1}^{r} [P_j(x)]^{d_j + \varepsilon f_j}.$$

The a's, b's, d's and f's are integers.
The P's are polynomials in the variables x_1, ..., x_n with rational coefficients. The polynomials are required to be positive inside the integration region, but may vanish on the boundaries of the integration region.

The integral J has a Laurent expansion

$$J = \sum_{j=j_0}^{\infty} c_j \varepsilon^j.$$

Theorem: The coefficients c_j of the Laurent expansion of the integral J are periods.

(Bogner, S.W., '07)

Definition of a period

A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

(Kontsevich, Zagier)

Domains defined by polynomial inequalities with rational coefficients are called semi-algebraic sets.

Example:

$$\pi = \iint_{x^2+y^2\leq 1} dx\, dy.$$

The set of all periods is countable.

Periods

Let $G \subset \mathbb{R}^n$ be a semi-algebraic set.
Let $f_1(x)$, $g_1(x)$, $f_2(x)$ and $g_2(x)$ be rational functions with rational coefficients.

If the integrals

$$J = \int_G d^n x \, \{ f_1(x) \, \ln g_1(x) + f_2(x) \, \ln g_2(x) \}$$

$$K = \int_G d^n x \, f(x) \, \ln g_1(x) \, \ln g_2(x)$$

are absolutely convergent, then they are periods.

Prior art: Igusa local zeta functions

Consider the following special case for a Feynman integral:

1. The graph has no external lines or all invariants s_T are zero.

2. All internal masses m_j are equal to 1.

3. All propagators occur with power 1, i.e. $\nu_j = 1$ for all j.

In this case the Feynman parameter integral reduces to

$$I_G = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \, \mathcal{U}^{-D/2}.$$

This integral is a Igusa local zeta function when viewed as a function of $D/2$.
The coefficients of the Laurent expansion are periods.

Belkale and Brosnan, '03

The algorithm of sector decomposition

Back to the general case:

$$J = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \left(\prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \right) \prod_{j=1}^{r} \left[P_j(x) \right]^{d_j + \varepsilon f_j}.$$

Algorithm:

Step 0: Convert all polynomials to homogeneous polynomials.
Step 1: Decompose into n primary sectors.
Step 2: Iterate sector decomposition until all polynomials are monomialised.
Step 3: Taylor expansion in the integration variables.
Step 4: Laurent expansion in ε.

Crucial: Termination of step 2.

Step 0: Convert to homogeneous polynomials

Convert all polynomials to homogeneous polynomials:

$$J = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \left(\prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \right) \prod_{j=1}^{r} [P_j(x)]^{d_j + \varepsilon f_j} .$$

Due to the presence of the delta-function we have

$$1 = x_1 + x_2 + ... + x_n.$$

Can multiply each term in each polynomial P_j by an appropriate power of $x_1 + ... + x_n$.

After step 0 we can assume that all polynomials are homogeneous.

Step 1: Generate primary sectors

Decompose the integral into n primary sectors as in

$$\int\limits_{x_j \geq 0} d^n x \;=\; \sum_{l=1}^{n} \int\limits_{x_j \geq 0} d^n x \prod_{i=1, i \neq l}^{n} \theta(x_l \geq x_i).$$

In the l-th primary sector substitute $x_j = x_l x'_j$ for $j \neq l$,
integrate out the variable x_l with the help of the delta-function.

After this step:

$$\int\limits_0^1 d^n x \prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \prod_{j=1}^{r} \left[P_j(x) \right]^{d_j + \varepsilon f_j},$$

- integral is now over the unit hypercube,
- the polynomials are positive semi-definite on the unit hypercube,
- zeros may only occur on coordinate subspaces,
- in general the polynomials P_j are no longer homogeneous.

Step 2: Iterate sector decomposition

Decompose the sectors iteratively into sub-sectors until each of the polynomials is monomialised, i.e. of the form

$$P(x) \quad = \quad C x_1^{m_1} \dots x_n^{m_n} \left(1 + P'(x)\right),$$

One iteration: Choose a subset $S = \{\alpha_1, ..., \alpha_k\} \subseteq \{1, \dots n\}$ according to a strategy. Decompose the k-dimensional hypercube into k sub-sectors according to

$$\int_0^1 d^n x \quad = \quad \sum_{l=1}^{k} \int_0^1 d^n x \prod_{i=1, i \neq l}^{k} \theta\left(x_{\alpha_l} \geq x_{\alpha_i}\right).$$

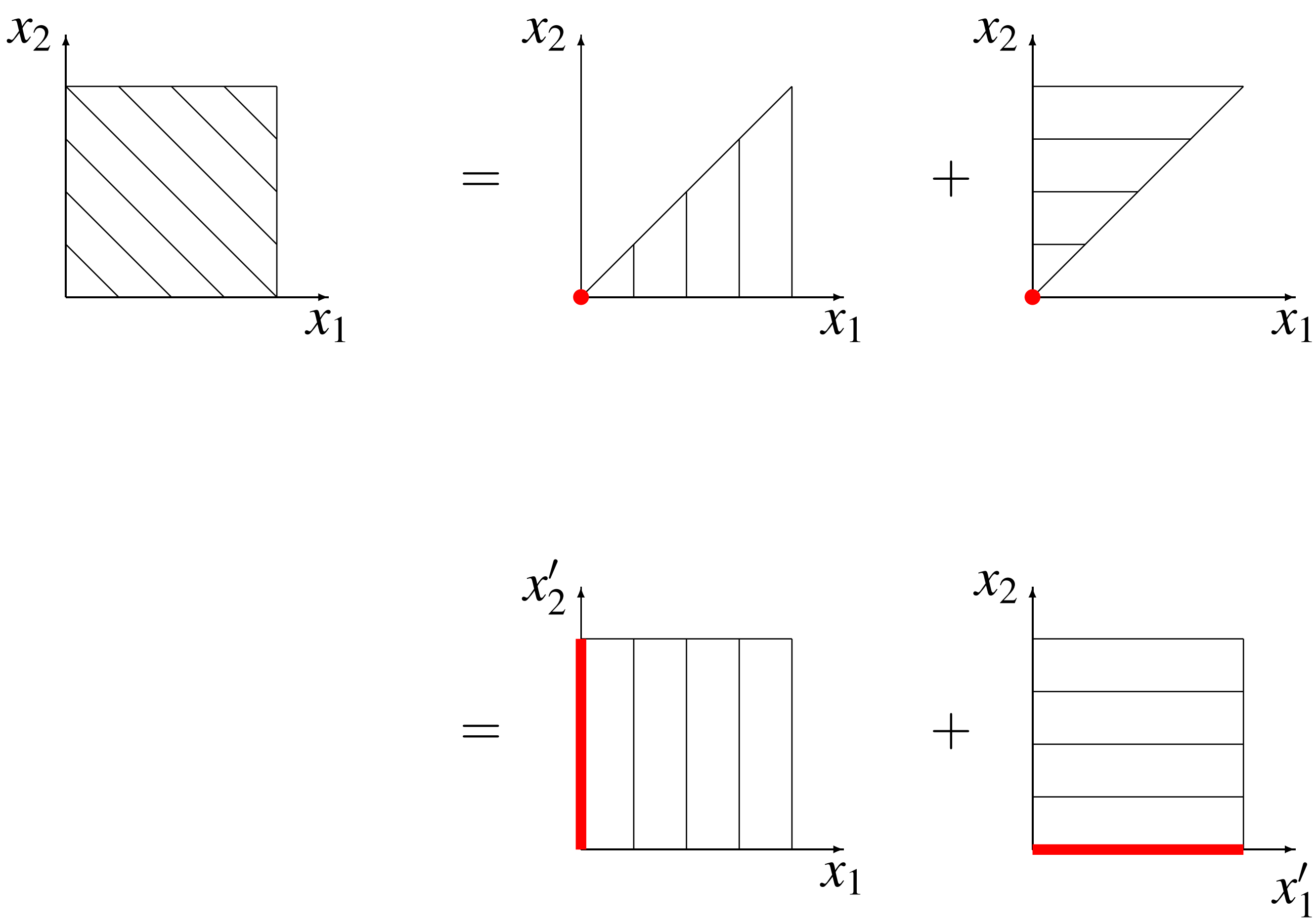
In the l-th sub-sector make for each element of S the substitution

$$x_{\alpha_i} \quad = \quad x_{\alpha_l} x'_{\alpha_i} \quad \text{for } i \neq l.$$

Iterate until all polynomials are monomialised.

Step 2: Blow-ups

Example: $S = \{1, 2\}$.
One iteration blows up the center $x_1 = x_2 = 0$.



Step 2: Strategies

How to choose S ? Have to avoid infinite recursion !

Example:

$$P(x_1, x_2, x_3) = x_1 x_3^2 + x_2^2 + x_2 x_3,$$
$$S = \{1, 2\}$$

First sub-sector: $x_1 = x_1'$, $x_2 = x_1' x_2'$, $x_3 = x_3'$.

$$P(x_1, x_2, x_3) = x_1' x_3'^2 + x_1'^2 x_2'^2 + x_1' x_2' x_3' = x_1' \left(x_3'^2 + x_1' x_2'^2 + x_2' x_3' \right) = x_1' P(x_1', x_3', x_2').$$

The choice $S = \{1, 2\}$ leads to an infinite recursion.

Step 2: Formulation of the problem

We have a product of polynomials

$$\prod_{j=1}^{r} P_j(x)$$

and seek a sequence of blow-ups, such that after a finite number of steps each polynomial is monomialised.

Reformulation:

$$\prod_{j=1}^{r} P_j(x) \quad = \quad 0$$

defines an algebraic variety. We look for a resolution of the singularities of an algebraic variety over a field of characteristic zero by a sequence of blow-ups.

Hironaka, 1964

Step 2: Hironaka's polyhedra game

Two players A and B are given a finite set M of points $m = (m_1, \ldots, m_n)$ in the first quadrant of $\mathbb{N}^n$. The positive convex hull of the set M is denoted by Δ.

1. Player A chooses a non-empty subset $S \subseteq \{1, \ldots, n\}$.

2. Player B chooses one element i out of this subset S.

3. All $(m_1, \ldots, m_n) \in M$ are replaced by new points $(m'_1, \ldots, m'_n)$:

$$m'_i = \sum_{j \in S} m_j - 1 \quad \text{and} \quad m'_j = m_j, \quad \text{if } j \neq i$$

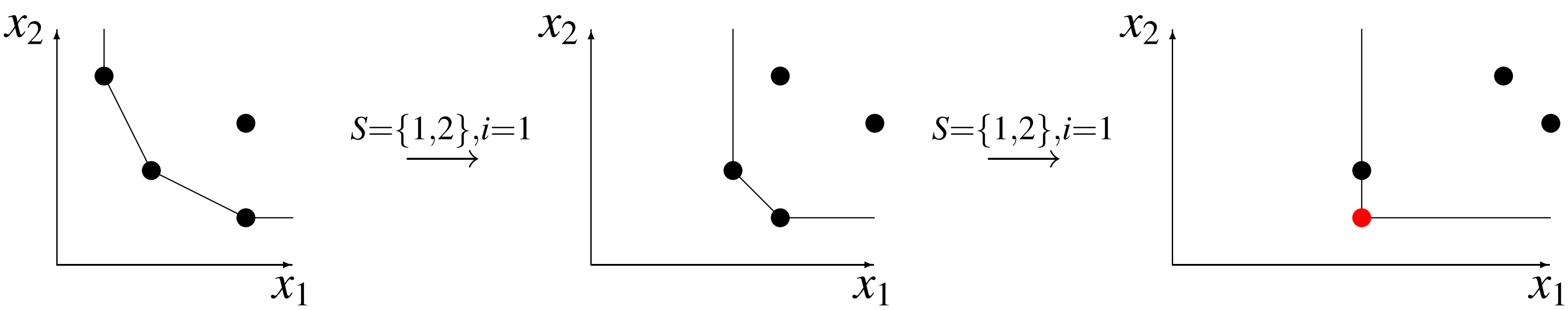
4. Set $M = M'$ and go back to step 1.

Player A wins the game if, after a finite number of moves, the polyhedron Δ is generated by one point, i.e. is of the form

$$\Delta \quad = \quad m + \mathbb{R}^n_+,$$

Step 2: Hironaka's polyhedra game

Example in two dimensions:



Player A wins the game!

Challenge: Find a winning strategy for player A.

Step 2: Winning strategies for the polyhedra game

Common to all winnig strategies is the definition of an invariant and a choice of S based on this invariant, such that any choice of player B will make this invariant decrease.

- Spivakovsky's strategy: The first solution to the polyhedra game.

  Spivakovsky 1983

- Zeillinger's strategy: The simplest one, but also the most inefficient one.

  Zeillinger 2006

- Encinas' and Hauser's strategy: Not restricted to principal ideals, with a lot of effort to make the proof understandable.

  Encinas and Hauser, 2003

See also: Villamayor, Bravo, Bierstone, Milman, ...

Step 3: Taylor expansion in the integration variables

We now have integrals of the form

$$\int_0^1 d^n x \prod_{i=1}^{n} x_i^{a_i+\varepsilon b_i} \prod_{j=1}^{r} \left[1+P'_j(x)\right]^{d_j+\varepsilon f_j},$$

For every x_j with $a_j < 0$ perform a Taylor expansion around $x_j = 0$:

$$\int_0^1 dx_j\, x_j^{a_j+b_j\varepsilon} I(x_j) = \int_0^1 dx_j\, x_j^{a_j+b_j\varepsilon} \left(\sum_{p=0}^{|a_j|-1} \frac{x_j^p}{p!} I^{(p)} + I^{(R)}(x_j) \right)$$

- The integration in the pole part can be carried out analytically.
- The remainder term is by construction integrable.

Step 3: Taylor expansion in the integration variables

At the end of step 3 we obtain a finite sum of integrals of the form

$$K(\varepsilon) = \frac{1}{g(\varepsilon)} \int_0^1 d^n x \, F(x, \varepsilon),$$

with

$$F(x,\varepsilon) = \sum_{j=1}^{N} f_j(x,\varepsilon), \qquad f_j(x,\varepsilon) = g_j(\varepsilon) \prod_{i=1}^{n} x_i^{a_i^j + \varepsilon b_i} \prod_{k=1}^{r} \left[P_k^j(x) \right]^{d_k^j + \varepsilon f_k}.$$

Here, $g(\varepsilon)$ and $g_j(\varepsilon)$ are polynomials in ε with integer coefficients. $P_k^j(x)$ is a polynomial with rational coefficients, non-vanishing on the unit hypercube. Further we have $a_i^j, b_i, d_i^j, f_i \in \mathbb{Z}$.

Step 3: Taylor expansion in the integration variables

$$\int_0^1 d^n x \, F(x, \varepsilon)$$

is convergent by construction for all ε in a neighbourhood of $\varepsilon = 0$. In one variable this integral is of the form

$$\int_0^1 dx \, x^{\varepsilon b} R(x, \varepsilon),$$

where the function $R(x, \varepsilon)$ does not contain any singularities on the integration domain and is therefore bounded. Therefore the integral is absolutely convergent for all ε with $|\varepsilon| < |1/b|$.

Step 4: Laurent expansion in ε

It remains to expand the integrals in ε: The expansion of the functions $1/g(\varepsilon)$ and $g_j(\varepsilon)$ yields rational numbers, for the other terms we have

$$
\begin{aligned}
x^{a+b\varepsilon} &= x^a \sum_{k=0}^{\infty} \frac{b^k}{k!} (\ln x)^k \varepsilon^k, \\
[P(x)]^{d+\varepsilon f} &= [P(x)]^d \sum_{k=0}^{\infty} \frac{f^k}{k!} (\ln (P(x)))^k \varepsilon^k.
\end{aligned}
$$

The integrals over $F_r(x)$ are absolutely convergent: In each variable we have integrals of the form

$$
\int_0^1 dx \, (\ln x)^k R_r(x), \quad k \in \mathbb{N}_0,
$$

where the function $R_r(x)$ does not contain any singularities on the integration domain and is therefore bounded.

Step 4: Laurent expansion in ε

Summary on sector decomposition:

- Each coefficient of the Laurent expansion is given as a finite sum of integrals.
- These integrals are absolutely convergent.
- All integrals are over the unit hypercube. This is clearly a semi-algebraic set.
- The integrands contain only rational functions with rational coefficients and logarithms thereof.

The coefficients of the Laurent expansion are periods and the theorem is proven.

Summary

- An open-source implementation of sector decomposition to compute numerically all coefficients of the Laurent expansion.

- A theorem on the Laurent expansion of Feynman integrals in the Euclidean region with all invariants rational:
  All coefficients are periods !

- “Upper limit” on the class of functions which can appear in multi-loop integrals.