From multileg loops to trees (by-passing Feynman's Tree Theorem)

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next-to-leading order (NLO) cross-sections

NLO @ LHC for 2→3 (many recent results)
and 2→4 (not yet a cross section)
(talk by T. Binoth)

$$\sigma^{NLO} = \int_{m+1} d\sigma^{R} + \int_{m} d\sigma^{V}$$

new feature wrt LO:
combine m with m+1

real radiation

virtual contribution

real radiation

kinematics: momentum conservation + observable dependent function

$$\int_{m+1} d\sigma^R = \int d\Phi^{(m+1)}(\{p_i\}) \quad \underbrace{M^{(m+1)}(\{p_i\}) \quad F^{(m+1)}(\{p_i\})}$$

split phase-space integrand in two parts:

$(\ldots)_{fin} + (\ldots)_{div}$

IR finite: computable numerically as LO

IR singular: analytically computable up to O(ε)

several well known/tested working methods (subtraction, dipole, slicing, mixed, ...)

virtual contribution

$$\int_m d\sigma^V = \int d\Phi^{(m)}(\{p_i\}) \quad \underbrace{\int d^d q \quad M^{(m)}(\{p_i\})} \quad F^{(m)}(\{p_i\})$$

loop integral: in multiparton processes (m≥5)
regarded as main practical bottleneck
many new developments in recent years

general goal

S. Catani @ HP^2 Zürich 2006
GR @ ILC Valencia 2006
T.Gleisberg PhD thesis

I transform loop integral into customary phase space integral for real radiation (loop ⇔ phase-space duality)

$$\int_{loop} d^d q \quad M^{(m)}(\{p_i\}, q) = \int d\Phi(q) \quad M^{(m+q)}(\{p_i\}, q)$$

$$d\Phi(q) \leftarrow d^d q \quad \delta_+(q^2)$$

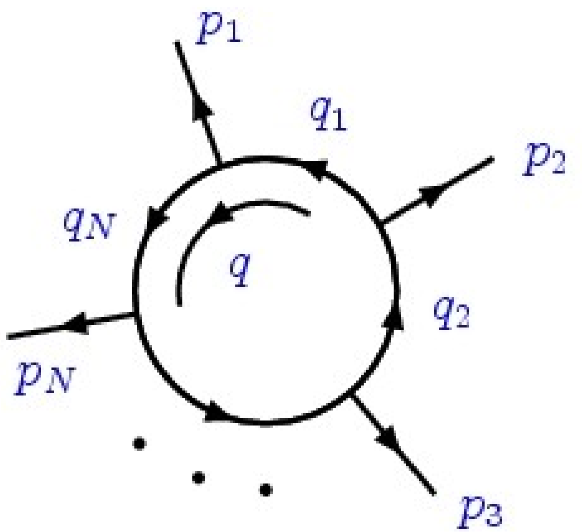
II then treat $\int_{m+q} (\dots)$
similarly to the real emission contribution $\int_{m+1} (\dots)$

III Monte Carlo integration

Outline

- The Feynman Tree Theorem
- A duality theorem between one-loop integrals and single-cut phase-space integrals
- Relating the FTT and the duality relation
- Massive integrals and unstable particles
- Gauge poles
- Duality at the amplitude level
- Final remarks

Notation



To simplify the presentation: massless internal lines only (more on massive particles later)

Scalar one-loop integral

$$L^{(N)}(p_1, \dots, p_N) = -i \int \frac{d^d q}{(2\pi)^d} \prod_{i=1}^{N} \frac{1}{q_i^2 + i0}$$

q^μ is the loop momentum (anti-clockwise)

internal lines

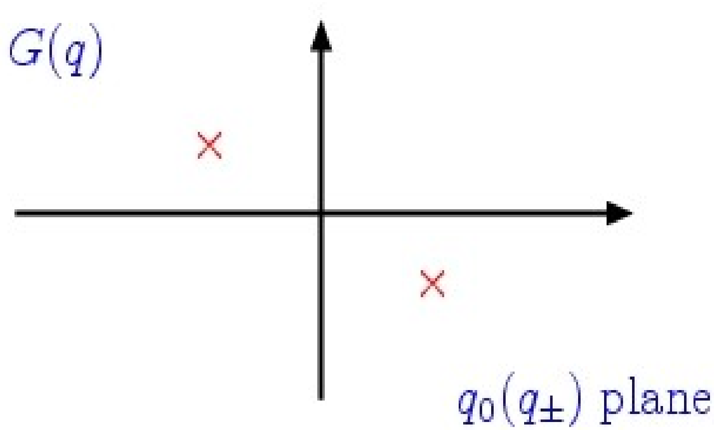
$$q_i = q + \sum_{k=1}^{i} p_k \;, \qquad \sum_{i=1}^{N} p_i = 0 \;, \qquad p_{N+i} = p_i \;.$$

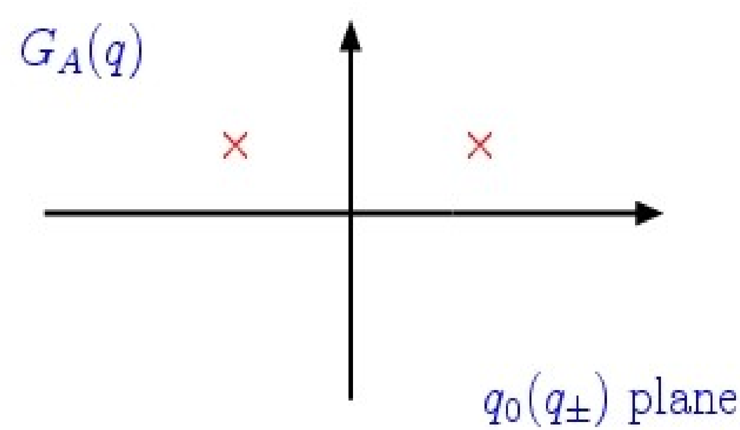
shorthand notation:

$$-i \int \frac{d^d q}{(2\pi)^d} \cdots \equiv \int_q \cdots \;, \qquad -i \int_{-\infty}^{+\infty} dq_0 \int \frac{d^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}} \cdots \equiv \int dq_0 \int_{\boldsymbol{q}} \cdots$$

$$\tilde{\delta}(q) \equiv 2\pi \; i \; \delta_+(q^2)$$

Feynman and Advanced propagators





Feynman propagator

$$G(q) \equiv \frac{1}{q^2+i0}$$

+i0: positive frequencies are propagated forward in time, and negative frequencies backward

Advanced propagator

$$G_A(q) \equiv \frac{1}{q^2-i0\,q_0}$$

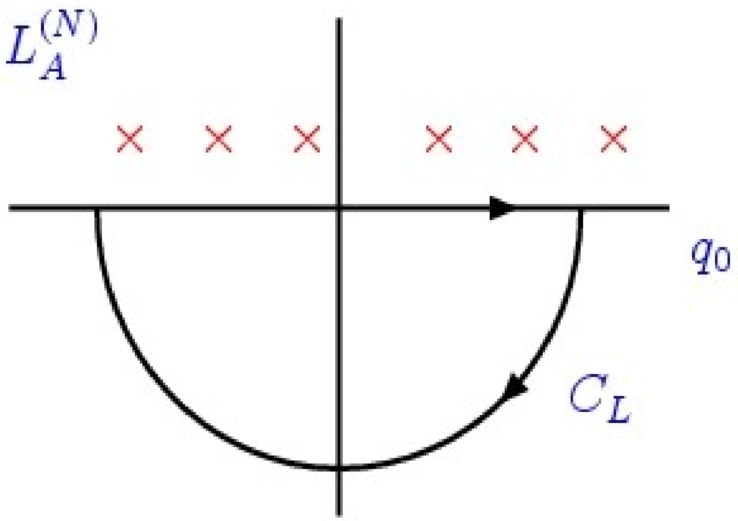
both poles displaced above the real axis (independently of the sign of the energy)

and are related by

$$\frac{1}{x\pm i0} = PV\left(\frac{1}{x}\right) \mp i\pi\,\delta(x)$$

$$G_A(q) = G(q) + \tilde{\delta}(q)$$

Feynman Tree Theorem



Advanced one-loop integral:
Feynman propagators replaced by advanced propagators

$$L_A^{(N)}(p_1, \dots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i)$$

Cauchy residue theorem

$$L_A^{(N)}(p_1, \dots, p_N) = 0$$

then

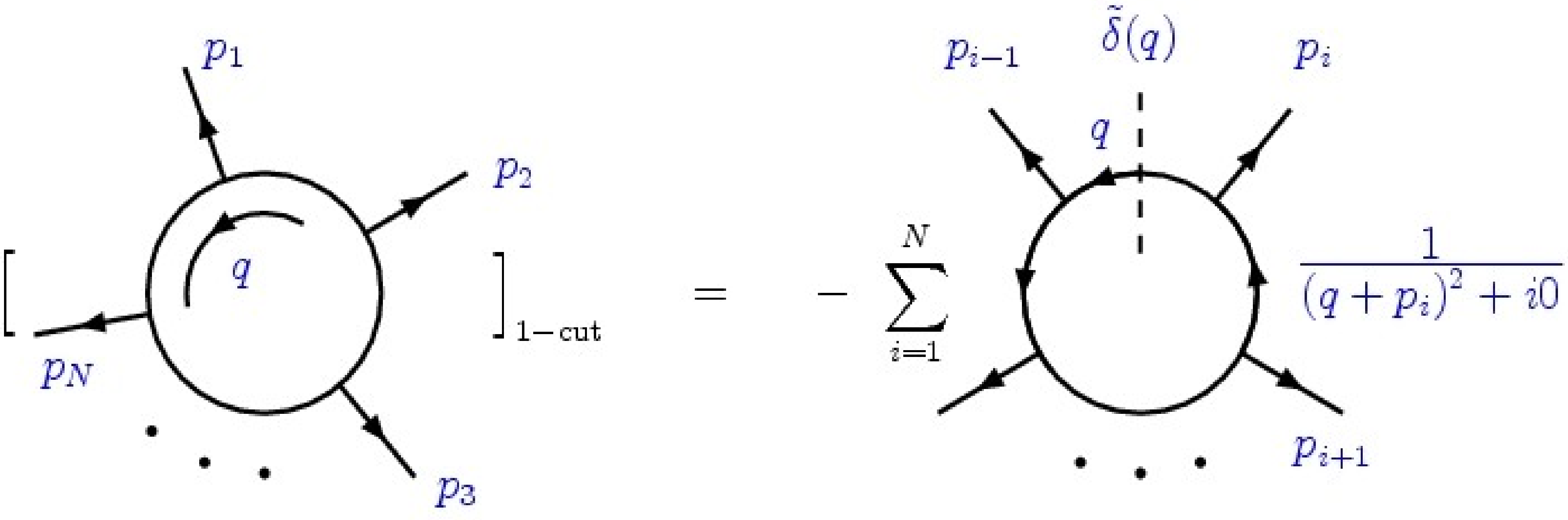
$$L_A^{(N)}(p_1, \dots, p_N) = \int_q \prod_{i=1}^{N} [G(q_i) + \tilde{\delta}(q_i)]$$
$$= L^{(N)} + L^{(N)}_{1-cut} + L^{(N)}_{2-cut} + \dots + L^{(N)}_{N-cut}$$

in four-dimensions, 4-cut maximum

Feynman Tree Theorem

$$L^{(N)}(p_1,\ldots,p_N) = -\left[L^{(N)}_{1-cut}(p_1,\ldots,p_N)+\ldots+L^{(N)}_{N-cut}(p_1,\ldots,p_N)\right]$$

The single-cut contribution



FTT for scattering amplitudes

For **relativistic, local and unitary** quantum field theories

$$\mathcal{A}^{(1-loop)} = - \left[\mathcal{A}^{(1-loop)}_{1-cut} + \mathcal{A}^{(1-loop)}_{2-cut} + \dots \right]$$

$\mathcal{A}^{(1\text{-}loop)}$ is a linear combination of one-loop integrals that differ from $L^{(N)}$ only by the inclusion of interaction vertices and, eventually, particle masses

- **particle masses:** real masses (unitary theories) do not affect the imaginary part of the poles

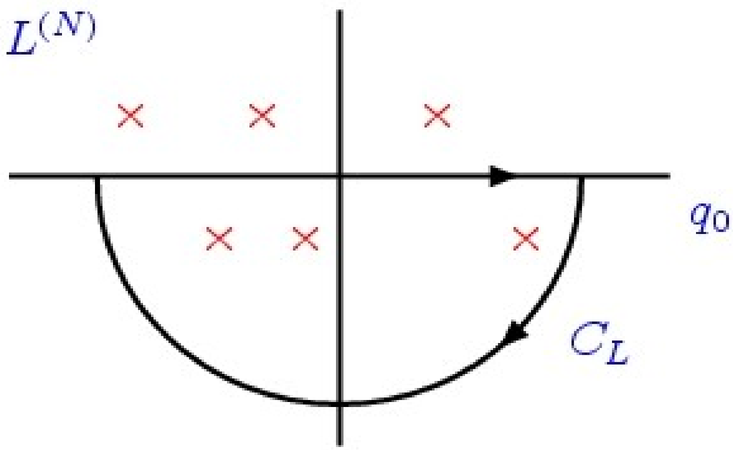
$$\tilde{\delta}(q_i) \to \tilde{\delta}(q_i, M_i) = 2\pi \; i \; \delta_+(q_i^2 - M_i^2)$$

- **interaction vertices:** introduce numerator factors

  in local theories at worst polynomials in the loop momentum ⇨ no additional singularities (more on gauge poles later)

  unitary constrains the convergence of the q_0 integration at infinity

Duality Theorem



Cauchy residue theorem

close the contour at ∞ on the lower half plane

⇨ select residues with **positive** definite energy

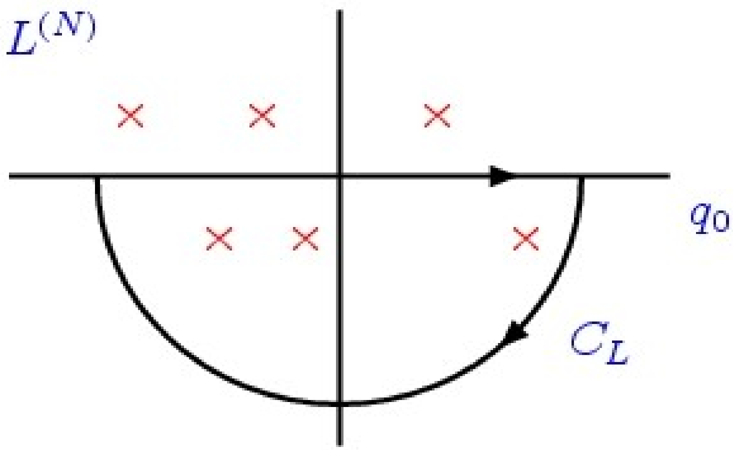
$$L^{(N)}(p_1,\dots,p_N) = -2\pi i \int_q \sum \mathrm{Res}_{\mathrm{Im}\, q_0<0}\left[\prod_{i=1}^{N} G(q_i)\right]$$

$$\mathrm{Res}_{ith-pole}\left[\prod_{j=1}^{N} G(q_j)\right] = \left[\mathrm{Res}_{ith-pole}\; G(q_i)\right]\left[\prod_{j\neq i}^{N} G(q_j)\right]_{ith-pole}$$

$$\mathrm{Res}_{ith-pole}\frac{1}{q_i^2+i0} = \int dq_0\, \delta_+(q_i^2)$$

- **equivalent to cut that line and set it on-shell**
- **one-loop integral represented as a linear combination of *N* single-cut phase-space integrals**
- **shift $q_i \to q$ in each term ⇔ single phase-space integral over *N* terms**

Duality Theorem



Cauchy residue theorem

$$L^{(N)}(p_1, \dots, p_N) = -2\pi\, i \int_{q} \sum Res_{\mathrm{Im}\, q_0 < 0} \left[\prod_{i=1}^{N} G(q_i) \right]$$

$$Res_{ith-pole} \left[\prod_{j=1}^{N} G(q_j) \right] = \left[Res_{\; ith-pole} \; G(q_i) \right] \left[\prod_{j \neq i}^{N} G(q_j) \right]_{ith-pole}$$

$$Res_{ith-pole} \frac{1}{q_i^2 + i0} = \int dq_0 \, \delta_+ (q_i^2)$$

$$\left[\prod_{j \neq i}^{N} \frac{1}{q_j + i0} \right]_{ith-pole} = \prod_{j \neq i}^{N} \frac{1}{q_j^2 - i0\, \eta (q_j - q_i)}$$

- the customary $+i0$ prescription is modified
- analytic continuation: $s_{ij} \to s_{ij}$ - $i0$ ***wrong***
- ***Lorentz covariant dual prescription***
- ***η is a future-like vector: $\eta_0 > 0$, $\eta^2 \geq 0$***

The calculation is elementary, but involves some **subtle points**

$$\left[\frac{1}{(q+k_j)^2+i0}\right]_{q^2=-i0,\, q_0=q_0^{(+)}} = \frac{1}{2q_0^{(+)}k_{j0}-2\boldsymbol{q}\cdot\boldsymbol{k_j}+k_j^2}$$

where $q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0}$

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where $$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}+O(i0^2)$$

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$$= \frac{1}{2qk_j+k_j^2-i0\,k_{j0}/|\boldsymbol{q}|}$$

$$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}$$

The calculation is elementary, but involves some **subtle points**

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$$= \frac{1}{2qk_j+k_j^2-i0\,k_{j0}/|\boldsymbol{q}|}$$

$$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}$$

- only the sign matters:

  $-\, i0\, k_{j0}/|\boldsymbol{q}| \rightarrow -\, i0\, k_{j0} \rightarrow -\, i0\, \eta k_j$ where $\eta^\mu = (\eta_0, 0)$ with $\eta_0 > 0$

- *different choices of the future-like vector η is equivalent to different choices of the coordinate system*

Duality theorem

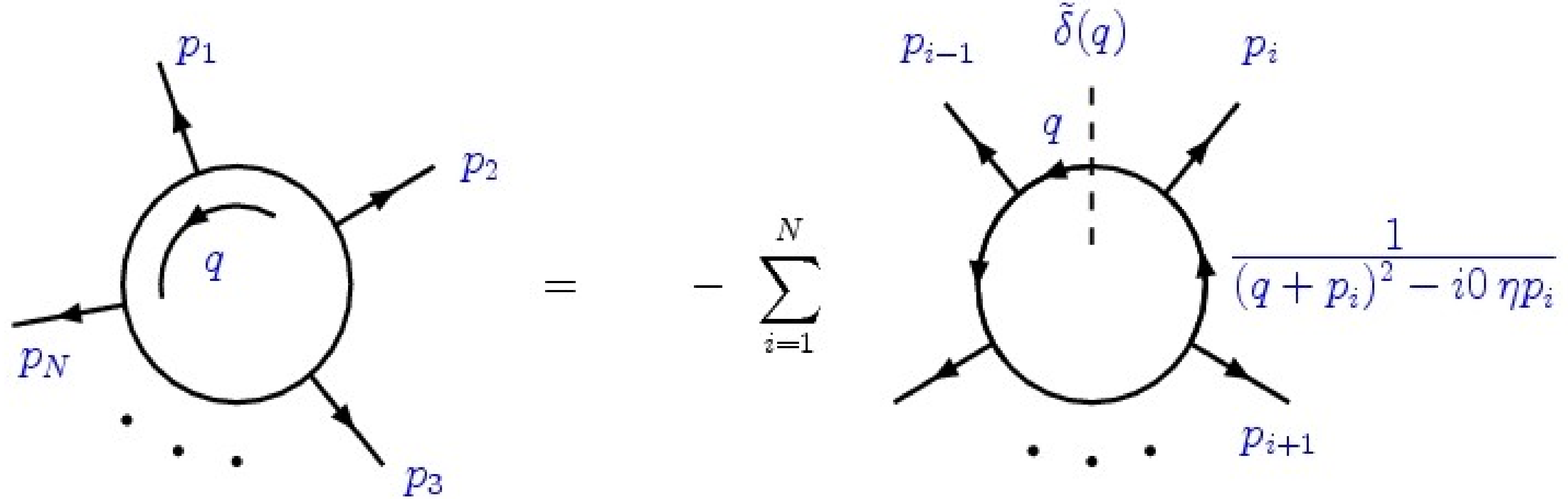
Duality relation between one-loop integrals and phase-space integrals

$$L^{(N)}(p_1,\dots,p_N) = -\tilde{L}^{(N)}(p_1,\dots,p_N)$$
$$= -\left[I^{(N-1)}(p_1,p_{12},\dots,p_{1,N-1}) + \textit{cyclic perms.}\right]$$

where

$$I^{(n)}(k_1,\dots,k_n) = \int_q \tilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2qk_j + k_j^2 - i0\eta k_j}$$

N one-particle phase-space integrals ⇔ one phase-space integral over *N* tree quantities



FTT-duality relation

- multiple-cut contributions (m≥2) are absent in the duality relation, only single-cut contributions are involved

- Feynman propagators ($+i0$) replaced by dual propagators ($-i0\ \eta k_i$)

- individual cut integrals depend on the future-like vector η^μ , it has to be the same for all, then it cancels

Relating FTT with duality

an algebraic proof

Feynman and dual propagators are related by

$$\tilde{\delta}(q)\frac{1}{2qk+k^2-i0\eta k} = \tilde{\delta}(q)\left[G(q+k)+\theta(\eta k)\tilde{\delta}(q+k)\right]$$

proof:

$$\frac{1}{x\pm i0}=PV\left(\frac{1}{x}\right)\mp i\pi\delta(x)$$

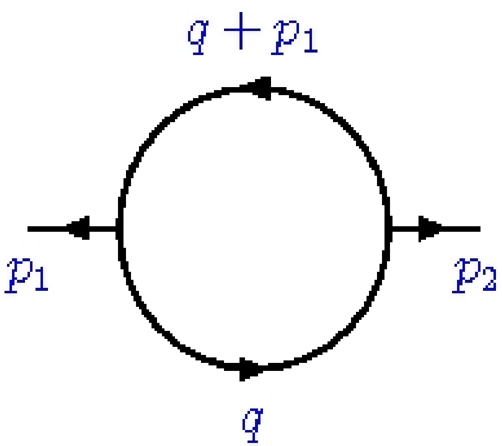
from $q^2=0$ and $q_0>0$ thus $\eta q>0$

plus $\theta(\eta k)$

$\eta(q+k)>0$ and $(q+k)^2=0$

$\delta((q+k)^2) \rightarrow \delta_+((q+k)^2)$

Two-point function



$$\tilde{L}^{(2)}(p_1,p_2) = \int_q \tilde{\delta}(q) \left[\left[G(q+p_1) + \theta(\eta\, p_1)\tilde{\delta}(q+p_1) \right] + \left[1 \Leftrightarrow 2 \right] \right]$$

$$= L^{(2)}_{1-cut}(p_1,p_2) + \left[\theta(\eta\, p_1) + \theta(\eta\, p_2) \right] L^{(2)}_{2-cut}(p_1,p_2)$$

momentum conservation

$p_1+p_2=0$ ⇨ **1** ✔

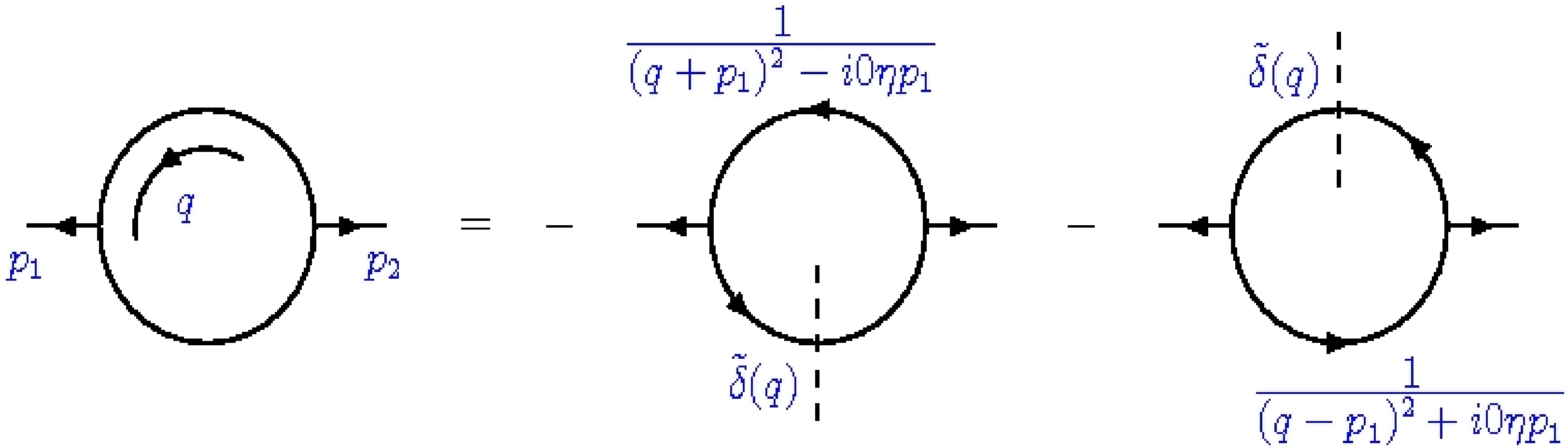
N-point function

The key ingredient is to proof (e.g. by induction)
the following algebraic identity

$$\theta(\eta\, p_1)\theta(\eta\, p_{12}) \;\cdots\; \theta(\eta\, p_{1,N-1}) \;+\; \textit{cyclic perms.} \;=\; 1$$

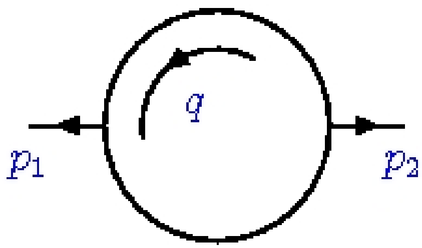
which follows from momentum conservation

Two-point function from duality



$$\tilde{L}^{(2)}(p_1, p_2) = I^{(1)}(p_1) + (p_1 \Leftrightarrow -p_1)$$

$$I^{(1)}(k) = -\frac{c_\Gamma}{2} \frac{(-k^2 - i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left[1 - i \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} sign(k^2 \eta k) \right]$$



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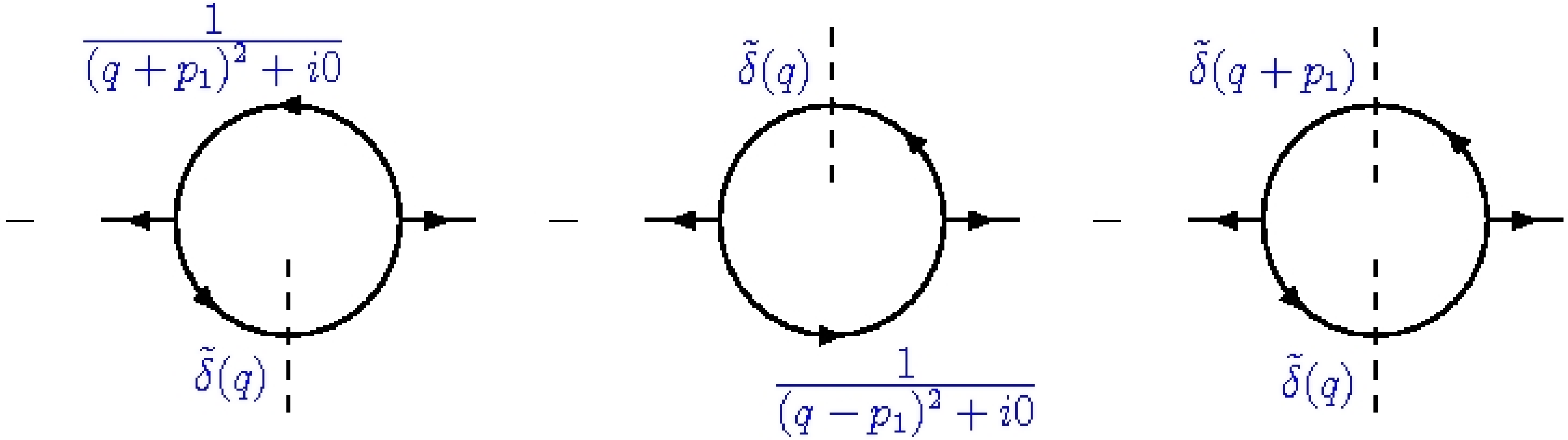
Two-point function from FTT

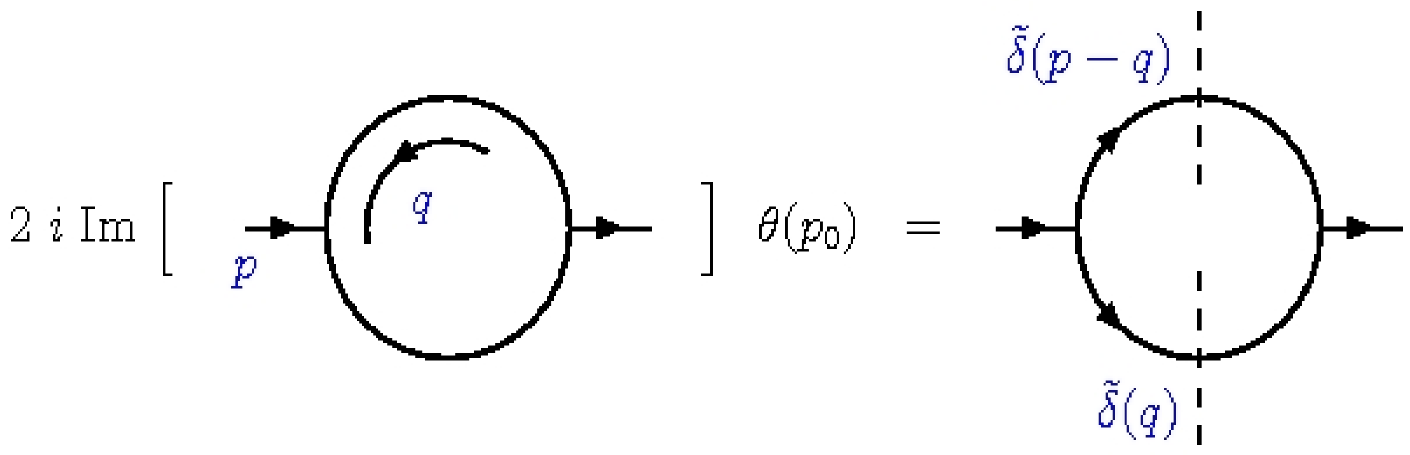


$$L^{(2)}_{1-cut}(p_1, p_2) \;=\; I^{(1)}_{1-cut}(p_1) \;+\; (p_1 \Leftrightarrow -p_1)$$

$$I^{(1)}_{1-cut}(k) \;=\; -\frac{c_\Gamma}{2}\frac{(-k^2-i0)^{-\epsilon}}{\epsilon(1-2\epsilon)}\left[1-i\;\frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)}\left[\theta(-k^2)+\theta(k^2)\,sign(k_0)\right]\right]$$

$$L^{(2)}_{2-cut}(p_1, p_2) \;=\; -i\,c_\Gamma\frac{(|p_1^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)}\frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)}\theta(-p_1^2)$$

Two-point function: Cutkosky



The double-cut contribution from the FTT is different from the **unitarity cut** contribution that gives the imaginary part

$$2\, i\, \mathrm{Im}[Bub(p^2)]\theta(p_0) = \int_q \tilde{\delta}(q)\tilde{\delta}(p-q) = i\, c_\Gamma \frac{(|p^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)} 2\sin(\pi\epsilon)\theta(p^2)\theta(p_0)$$

due to the different positive-energy flow in the internal lines

Massive integrals, complex masses and unstable particles

Real masses: do not affect the dual prescription $\tilde{\delta}(q_i) \rightarrow \tilde{\delta}(q_i, M_i)$

$$\frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)}$$

Massive integrals, complex masses and unstable particles

Real masses: do not affect the dual prescription $\tilde{\delta}(q_i) \rightarrow \tilde{\delta}(q_i, M_i)$

$$\frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)}$$

Unstable particles: Dyson summation produces finite-width effects that lead to the introduction of finite imaginary contributions in the propagators. In the complex mass scheme

$$G_C(q;s) = \frac{1}{q^2 - s}$$

$$s = \mathrm{Re}\ s + i\ \mathrm{Im}\ s \quad \textit{with} \quad \mathrm{Re}\ s > 0 > \mathrm{Im}\ s$$

produces poles in the q_0 plane that are located far from the real axis

Unstable particles

Duality relation

$$\tilde{L}^{(N)}(p_1,\ldots,p_N) \to \tilde{L}^{(N)}(p_1,\ldots,p_N) + \tilde{L}_C^{(N)}(p_1,\ldots,p_N)$$

From the poles of the complex-mass propagators

where

$$\tilde{L}_C^{(N)}(p_1,\ldots,p_N) = \int_q \sum_{i\in C} \tilde{\delta}(q_i, s_i) \left[\prod_{j\neq i} \ldots\right]$$
$$= \int \frac{d^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}} \sum_{i\in C} \frac{1}{2\sqrt{\boldsymbol{q}_i^2+s_i}} \left[\prod_{j\neq i} \ldots\right]_{q_{i0}=\sqrt{\boldsymbol{q}_i^2+s_i}}$$

pole has a finite negative imaginary part ⇨ the +i0 prescription of Feynman propagators can be removed

- FTT: modify the 1-cut contribution, but do not produce additional m-cut contributions
- complex mass $s(q^2)$, but always at a finite imaginary distance from real axis

Gauge poles

Quantization of gauge theories requires a gauge-fixing procedure

fictitious particles: Faddeev-Popov ghosts in unbroken non-Abelian gauge theories, or would-be Goldstone bosons in spontaneously broken gauge theories
➪ ***cut exactly as physical particles***

gauge bosons: polarization tensor

't Hooft-Feynman gauge ✔

$$d^{\mu\nu} = -g^{\mu\nu} + (\xi-1)\; l^{\mu\nu}(q)\; G_G(q)$$

$l^{\mu\nu}(q)$ propagates longitudinal polarizations, harmless polynomial dependence on q

- ***Spontaneously-broken gauge theories***

$$G_G(q) = \frac{1}{\xi(q^2+i0)-M^2}$$

unitary gauge (ξ=0) ✔

- ***Un-broken gauge theories***

covariant gauge $G_G(q) = \frac{1}{q^2+i0}$ second order pole ✘

physical gauge $G_G(q) = \frac{1}{(n\cdot q)^k}\,, \quad k=1,2$ if $n\cdot\eta=0$ ✔

Loop-tree duality for amplitudes

In analogy with the FTT, in unitary and local field theories

$$\mathcal{A}^{(1-loop)} = - \tilde{\mathcal{A}}^{(1-loop)}$$

- starting from $\mathcal{A}^{(1\text{-loop})}$, consider all single cuts
- replace uncut propagators by dual propagators

$$\mathcal{A}^{(1-loop)} \simeq - \int_q \sum_P \tilde{\delta}(q; M_P) \sum_{dof(P)} \mathcal{A}_P^{(tree)}$$

Green's functions

Off-shell Green's function with *N* external legs

$$\mathcal{A}_N^{(1-loop)}(\ldots) = +\frac{1}{2}\int \frac{d^d q}{(2\pi)^{d-1}} \sum_P \delta_+(q^2 - M_P^2)\,\sigma(P)\,\tilde{\mathcal{A}}_{N+2}^{(tree)}(P(q) \leftarrow P(q), \ldots)$$

- $\sigma(P)=\pm 1$ Bose-Fermi statistics factor
- Σ_P sums over particles and antiparticles

Tree-level amplitude for the forward scattering process $P(q)\to P(q)$ in the field of N external legs

e.g.

$$\mathcal{A}_{N+2}^{(tree)}(g(q) \leftarrow g(q), \ldots) = \sum_{\lambda} \sum_{a,b} \left(\epsilon_{a,\mu}^{(\lambda)}(q)\right)^* \left[\mathcal{A}_{N+2}(g(q), g(-q), \ldots)\right]_{ab}^{\mu\nu} \epsilon_{b,\nu}^{(\lambda)}(q)$$

Scattering amplitudes: only relevant point is the on-shell limit of the corresponding Green's function (wave function factors of the external lines)

Summary

- Derived a duality relation between one-loop integrals and **single-cut** phase space integrals.

- Duality relation realized by a modification of the customary $+i0$ prescription of the Feynman propagators.

- The new prescription, written in a Lorentz covariant form, compensates for the absence of multiple-cut contributions that appear in the FTT.

- Valid for any relativistic, local and unitary field theory, in arbitrary space-time dimensions.

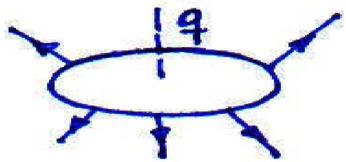
- Suitable for analytical calculations of one-loop scattering amplitudes, and for numerical evaluation of cross-sections at NLO

- natural extension to two-loops, under investigation

IR classification of dual integrals

IR behaviour depends on the two external momenta joined by the cut line

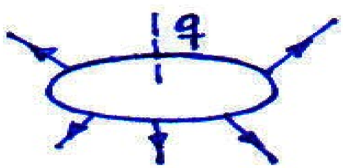
$p_{i-1}^2 \neq 0$ $\quad$ $p_i^2 \neq 0$



$$I^{(N-1)}_{finite} = \int_q \delta_+(q^2) \left[\prod_{j=1}^{N-1} \frac{1}{2\,q\cdot k_j + k_j^2} \right]$$

IR finite (numerically integrable in d=4)

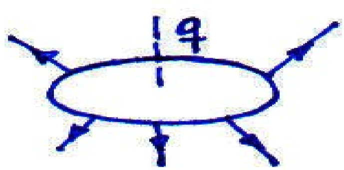
$p_{i-1}^2 \neq 0$ $\quad$ $p_i^2 = p^2 = 0$



$$I^{(N-2)}_{collinear} = \int_q \delta_+(q^2) \frac{1}{p\cdot q} \left[\prod_{j=1}^{N-2} \frac{1}{2\,q\cdot k_j + k_j^2} \right]$$

IR divergent in collinear region *q // p*,
single 1/ε poles

$p_{i-1}^2 = \bar{p}^2 = 0$ $\quad$ $p_i^2 = p^2 = 0$



$$I^{(N-3)}_{soft} = \int_q \delta_+(q^2) \frac{p\cdot \bar{p}}{p\cdot q\; \bar{p}\cdot q} \left[\prod_{j=1}^{N-3} \frac{1}{2\,q\cdot k_j + k_j^2} \right]$$

IR divergent in collinear regions *q // p and q // p*
double $1/\varepsilon^2$ and single 1/ε poles