$e^+e^- \rightarrow 3$ jets and event shapes at NNLO

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in collaboration with T. Gehrmann, E.W.N. Glover, G. Heinrich

Loops and Legs in Quantum Field Theory

Sondershausen,Germany, 21.04.08

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$e^+e^- \to 3$ jets and event shapes

Classical QCD observable

- testing ground for QCD: perturbation theory, power corrections and logarithmic resummation
- jet observables enable a precise determination of the strong coupling constant α_s
- At present: current error on α_s from jet observables dominated by theoretical uncertainty:
  S. Bethke, 2006

$$\alpha_s(M_Z) = 0.121 \pm 0.001(\text{experiment}) \pm 0.005(\text{theory})$$

- theoretical uncertainty largely from missing higher orders
- current status: NLO plus NLL resummation
- $\rightarrow$ NNLO corrections to jet observables are needed !

Jet Observables

Observing ”free” quarks and gluons at colliders

QCD describes quarks and gluons; experiments observe hadrons

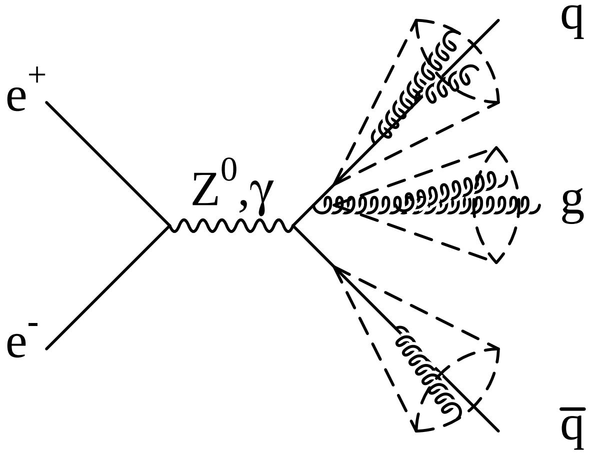
- describe parton $\longrightarrow$ hadron transition (fragmentation)
- define appropriate final states, independent of particle type in final state (jets)

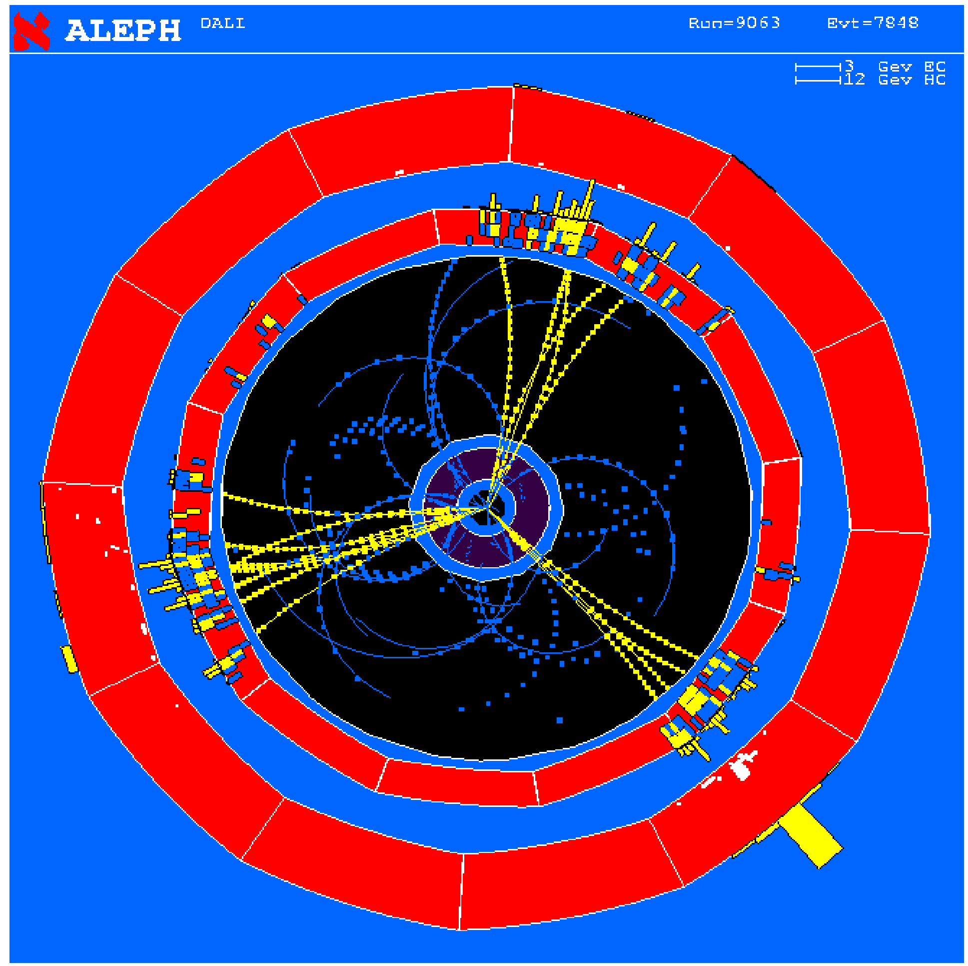
Jets

- experimentally: hadrons with common momentum direction
- theoretically: partons with common momentum direction
- Event : characterized by the number of jets it contains, determined with a jet algorithm

Jet Observables

$e^+e^- \to 3$ jets
event at LEP

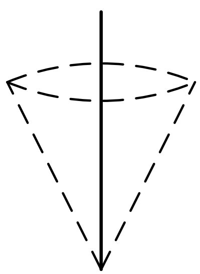




Jets in Perturbative QCD

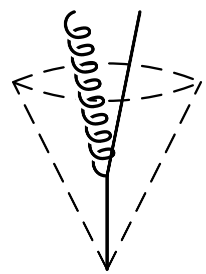
Jet Description

- Partons are combined into jets using the same jet algorithm as in experiment



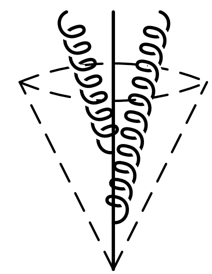
LO

each parton forms 1 jet on its own



NLO

2 partons in 1 jet, 1 parton experimentally unresolved



NNLO

3 partons in 1 jet, 2 partons experimentally unresolved

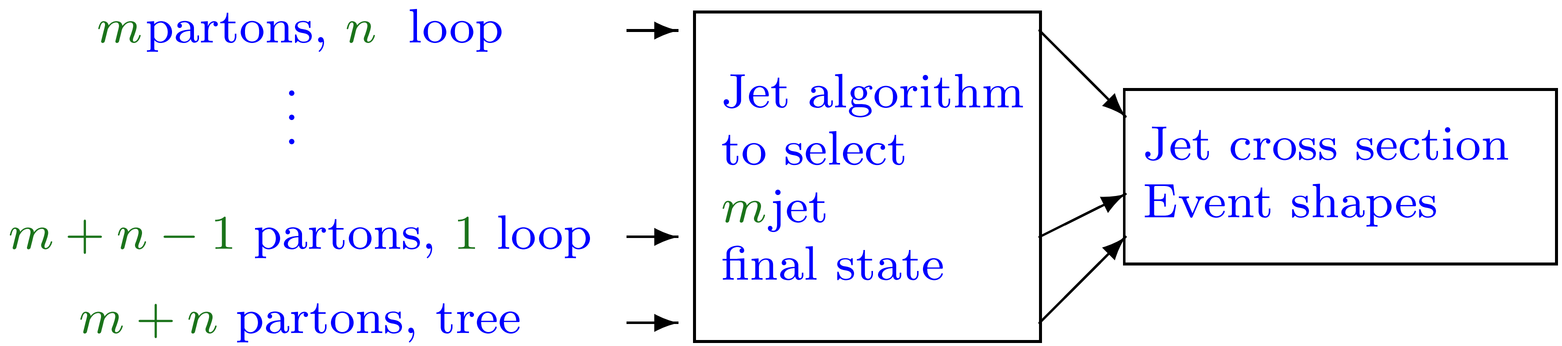
Current state-of-the-art: NLO plus resummation of all-order logarithms (NLLA)
Need for higher orders:

- reduce error on α_s
- better matching of parton level and hadron level

Jets in Perturbation Theory

General structure:

m jets, n–th order in perturbation theory



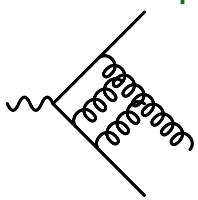
- Jet algorithm acts differently on different partonic final states
- Divergencies from soft and collinear real and virtual contributions must be extracted before application of jet algorithm $\longrightarrow$ subtraction formalism needed

consider $e^+e^- \longrightarrow 3$ jets as an example

Ingredients to NNLO $e^+e^- \to$ 3-jet

- Two-loop matrix elements

  $|\mathcal{M}|^2_{\text{2-loop},3 \text{ partons}}$

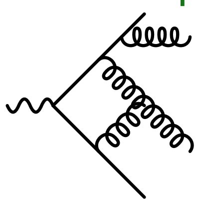
  

  explicit infrared poles from loop integrals

  L. Garland, T. Gehrmann, N. Glover, A. Koukoutsakis, E. Remiddi; S. Moch, P. Uwer, S. Weinzierl

- One-loop matrix elements

  $|\mathcal{M}|^2_{\text{1-loop},4 \text{ partons}}$

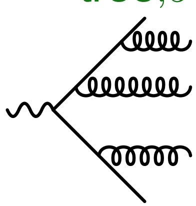
  

  explicit infrared poles from loop integral and implicit infrared poles due to single unresolved radiation

  Z. Bern, L. Dixon, D. Kosower, S. Weinzierl; J. Campbell, D.J. Miller, E.W.N. Glover

- Tree level matrix elements

  $|\mathcal{M}|^2_{\text{tree},5 \text{ partons}}$

  implicit infrared poles due to double unresolved radiation

  K. Hagiwara, D. Zeppenfeld; F.A. Berends, W.T. Giele, H. Kuijf; N. Falck, D. Graudenz, G. Kramer

Infrared Poles cancel in the sum

NNLO Subtraction

Structure of NNLO m-jet cross section:

$$
\begin{aligned}
d\sigma_{NNLO} = & \int_{d\Phi_{m+2}} \left(d\sigma^{R}_{NNLO} - d\sigma^{S}_{NNLO} \right) \\
& + \int_{d\Phi_{m+1}} \left(d\sigma^{V,1}_{NNLO} - d\sigma^{VS,1}_{NNLO} \right) \\
& + \int_{d\Phi_{m}} d\sigma^{V,2}_{NNLO} + \int_{d\Phi_{m+2}} d\sigma^{S}_{NNLO} + \int_{d\Phi_{m+1}} d\sigma^{VS,1}_{NNLO} ,
\end{aligned}
$$

- $d\sigma^{S}_{NNLO}$: real radiation subtraction term for $d\sigma^{R}_{NNLO}$
- $d\sigma^{VS,1}_{NNLO}$: one-loop virtual subtraction term for $d\sigma^{V,1}_{NNLO}$
- $d\sigma^{V,2}_{NNLO}$: two-loop virtual corrections

- Subtraction terms constructed using the antenna subtraction method at NNLO
  T. Gehrmann, N. Glover, AG.
- Each line above is finite numerically and free of infrared ϵ-poles
  $\longrightarrow$ numerical programme (EERAD3) T. Gehrmann, N. Glover, G. Heinrich, AG.

Three-jet cross section at NNLO

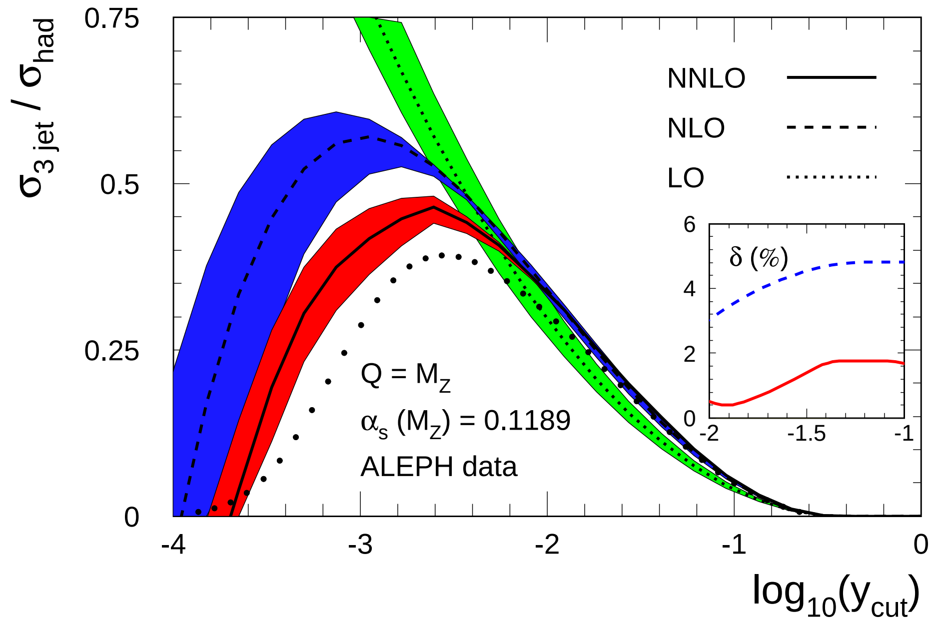
NNLO corrections: jet rates

Three-jet fraction in Durham jet algorithm

$$y_{i,j,D} = \frac{2\min(E_i^2, E_j^2)\,(1-\cos\theta_{ij})}{E_{vis}^2}$$

- vary $\mu = [M_Z/2\,;\,2\,M_Z]$
- determine minimal and maximal values

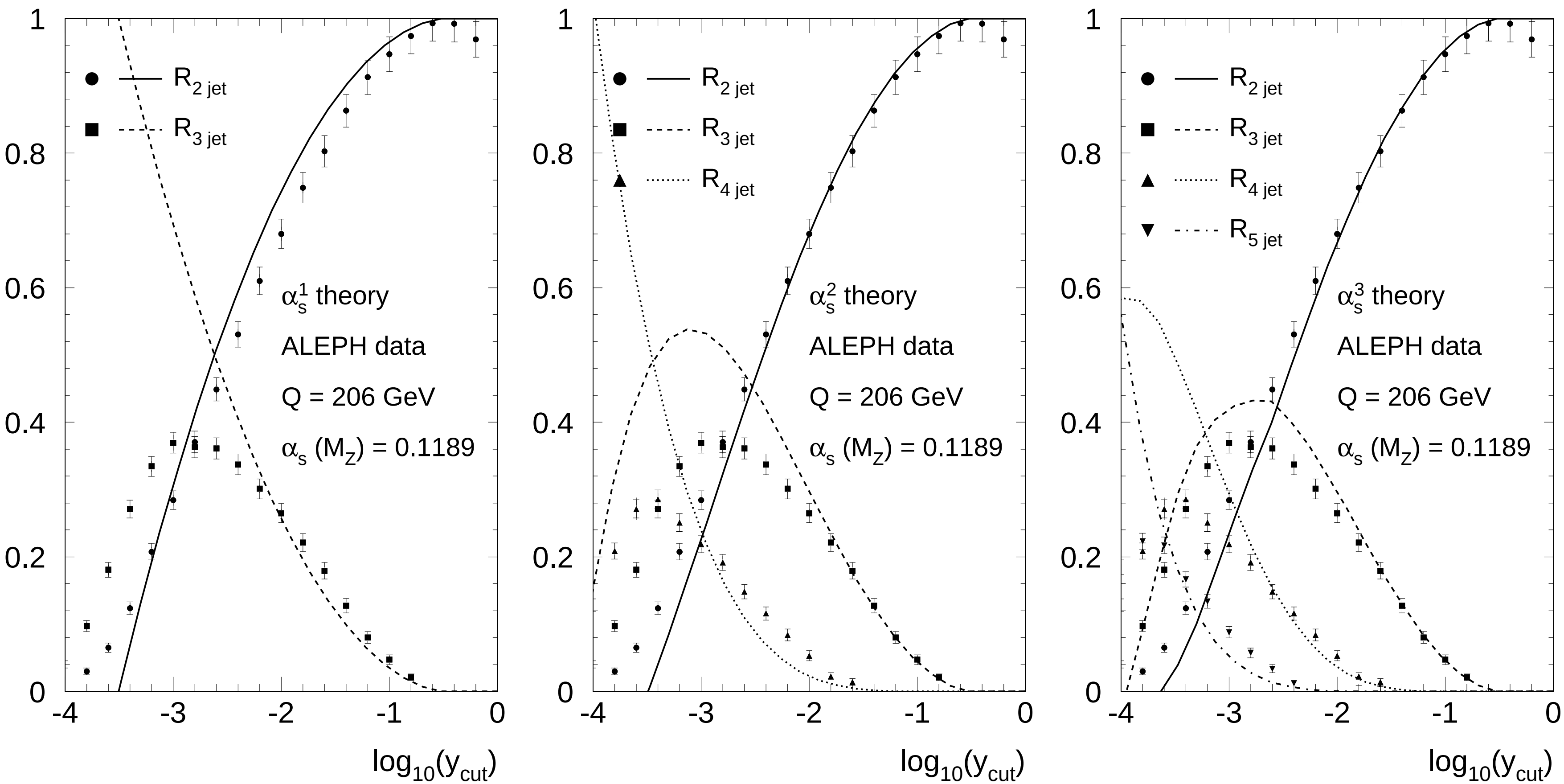
$$\delta = \frac{\max(\sigma) - \min(\sigma)}{2\sigma(\mu = M_Z)}$$



- NNLO corrections small
- substantial reduction of scale dependence
- better description towards lower jet resolution

Three-jet cross section at NNLO

NNLO corrections: jet rates



- substantial improvement towards lower $y_{\rm cut}$
- two-jet rate now NNNLO

$e^+e^- \to 3$ jets and event shapes

Event shape variables

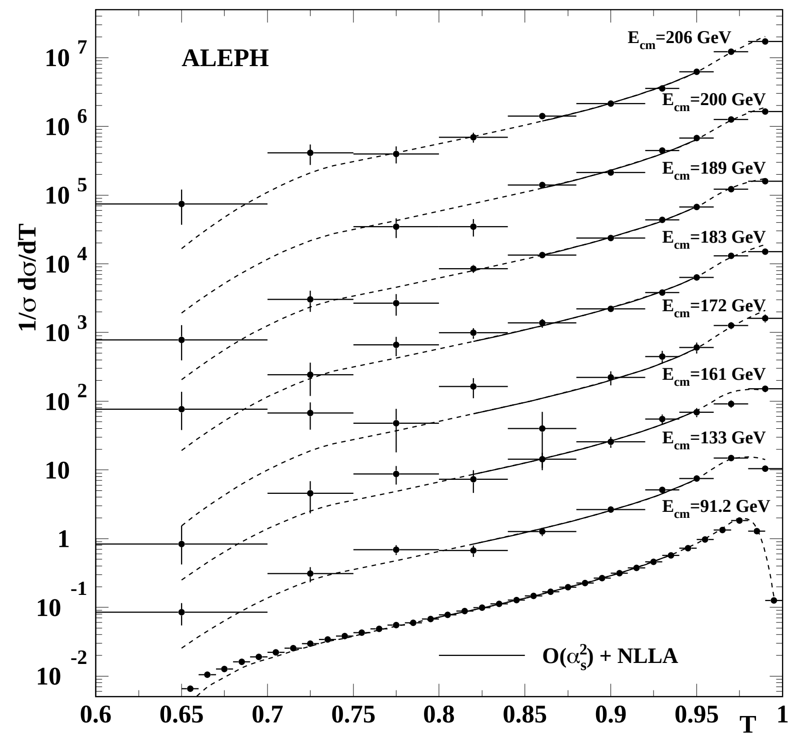
- characterize the geometrical properties of final state events, are infrared safe
- easily accesible experimentally

e.g. Thrust in e^+e^-

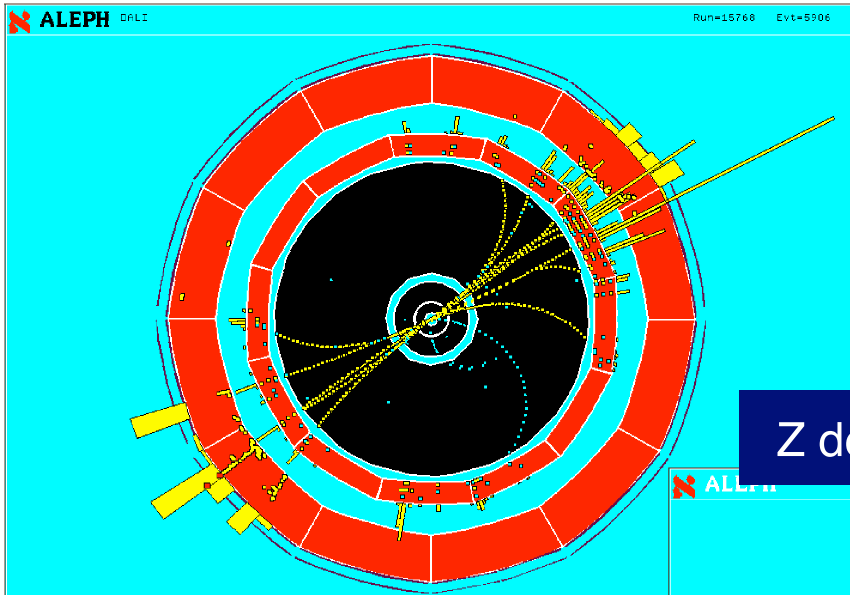
$$T = \max_{\vec{n}} \frac{\sum_{i=1}^{n} |\vec{p_i} \cdot \vec{n}|}{\sum_{i=1}^{n} |\vec{p_i}|}$$

limiting values:

- back-to-back (two-jet) limit: $T = 1$
- spherical limit: $T = 1/2$
- poorly convergent perturbative expansion for $T \longrightarrow 1$
  $\longrightarrow$ resummation needed



Event shape variables

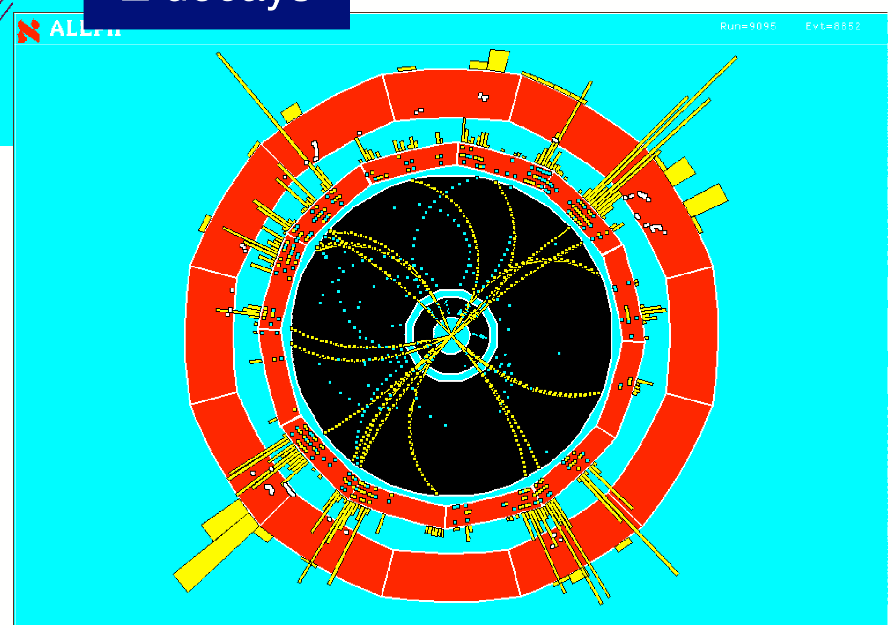


Thrust → 1

Thrust in e^+e^-

$$T = \max_{\vec{n}} \frac{\sum_{i=1}^{n} |\vec{p_i} \cdot \vec{n}|}{\sum_{i=1}^{n} |\vec{p_i}|}$$

Z decays



Thrust → 1/2

Event shapes variables

Standard Set of LEP

- Thrust (E. Farhi)

$$T = \max_{\vec{n}} \left(\sum_{i=1}^{n} |\vec{p_i} \cdot \vec{n}| \right) / \left(\sum_{i=1}^{n} |\vec{p_i}| \right)$$

- Heavy jet mass (L. Clavelli, D. Wyler)

$$\rho = M_i^2/s = \frac{1}{E_{\rm vis}^2} \left(\sum_{k \in H_i} |\vec{p_k}| \right)^2$$

- C-parameter: eigenvalues of the tensor (G. Parisi)

$$\Theta^{\alpha\beta} = \frac{1}{\sum_k |\vec{p_k}|} \frac{\sum_k p_k^\alpha p_k^\beta}{\sum_k |\vec{p_k}|}$$

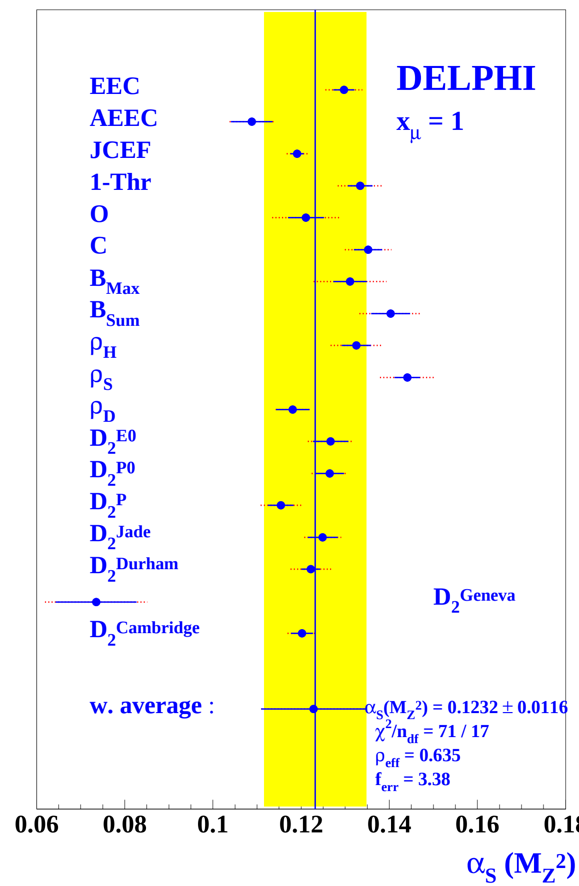
- Jet broadenings (S. Catani, G. Turnock, B. Webber)

$$B_i = \left(\sum_{k \in H_i} |\vec{p_k} \times \vec{n}_T| \right) / \left(2 \sum_k |\vec{p_k}| \right)$$

$$B_W = \max(B_1, B_2) \qquad B_T = B_1 + B_2$$

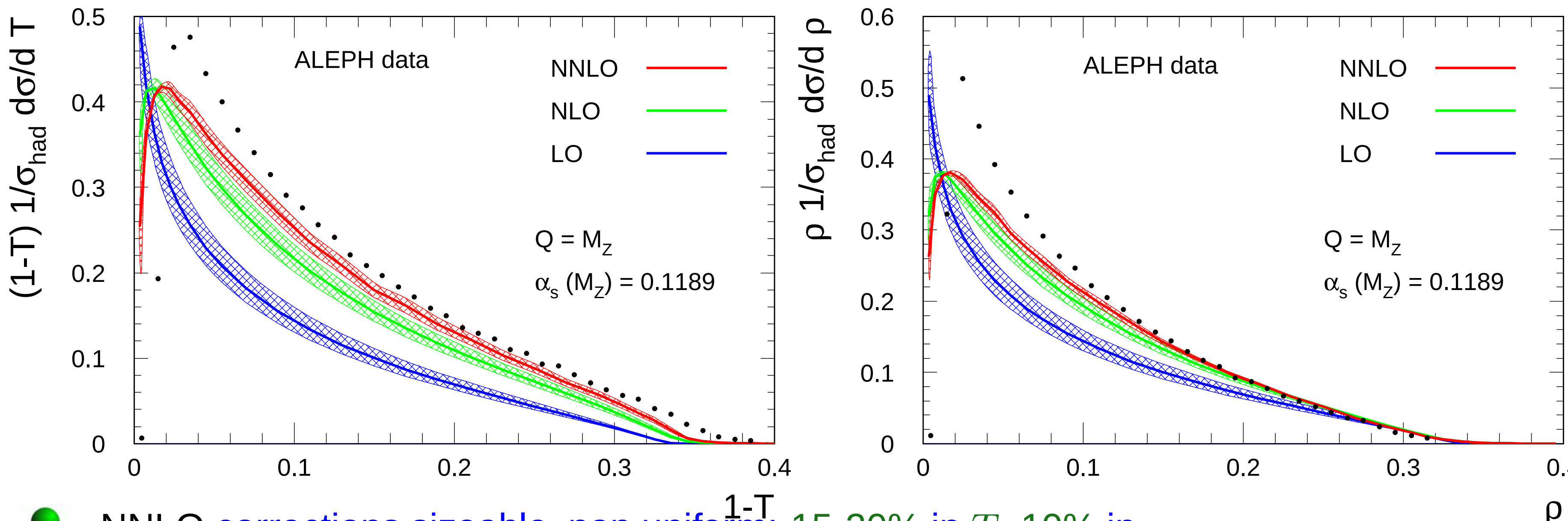
- $3j \to 2j$ transition parameter in Durham algorithm y_{23}^D

  S.Catani, Y.L.Dokshitzer, M.Olsson, G.Turnock, B.Webber



Event shapes at NNLO

NNLO thrust and heavy mass distributions

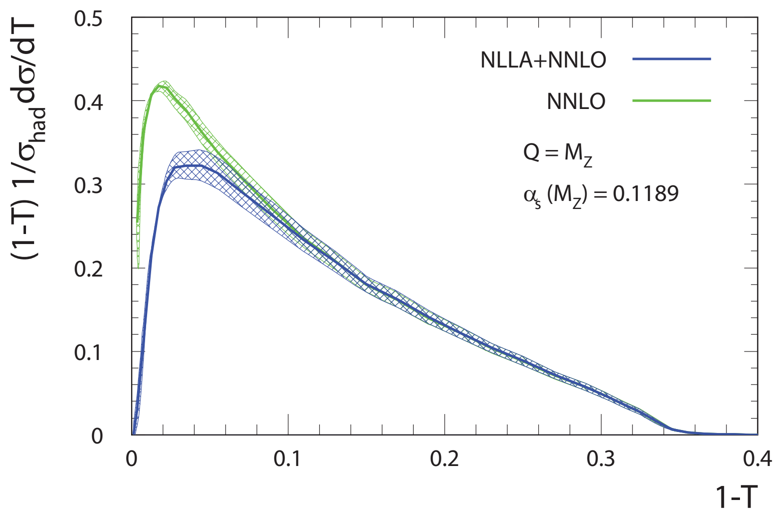


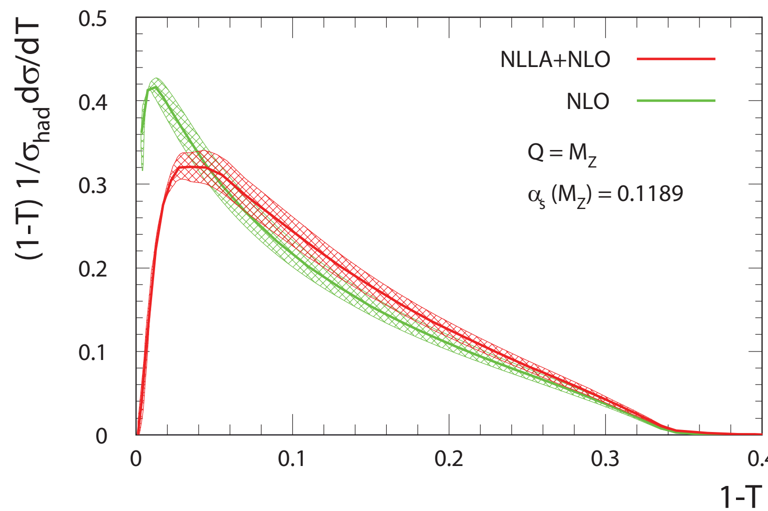
- NNLO corrections sizeable, non uniform: 15-20% in T, 10% in ρ
- theory uncertainty reduced by about 40 %
- large $1-T, \rho > 0.33$: kinematically forbidden at LO
- small $1-T, \rho$: two-jet region, need matching onto NLL resummation
  T. Gehrmann, G. Luisoni, H. Stenzel
- need to include hadronization corrections

Event shapes at NLLA+NNLO

Matching onto resummation

T. Gehrmann, G. Luisoni, H. Stenzel

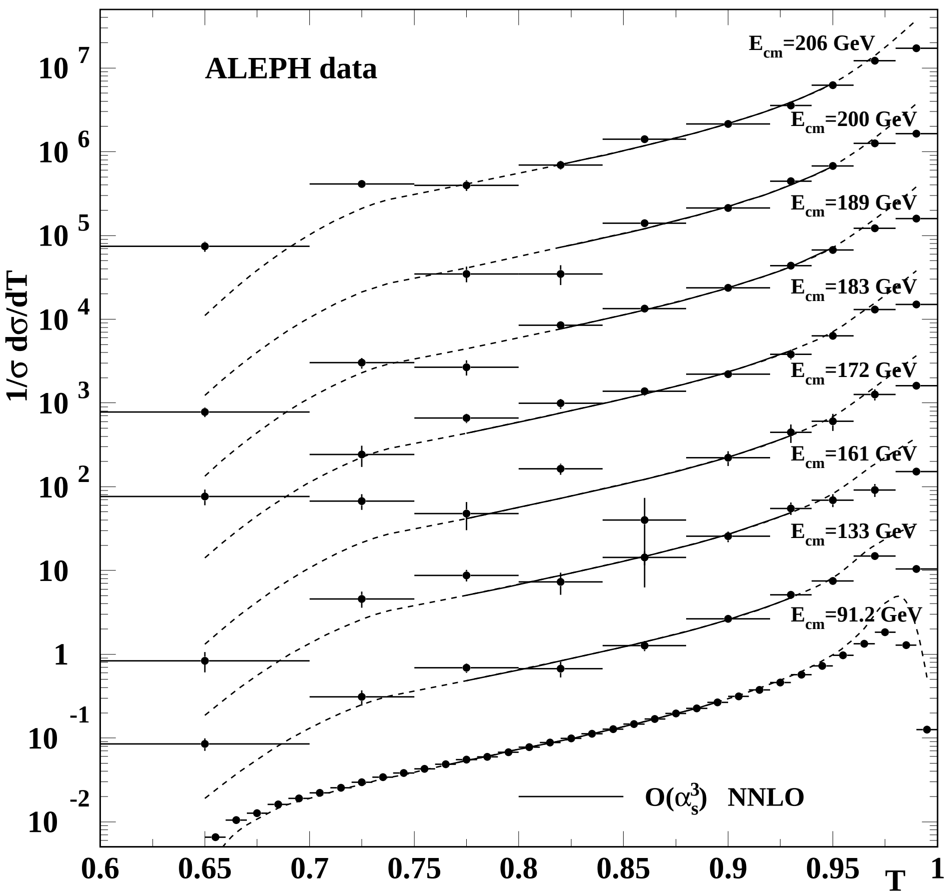




- resummation to NLLA (S. Catani, L. Trentadue, G. Turnock, B. Webber; Y.L. Dokshitzer, A. Lucenti, G. Marchesini, G.P. Salam; A. Banfi, G. Zanderighi)
- NLO and NLLA+NLO differ in normalisation throughout the full kinematical range
- difference between NNLO and NLLA+NNLO restricted to the two-jet region
- improved scale-dependence in three-jet region
- scale-dependence of NLLA dominant $\longrightarrow$ need higher orders in resummation
  T. Becher, M. Schwartz: thrust beyond NLLA

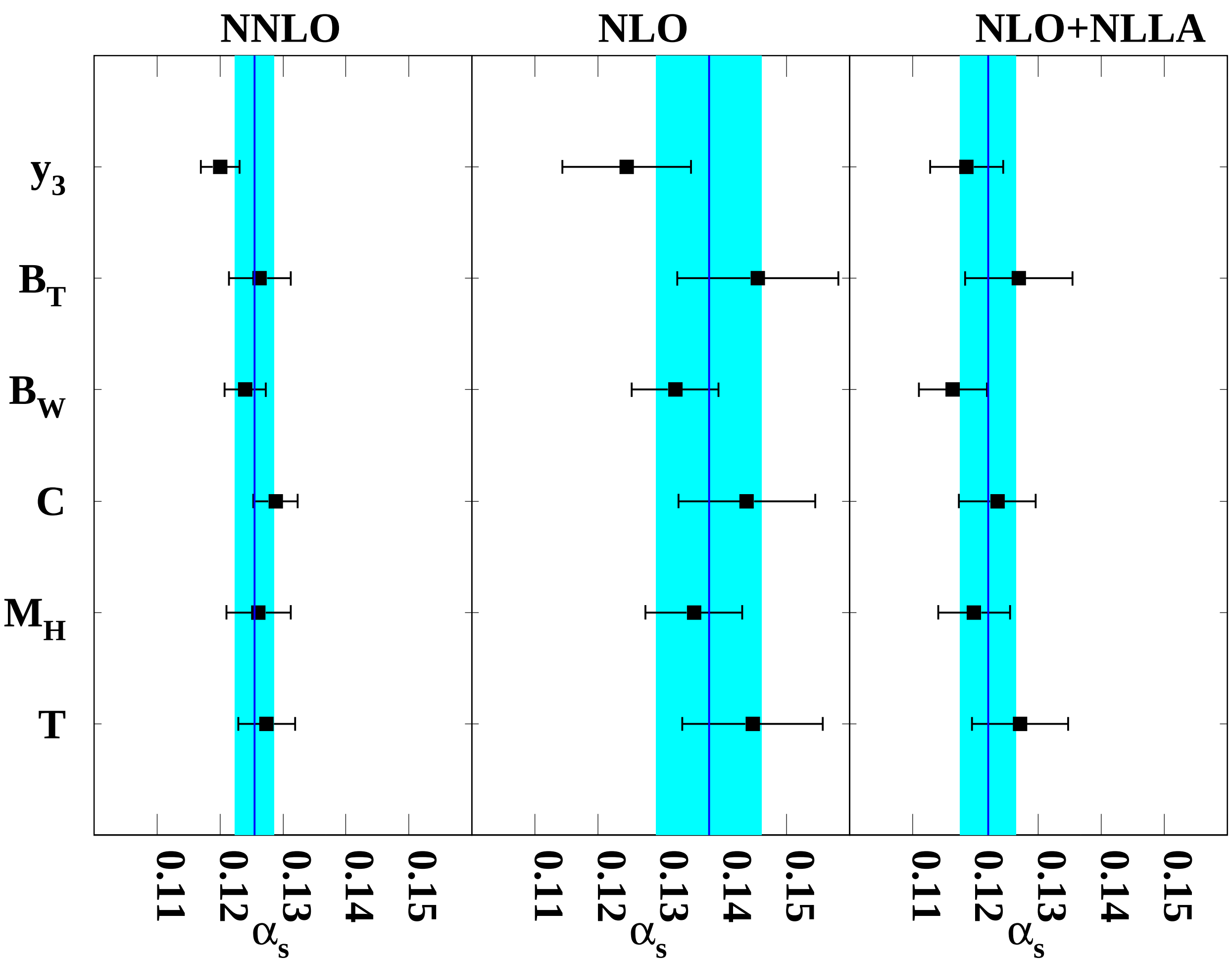
Comparison with data

High precision data from all LEP experiments, compare here to ALEPH



- include quark mass effects to NLO
  P. Nason, C. Oleari
  W. Bernreuther, A. Brandenburg, P. Uwer
  G. Rodrigo, A. Santamaria
- include hadronization corrections
  HERWIG: B. Webber et al.
  ARIADNE: T. Sjostrand et al.
- try new fit of α_s, based on ALEPH analysis
  G. Dissertori, T. Gehrmann,
  G. Heinrich, H. Stenzel, AG

Extraction of α_s



- scale uncertainty reduced by factor 2 compared to NLO; factor 1.3 compared to NLLA+NLO
- scatter among values from different observables reduced very substantially at NNLO $\longrightarrow$ genuine NNLO effect

Result for all ALEPH event shapes of LEP1/LEP2

$$\alpha_s(M_Z) = 0.1240 \pm 0.0008(stat) \pm 0.0010(exp) \pm 0.0011(had) \pm 0.0029(theo)$$

Summary and Conclusions

- Completed the calculation of the NNLO corrections to $e^+e^- \longrightarrow 3$ jets using the antenna subtraction method
- Presented results for the 3-jet cross section (Durham algorithm)
  - improvement towards lower y_{cut}
  - reduced scale dependence
- Presented results for event shapes in e^+e^- annihilation
  - size of the NNLO corrections sizeable but not uniform
  - improved theoretical uncertainty
  - considerably better consistency between observables
  - new NNLO extraction of α_s, more phenomenology to come
- next steps
  - α_s from NLLA+NNLO
  - jet rate studies with different jet algorithms ...
- Precision calculations for jet observables at LHC in progress

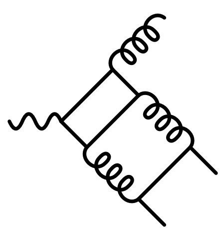
Back-up slides

Colour structure of NNLO 3-jet

Decomposition into leading and subleading colour terms

$$\sigma_{NNLO} = (N^2-1)\Big[N^2\, A_{NNLO} + B_{NNLO} + \frac{1}{N^2}\, C_{NNLO} + N N_F\, D_{NNLO}$$
$$+\frac{N_F}{N}\, E_{NNLO} + N_F^2\, F_{NNLO} + N_{F,\gamma}\left(\frac{4}{N}-N\right)\, G_{NNLO}\Big]$$

- last term: closed quark loop coupling to vector boson, numerically tiny



$$N_{F,\gamma} = \frac{\left(\sum_q e_q\right)^2}{\sum_q e_q^2}$$

- most subleading colour: C_{NNLO}, E_{NNLO}, F_{NNLO}, (G_{NNLO})
  QED-type contributions: gluons $\longrightarrow$ photons
- simplest term: F_{NNLO}, only 3 parton and 4 parton contributions

Colour-ordered antenna functions

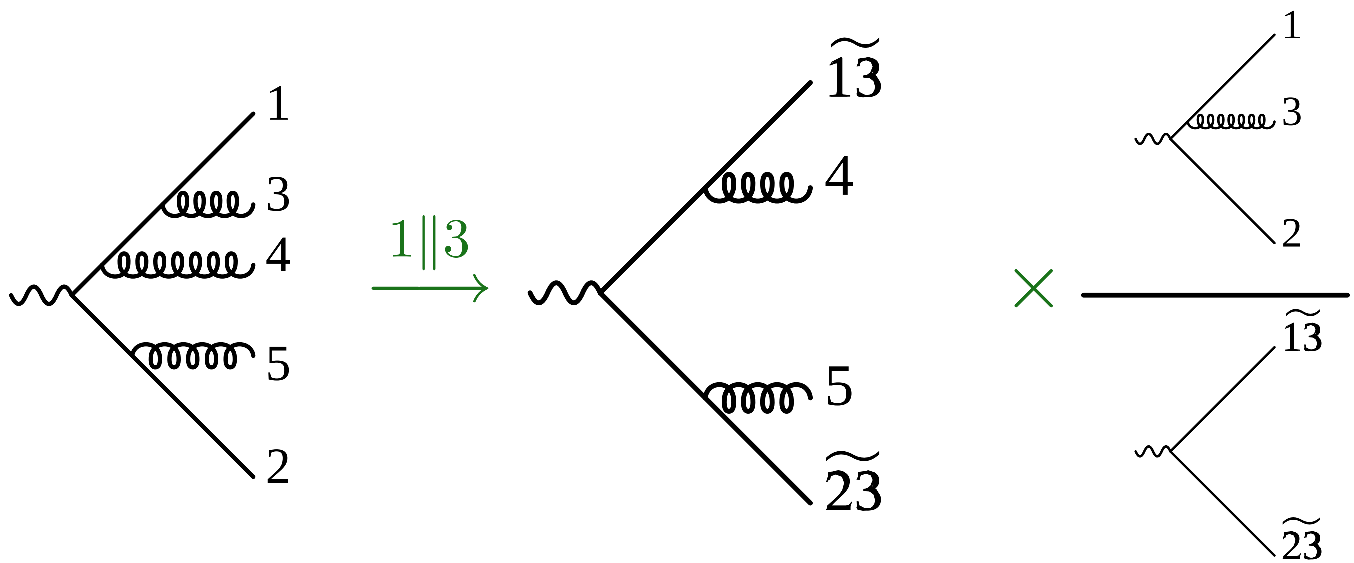
Antenna Functions

- colour-ordered pair of hard partons (radiators) with radiation in between
  - hard quark-antiquark pair
  - hard quark-gluon pair
  - hard gluon-gluon pair
- three-parton antenna $\longrightarrow$ one unresolved parton
- four-parton antenna $\longrightarrow$ two unresolved partons
- can be at tree level or at one loop
- all three-parton and four-parton antenna functions can be derived from physical matrix elements, normalised to two-parton matrix elements
  - $q\bar{q}$ from $\gamma^* \to q\bar{q} + X$
  - qg from $\tilde{\chi} \to \tilde{g}g + X$
  - gg from $H \to gg + X$

Antenna functions

Quark-antiquark

consider subleading colour (gluons photon-like)



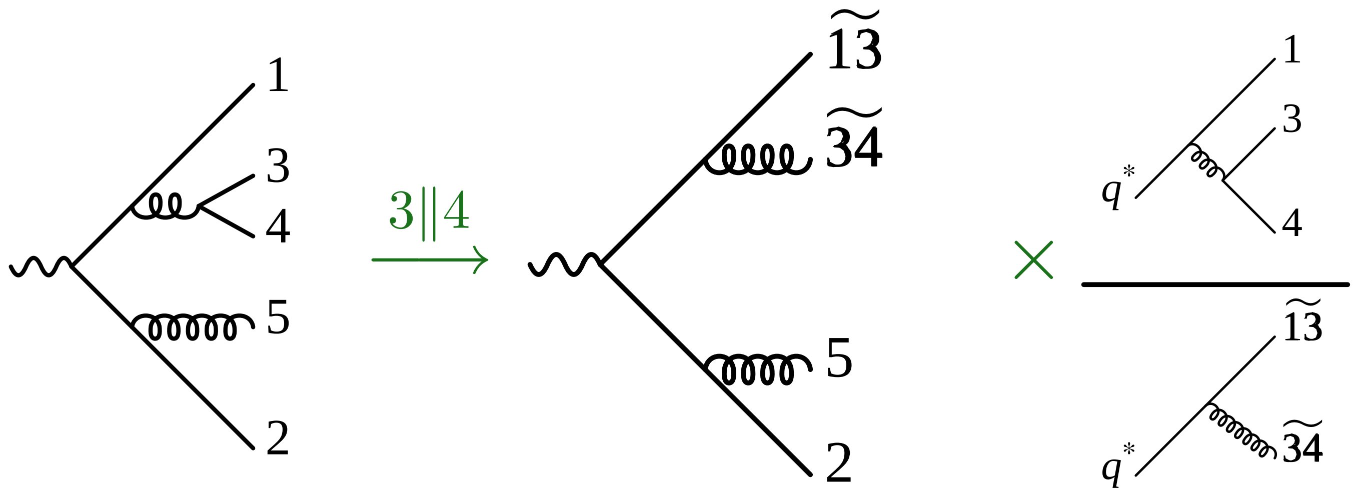
$$|M_{q\bar{q}ggg}|^2(1,3,4,5,2) \xrightarrow{1\|3} |M_{q\bar{q}gg}|^2(\widetilde{13},4,5,\widetilde{23}) \times X_{132}$$

with

$$X_{132} = \frac{|M_{q\bar{q}g}|^2}{|M_{q\bar{q}}|^2} \equiv A_3^0(1_q, 3_g, 2_{\bar{q}})$$

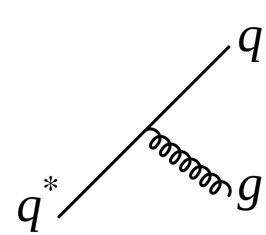
Antenna functions

Quark-gluon



$$|M_{q\bar{q}q\bar{q}g}|^2(1,3,4,5,2) \xrightarrow{3\|4} |M_{q\bar{q}gg}|^2(\widetilde{13},\widetilde{34},5,2) \times X_{134}$$

with hard radiators:
quark $(\widetilde{13})$ and gluon $(\widetilde{34})$



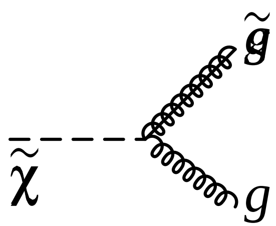
q^*	:	spin $1/2$, colour triplet
$q(\widetilde{13})$	:	spin $1/2$, colour triplet
$g(\widetilde{34})$	:	spin 1, colour octet

Off-shell matrix element: violates $SU(3)$ gauge invariance

Antenna functions

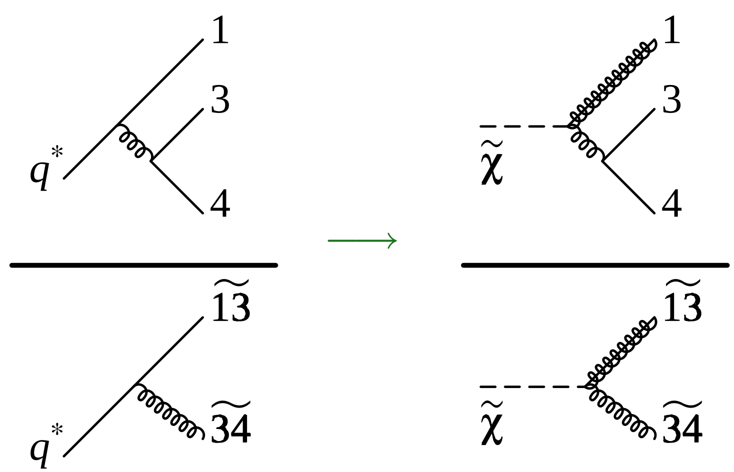
Quark-gluon

Construct colour-ordered qg antenna function from $SU(3)$ gauge-invariant decay:
neutralino $\rightarrow$ gluino + gluon (T. Gehrmann, E.W.N. Glover, AG)



$\tilde{\chi}$: spin 1/2, colour singlet
$\tilde{g}$: spin 1/2, colour octet
g : spin 1, colour octet

Gluino $\tilde{g}$ mimics quark and antiquark (same Dirac structure), but is octet in colour space



$\tilde{\chi} \rightarrow \tilde{g}g$ described by effective Lagrangian
H. Haber, D. Wyler

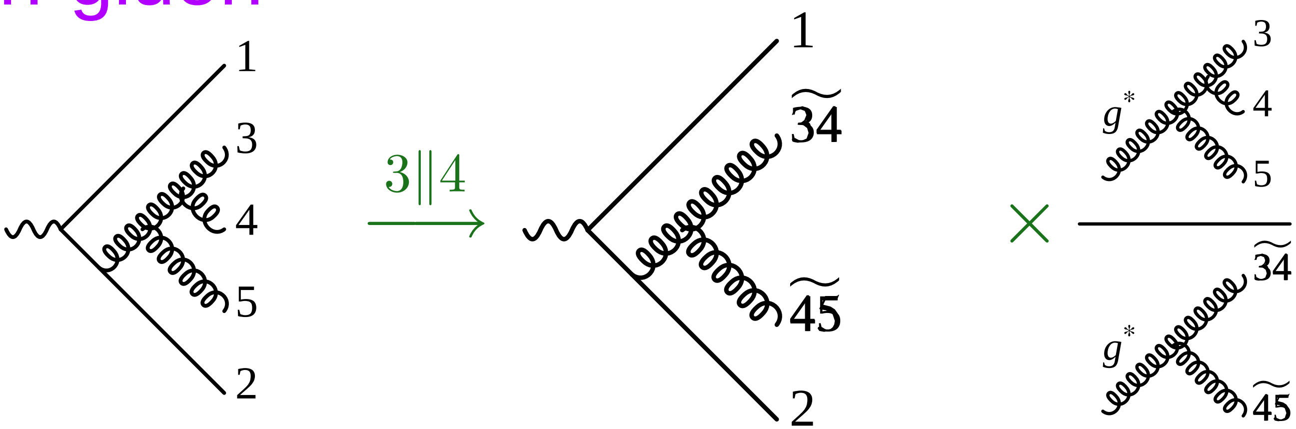
$$\mathcal{L}_{\rm int} = i\eta\overline{\psi}^a_{\tilde{g}}\sigma^{\mu\nu}\psi_{\tilde{\chi}}F^a_{\mu\nu} + ({\rm h.c.})$$

Antenna function

$$X_{134} = \frac{|M_{\tilde{g}q'\bar{q}'}|^2}{|M_{\tilde{g}g}|^2} \equiv E_3^0(1_q, 3_{q'}, 4_{\bar{q}'})$$

Antenna functions

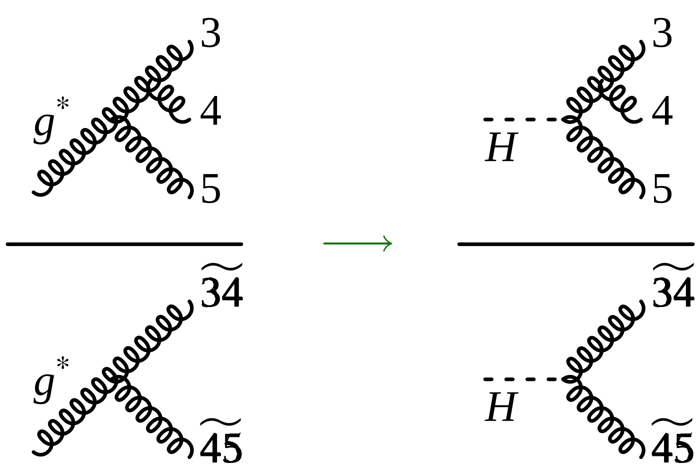
Gluon-gluon



$$|M_{q\bar{q}ggg}|^2(1,3,4,5,2) \stackrel{3\|4}{\longrightarrow} |M_{q\bar{q}gg}|^2(1,\widetilde{34},\widetilde{45},2) \times X_{345}$$

$H \rightarrow gg$ described by effective Lagrangian

F. Wilczek; M. Shifman, A. Vainshtein, V. Zakharov



$$\mathcal{L}_{\rm int} = \frac{\lambda}{4} H F^a_{\mu\nu} F^{\mu\nu}_a$$

Antenna function

$$X_{345} = \frac{|M_{ggg}|^2}{|M_{gg}|^2} \equiv F_3^0(3_g, 4_g, 5_g)$$

Antenna functions

	tree level	one loop
quark-antiquark		
$qg\bar{q}$	$A_3^0(q,g,\bar{q})$	$A_3^1(q,g,\bar{q}),\ \tilde{A}_3^1(q,g,\bar{q}),\ \hat{A}_3^1(q,g,\bar{q})$
$qgg\bar{q}$	$A_4^0(q,g,g,\bar{q}),\ \tilde{A}_4^0(q,g,g,\bar{q})$	
$qq'\bar{q}'\bar{q}$	$B_4^0(q,q',\bar{q}',\bar{q})$	
$qq\bar{q}\bar{q}$	$C_4^0(q,q,\bar{q},\bar{q})$	
quark-gluon		
qgg	$D_3^0(q,g,g)$	$D_3^1(q,g,g),\ \hat{D}_3^1(q,g,g)$
$qggg$	$D_4^0(q,g,g,g)$	
$qq'\bar{q}'$	$E_3^0(q,q',\bar{q}')$	$E_3^1(q,q',\bar{q}'),\ \tilde{E}_3^1(q,q',\bar{q}'),\ \hat{E}_3^1(q,q',\bar{q}')$
$qq'\bar{q}'g$	$E_4^0(q,q',\bar{q}',g),\ \tilde{E}_4^0(q,q',\bar{q}',g)$	
gluon-gluon		
ggg	$F_3^0(g,g,g)$	$F_3^1(g,g,g),\ \hat{F}_3^1(g,g,g)$
$gggg$	$F_4^0(g,g,g,g)$	
$gq\bar{q}$	$G_3^0(g,q,\bar{q})$	$G_3^1(g,q,\bar{q}),\ \tilde{G}_3^1(g,q,\bar{q}),\ \hat{G}_3^1(g,q,\bar{q})$
$gq\bar{q}g$	$G_4^0(g,q,\bar{q},g),\ \tilde{G}_4^0(g,q,\bar{q},g)$	
$q\bar{q}q'\bar{q}'$	$H_4^0(q,\bar{q},q',\bar{q}')$	

Numerical Implementation

Parton-level event generator

Starting point: $e^+e^- \to 4$ jets at NLO (EERAD2: J. Campbell, M. Cullen, N. Glover)

- contains already 4-parton and 5-parton matrix elements
- is based on NLO antenna subtraction

additions:

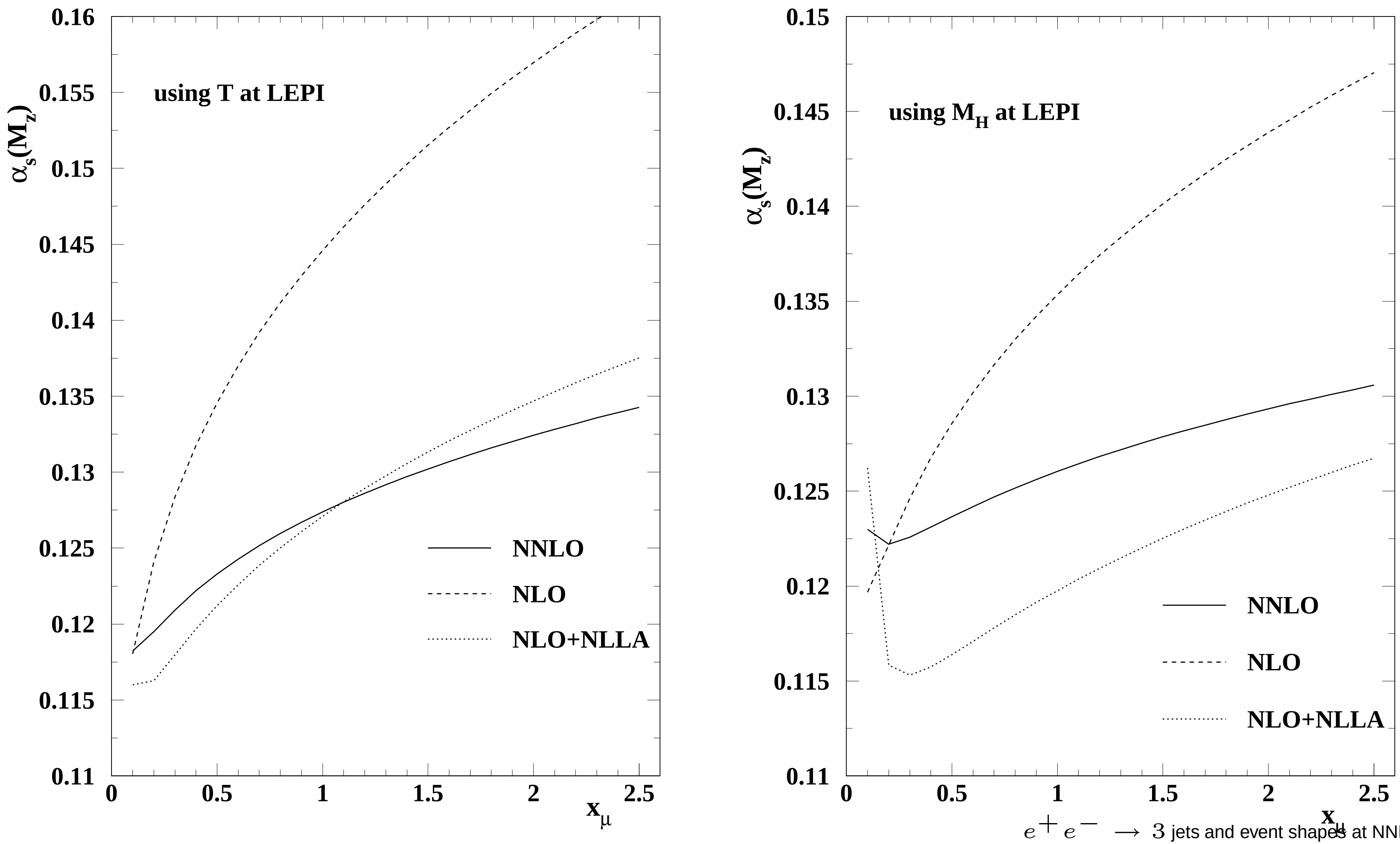
- NNLO subtraction terms (5-parton channel)
- 1-loop-single unresolved integrated subtraction term (4-parton channel)
- 2-loop matrix element (3-parton channel)

checks:

- analytic cancellation of infrared poles
- local cancellations along phase space trajectories approaching singular limits
- confirmation of our result by independent calculation of all logarithmically enhanced terms in the thrust distribution using the SCET formalism
  very recently: T. Becher, M. Schwartz

Extraction of α_s

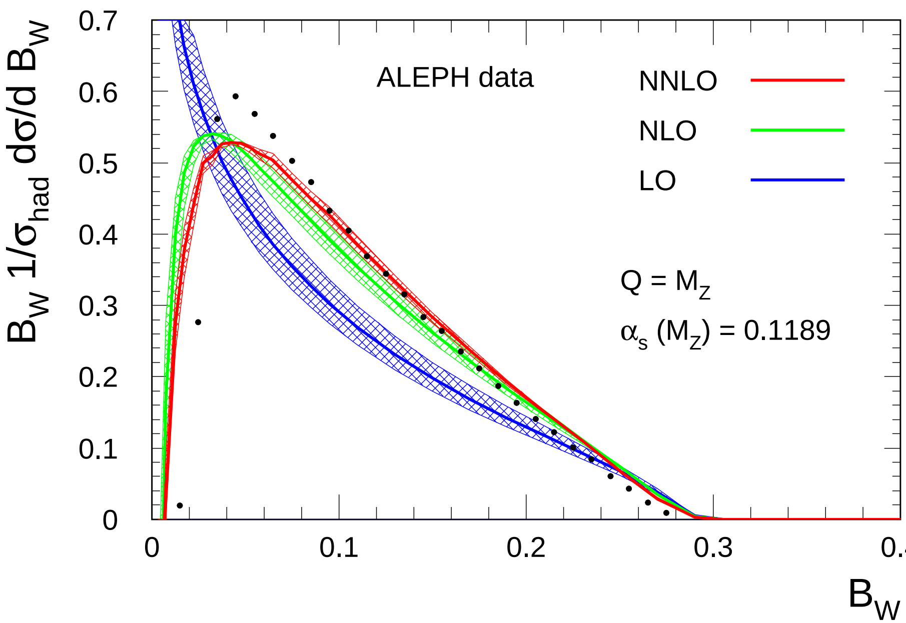
Uncertainty from renormalisation scale



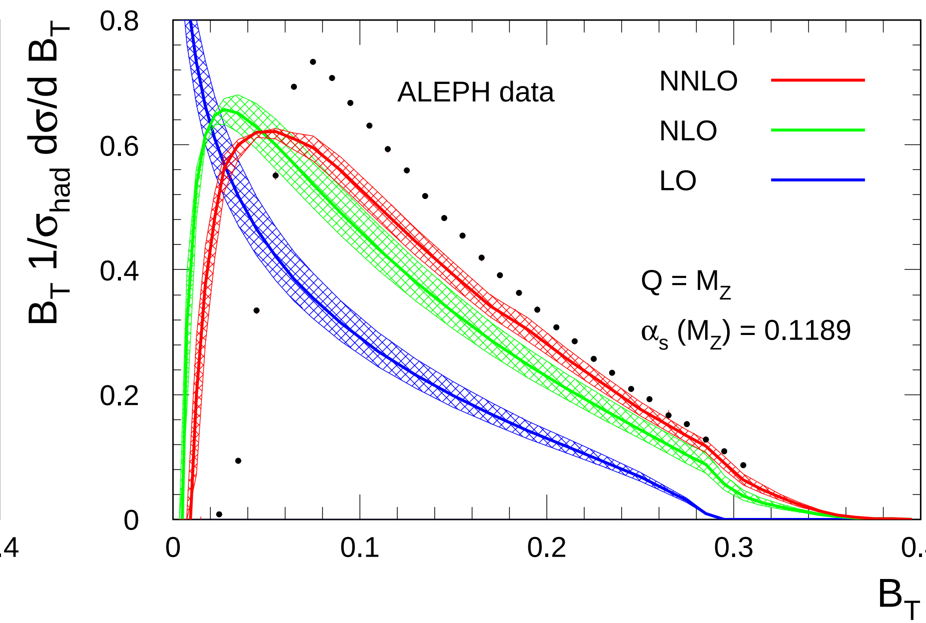
Event shapes at NNLO

NNLO corrections: broadenings

wide jet boadening B_W



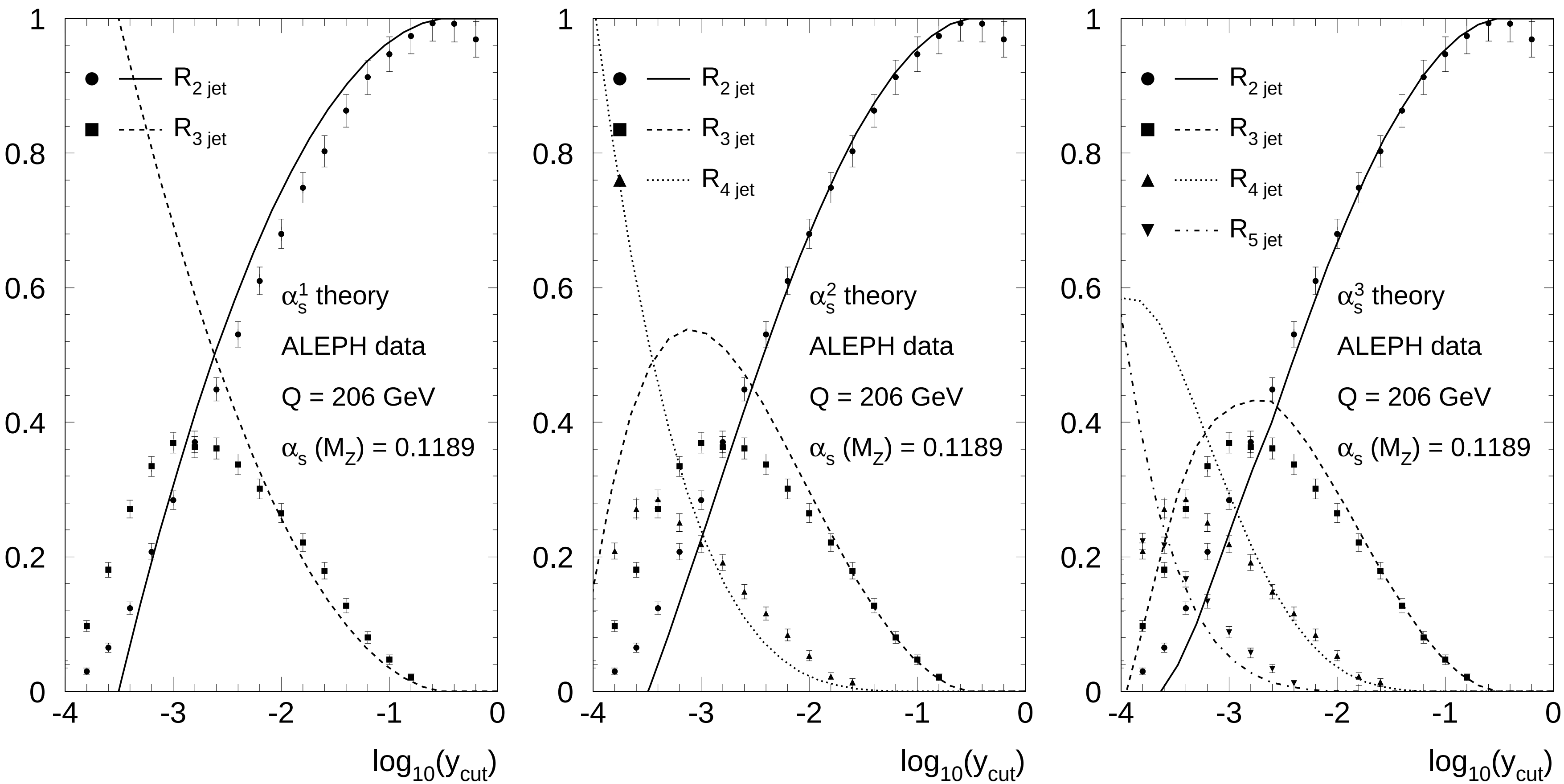
total jet boadening B_T



- NNLO corrections for B_W smaller than for B_T
- again require matching onto NLL resummation and hadronization corrections
- observe: small corrections for Y_3; large corrections for C
- reduction of dependence on renormalisation scale by 30–60%

Three-jet cross section at NNLO

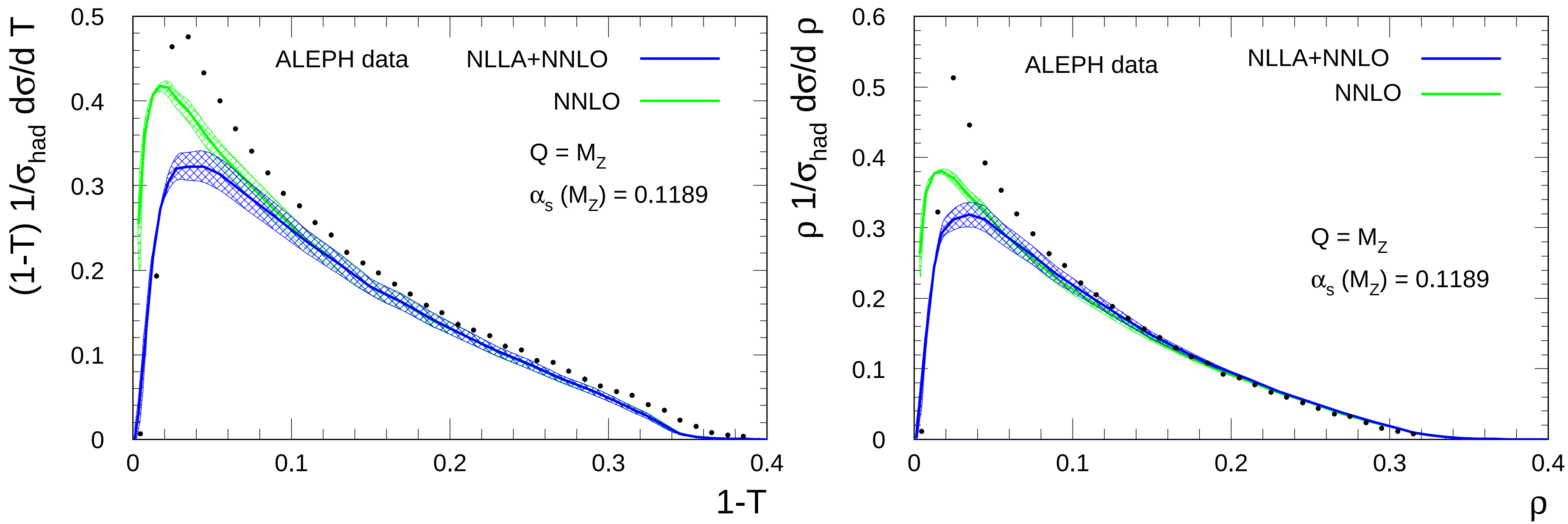
NNLO corrections: jet rates



- substantial improvement towards lower $y_{\rm cut}$
- two-jet rate now NNNLO

Event shapes at NNLO+NLLA

NNLO+NLLA thrust and heavy mass



- (NNLO +NLLA) compared to (NNLO) prediction
  - slightly better description towards the 2-jet limit
  - In the 3-jet region, two predictions in agreement
  - further improvement needed: by including hadronization corrections

NLO Subtraction

Structure of NLO m-jet cross section (subtraction formalism):

Z. Kunszt, D. Soper

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^R_{NLO} - \mathrm{d}\sigma^S_{NLO} \right) + \left[\int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^S_{NLO} + \int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^V_{NLO} \right]$$

- $\mathrm{d}\sigma^S_{NLO}$: local counter term for $\mathrm{d}\sigma^R_{NLO}$
- $\mathrm{d}\sigma^R_{NLO} - \mathrm{d}\sigma^S_{NLO}$: free of divergences, can be integrated numerically

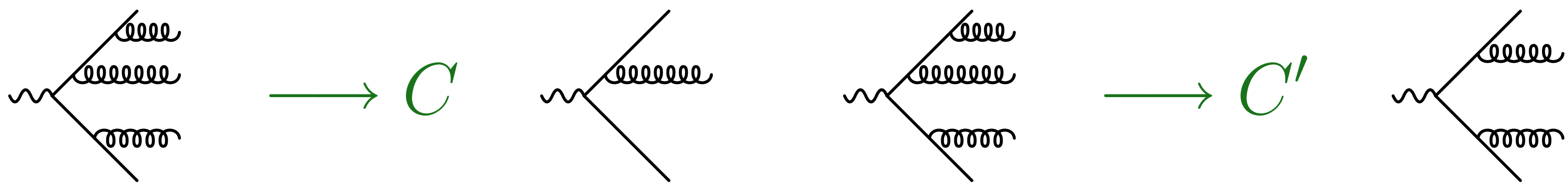
General methods at NLO

- Dipole subtraction
  S. Catani, M. Seymour; NNLO: S. Weinzierl
- $\mathcal{E}$-prescription
  S. Frixione, Z. Kunszt, A. Signer;
  NNLO: S. Frixione, M. Grazzini; V. Del Duca, G. Somogyi, Z. Trocsanyi
- Antenna subtraction (derived from physical matrix elements)
  D. Kosower; J. Campbell, M. Cullen, N. Glover;
  NNLO: T Gehrmann, E.W.N. Glover, AG

Real Corrections at NNLO

Infrared subtraction terms

$m+2$ partons $\to m$ jets: $m+2 \to m+1$ pseudopartons $\to m$ jets:



- Double unresolved configurations:
  - triple collinear
  - double single collinear
  - soft/collinear
  - double soft
- Single unresolved configurations:
  - collinear
  - soft

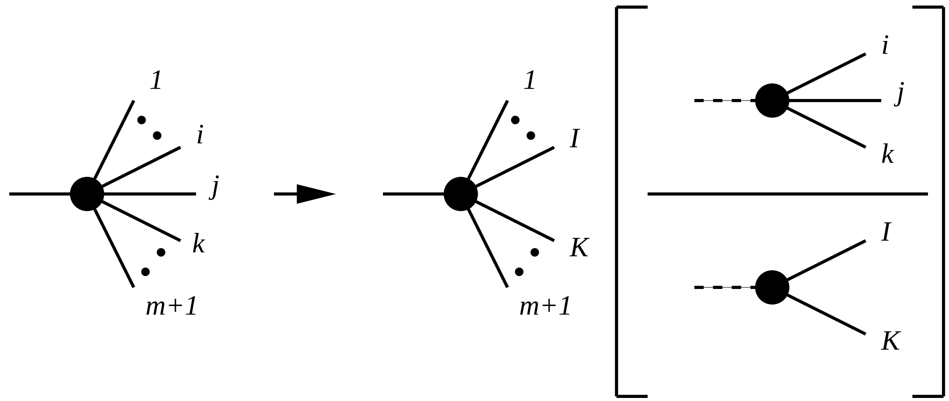
J. Campbell, E.W.N. Glover; S. Catani, M. Grazzini

Issue: find subtraction functions which

- approximate full $m+2$ matrix element in all singular limits
- are sufficiently simple to be integrated analytically

NLO Antenna Subtraction

Building block of $\mathrm{d}\sigma^S_{NLO}$: NLO-Antenna function X^0_{ijk} and phase space $\mathrm{d}\Phi_{X_{ijk}}$



$$\mathrm{d}\sigma^S_{NLO} = \mathcal{N} \sum_{m+1} \sum_j \mathrm{d}\Phi_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1};q) \cdot \mathrm{d}\Phi_{X_{ijk}}(p_i,p_j,p_k;\tilde{p}_I+\tilde{p}_K)$$

$$\times \quad X^0_{ijk}\,|\mathcal{M}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})|^2\, J_m^{(m)}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})$$

Integrated subtraction term (analytically)

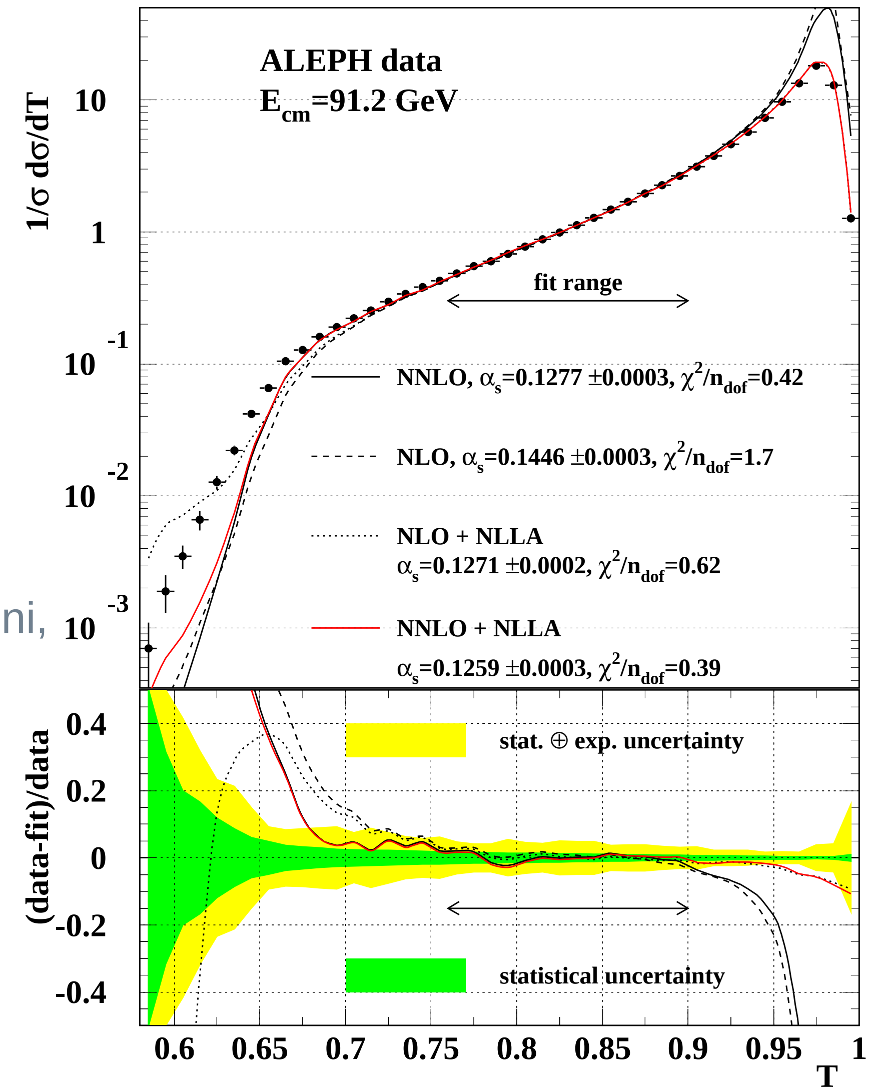
$$|\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijk}}\, X^0_{ijk}$$

can be combined with $\mathrm{d}\sigma^V_{NLO}$

Outlook

Next steps:

- fit α_s using NLLA+NNLO
  G. Luisoni, H. Stenzel, T. Gehrmann
- study jet rates in different algorithms
- study moments of event shapes
- revisit analytic power corrections
  Y.L. Dokshitzer, A. Lucenti, G. Marchesini, G.P. Salam
- electroweak corrections
- resummation and beyond at NLLA



Double Real Subtraction

Tree-level real radiation contribution to m jets at NNLO

$$d\sigma^{R}_{NNLO} = \mathcal{N} \sum_{m+2} d\Phi_{m+2}(p_1, \ldots, p_{m+2}; q) \frac{1}{S_{m+2}}$$

$$\times \quad |\mathcal{M}_{m+2}(p_1, \ldots, p_{m+2})|^2 \; J_m^{(m+2)}(p_1, \ldots, p_{m+2})$$

- $d\Phi_{m+2}$: full $m+2$-parton phase space
- $J_m^{(m+2)}$: ensures $m+2$ partons $\rightarrow m$ jets
  $\longrightarrow$ two partons must be experimentally unresolved

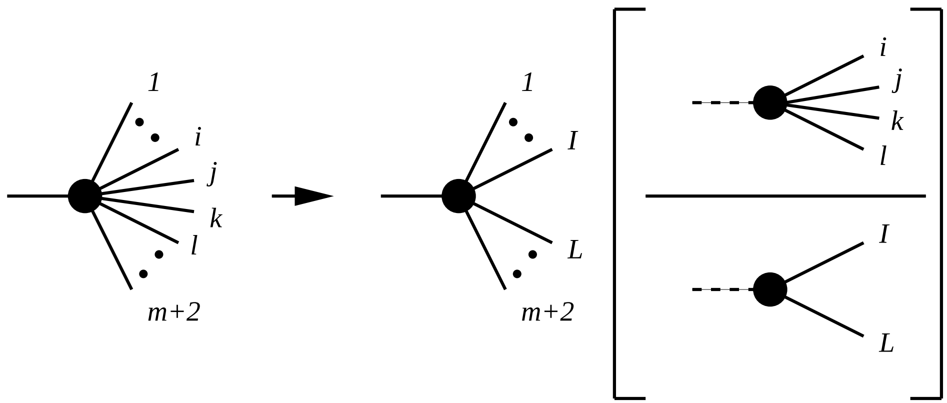
Up to two partons can be theoretically unresolved (soft and/or collinear)

Building blocks of subtraction terms: based on colour-ordered amplitudes

- products of two three-parton antenna functions
- single four-parton antenna function

Double Real Subtraction

Two colour-connected unresolved partons



$$X^0_{ijkl} = S_{ijkl,IL} \frac{|M^0_{ijkl}|^2}{|M^0_{IL}|^2}$$

$$\mathrm{d}\Phi_{X_{ijkl}} = \frac{\mathrm{d}\Phi_4}{P_2}$$

$$\mathrm{d}\sigma^S_{NNLO} = \mathcal{N} \sum_{m+2} \sum_j \mathrm{d}\Phi_m(p_1, \ldots, \tilde{p}_I, \tilde{p}_L, \ldots, p_{m+2}; q) \cdot \mathrm{d}\Phi_{X_{ijkl}}(p_i, p_j, p_k, p_l; \tilde{p}_I + \tilde{p}_L)$$

$$\times \quad X^0_{ijkl}\, |\mathcal{M}_m(p_1, \ldots, \tilde{p}_I, \tilde{p}_L, \ldots, p_{m+2})|^2\, J_m^{(m)}(p_1, \ldots, \tilde{p}_I, \tilde{p}_L, \ldots, p_{m+2})$$

Integrated subtraction term (analytically)

$$|\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijkl}}\, X^0_{ijkl} \sim |\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_4 |M^0_{ijkl}|^2$$

Four-particle inclusive phase space integrals are known

T. Gehrmann, G. Heinrich, AG

Colour-ordered antenna functions

Antenna functions

- colour-ordered pair of hard partons (radiators) with radiation in between
  - hard quark-antiquark pair
  - hard quark-gluon pair
  - hard gluon-gluon pair
- three-parton antenna $\longrightarrow$ one unresolved parton
- four-parton antenna $\longrightarrow$ two unresolved partons
- can be at tree level or at one loop
- all three-parton and four-parton antenna functions can be derived from physical matrix elements
  - $q\bar{q}$ from $\gamma^* \to q\bar{q} + X$
  - qg from $\chi \to \tilde{g}g + X$
  - gg from $H \to gg + X$
- can be integrated analytically over the antenna phase spaces

$e^+e^- \to 3$ jets at NNLO

Structure of $e^+e^- \to 3$ jets program:

EERAD3: T. Gehrmann, E.W.N. Glover, G. Heinrich, AG