Four loop vacuum bubbles for two-dimensional four-fermi theory renormalization

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Background to problem

- 4-fermi theories in two dimensions are of interest due to connection with various areas
- Gross-Neveu model is asymptotically free and multiplicatively renormalizable in two dimensions
- It possesses an *exact* S-matrix and exact mass gap and also underlies several problems in condensed matter physics
- For example, it is equivalent to the random bond Ising model at criticality, which requires the renormalization group functions to a high loop order
- Another model is the non-abelian Thirring model (NATM) which is asymptotically free
- The structure of its renormalization group functions is an open question
- Not clear whether the new Casimir $d_F^{abcd} d_F^{abcd}$ occurs at four loops similar to QCD

$$d_F^{abcd} = \frac{1}{6} \mathrm{Tr} \left(T^a T^{(b} T^c T^{d)} \right)$$

- As a first step, aim is to compute mass anomalous dimensions at four loops in $\overline{\mathrm{MS}}$ in the Gross-Neveu model

Gross-Neveu model

- Bare two dimensional Lagrangian with $SU(N)$ symmetry is

$$L = i\bar{\psi}_0^i \partial\!\!\!/ \psi_0^i - m_0 \bar{\psi}_0^i \psi_0^i + \frac{1}{2} g_0 (\bar{\psi}_0^i \psi_0^i)^2$$

- Renormalization group functions are known in $\overline{\text{MS}}$ at various orders

$$\begin{aligned}
\gamma(g) &= (2N-1)\frac{g^2}{8\pi^2} - (N-1)(2N-1)\frac{g^3}{16\pi^3} \\
&\quad + (4N^2 - 14N + 7)(2N-1)\frac{g^4}{128\pi^4} + O(g^5) \\
\gamma_m(g) &= -(2N-1)\frac{g}{2\pi} + (2N-1)\frac{g^2}{8\pi^2} \\
&\quad + (4N-3)(2N-1)\frac{g^3}{32\pi^3} + O(g^4) \\
\beta(g) &= (d-2)g - (N-1)\frac{g^2}{\pi} + (N-1)\frac{g^3}{2\pi^2} \\
&\quad + (N-1)(2N-7)\frac{g^4}{16\pi^4} + O(g^5)
\end{aligned}$$

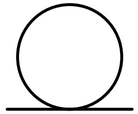
- $\gamma(g)$ and $\gamma_m(g)$ vanish for $N = \frac{1}{2}$ which is free field theory
- Four loop wave function computed in massless model, [Vyazovskii & Vasiliev]
- Only two difficult graphs to evaluate, but calculation is infrared safe
- For β-function and mass anomalous dimension a non-zero mass is needed to avoid infrared singularities
- For completeness the NATM Lagrangian is

$$L = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i - \frac{1}{2}g(\bar{\psi}^i\gamma^\mu T^a\psi^i)^2$$

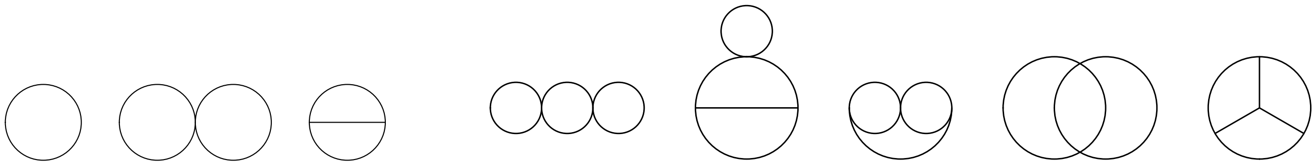
- To compute the Gross-Neveu mass anomalous dimension will use dimensional regularization in $d = 2 - \epsilon$ dimensions

Mass anomalous dimension

- Mass anomalous dimension also requires $m \neq 0$ as snails are crucial



- Method is to compute 2-point function but with nullified external momenta
- Hence require computation of massive vacuum bubbles to four loops relative to two dimensions
- To three loops basic topologies are known and easy to determine except for Benz which does not arise for 2-point function



- Fermionic nature of model means lines are decorated with scalar products of internal momenta

Vacuum bubbles to three loops

- Analogous problem in four dimensions is understood and encoded for example in MATAD
- Evaluation of basic integrals in two dimensions is simpler given that lower dimension produces integrals which are less divergent by power counting
- Structure of expressions is similar
- For example

$$\Delta(0) \;=\; i^2 \int_{kl} \frac{1}{[k^2 - m^2][l^2 - m^2][(k-l)^2 - m^2]}$$

is finite, where $\int_k = \int \frac{d^d k}{(2\pi)^d}$, and

$$\Delta(0) \;=\; -\,\frac{9 s_2}{16\pi^2 m^2} \;+\; O(\epsilon)$$

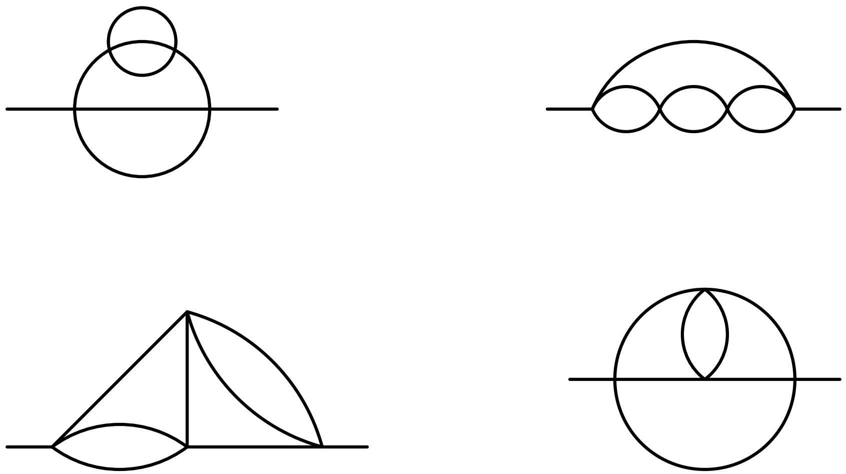
where $s_2 = (2\sqrt{3}/9)\mathrm{Cl}_2(2\pi/3)$ and $\mathrm{Cl}_2(x)$ is the Clausen function

- Numerator reduction achieved via usual techniques

$$kl \;=\; \frac{1}{2}\left[k^2 \;+\; l^2 \;-\; [(k-l)^2 - m^2] \;-\; m^2\right]$$

Four loops

- There are 1 one loop, 2 two loop, 7 three loop and 36 four loop graphs to be determined
- At four loop there are 18 distinct topologies with 14 involving snails in one form or another which are elementary to deduce
- Difficult graphs are

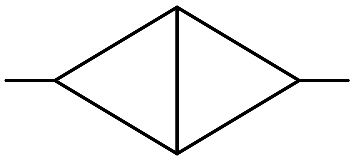


- Two main techniques to achieve tensor reduction

- First, where appropriate

$$i\int_l \frac{l^\mu l^\nu}{[l^2-m^2][(k-l)^2-m^2]} = \frac{1}{(d-1)} i\int_l \frac{1}{[l^2-m^2][(k-l)^2-m^2]} \times\left[\eta^{\mu\nu}\left(l^2-\frac{kl^2}{k^2}\right)-\frac{k^\mu k^\nu}{k^2}\left(l^2-d\frac{kl^2}{k^2}\right)\right]$$

- This potentially introduces infrared infinities but can show overall everything is infrared safe
- Other is the application of Tarasov's algorithm as encoded in the TARCER package
- TARCER performs tensor reduction for the basic 2-loop self energy



- This allows for all problem scalar products in numerator of difficult graphs to be written in terms of master scalar integrals

- Tensor reduction produces integrals like $i\int_k (k^2)^2 J^3(k^2)$
- Evaluate by either integration by parts or differentiation with respect to the mass

$$
\begin{aligned}
i\int_k (k^2)^2 J^3(k^2) &= 2I^4 + \frac{(7d-13)m^2 I}{(2d-5)}\, i\int_k J^2(k^2) \\
&\quad + \frac{2(d-1)(d-3)}{(2d-5)(d-2)} m^4\, i\int_k J^3(k^2)
\end{aligned}
$$

where $I = i\int_k \frac{1}{[k^2-m^2]}$ and $J(p^2) = i\int_k \frac{1}{[k^2-m^2][(k-p)^2-m^2]}$ with

$$
J(p^2) = -\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \left(\frac{4m^2-p^2}{4}\right)^{d/2-2} {}_2F_1\left(2-\frac{d}{2}, \frac{1}{2}; \frac{3}{2}; \frac{p^2}{p^2-4m^2}\right)
$$

- Introduces pole in two dimensions from d-dimensional dependence but this multiplies a finite integral (see later)

Mass anomalous dimension?

- Standard renormalization produces the final $\overline{\rm MS}$ divergence for the 2-point function at four loops as

$$\ldots + \left[\frac{5N^3}{192} + \frac{3N^2}{64} - \frac{5N}{48} + \frac{49}{1536} - \zeta(3) \left(\frac{N^3}{32} - \frac{N^2}{4} + \frac{41N}{128} - \frac{27}{256} \right) \right] \frac{g^4}{\pi^4 \epsilon}$$

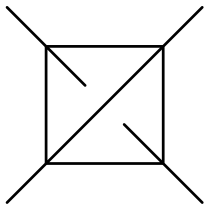
- Remaining part of divergence structure consistent with renormalization group expectations and large N critical exponents at $O(1/N^2)$
- Clearly non-vanishing when $N = \frac{1}{2}$ but calculation *is* correct
- Resolution is that Gross-Neveu model is renormalizable in two dimensions but when dimensionally regularized it ceases to be *multiplicatively* renormalizable
- Additional evanescent operator $\mathcal{O}_3$ generated at three loops in 4-point function renormalization with associated renormalization constant Z_{33} where

$$\mathcal{O}_n \; = \; \frac{1}{2} \bar{\psi}^i \Gamma_{(n)}^{\mu_1 \ldots \mu_n} \psi^i \, \bar{\psi}^i \Gamma_{(n) \; \mu_1 \ldots \mu_n} \psi^i$$

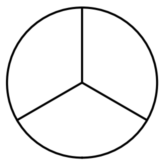
and $\Gamma_{(n)}^{\mu_1 \mu_2 \ldots \mu_n} = \gamma^{[\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_n]}$ is the γ-matrix basis in d-dimensions

4-point function renormalization

- Re-evaluate 4-point function renormalization at three loops as there are two different values for Z_{33} and the associated β-function $\beta_3(g)$
- One was determined in massless model with two momenta nullified; other was in massive model
- Both assumed the following graph was finite



- In massive model the tensor reduction gives the following contribution in the $\Gamma_{(3)} \otimes \Gamma_{(3)}$ channel

$(d-2)^{-1}$ 

- Cannot be evaluated by integration by parts; massive Benz integral is finite in two dimensions

- Relate to massive tetrahedron vacuum bubble of Broadhurst via Tarasov's $d \to (d+2)$ formalism as encoded in TARCER for two loop self energy
- Produces relation for Benz topology

$$
\begin{aligned}
&\mathrm{Be}(1,1,1,1,1,1,m^2,m^2,m^2,m^2,m^2,m^2,d) \\
&= -\frac{1}{12m^4} i \int_k J^2(k^2) \; - \; \frac{3}{4m^2} i \int_k \frac{J^2(k^2)}{[k^2-m^2]} \\
&\quad + \frac{\pi d(d-1)(d-2)}{m^6} \Big[\mathrm{Be}(1,1,1,1,1,1,m^2,m^2,m^2,m^2,m^2,m^2,d+2) \\
&\qquad\qquad - \mathrm{Be}(1,1,1,1,1,1,m^2,m^2,m^2,m^2,m^2,0,d+2) \Big]
\end{aligned}
$$

- The two dimensional integrals are finite and can be evaluated in two dimensions directly using properties of hypergeometric function

$$
i \int_k J^2(k^2) \Big|_{d=2} = -\frac{7\zeta(3)}{64\pi^3 m^2} \;\; , \;\; i \int_k \frac{J^2(k^2)}{[k^2-m^2]} \Big|_{d=2} = \frac{11\zeta(3)}{576\pi^3 m^4}
$$

and

$$
i \int_k J^3(k^2) \Big|_{d=2} = \frac{3\zeta(3)}{256\pi^4 m^4}
$$

- To order in ϵ we are interested in, the massive three loop tetrahedron in $d = 2 - \epsilon$ dimensions is

$$\mathrm{Be}(1,1,1,1,1,1,m^2,m^2,m^2,m^2,m^2,m^2,d) \; = \; - \; \frac{\zeta(3)}{192\pi^3 m^6} \; + \; O(\epsilon)$$

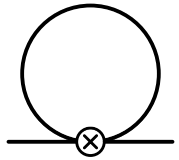
- Consequently

$$\beta_3(g) \; = \; \left[\frac{3\zeta(3)}{64} \; - \; \frac{1}{16} \right] \frac{g^3}{\pi^3} \; + \; O(g^4)$$

which differs from previous two expressions

Return to problem

- Upshot is that the following graph has to be included in the renormalization of the 2-point function



- This produces the (intermediate) mass anomalous dimension

$$\begin{aligned}\tilde{\gamma}_m(g) &= -(2N-1)\frac{g}{2\pi} + (2N-1)\frac{g^2}{8\pi^2} + (4N-3)(2N-1)\frac{g^3}{32\pi^3} \\ &+ \big[(48N^3 - 384N^2 + 492N - 138)\zeta(3) \\ &\quad - 40N^3 - 72N^2 + 160N - 81\big]\frac{g^4}{384\pi^4} + O(g^5)\end{aligned}$$

- Again agrees with $O(1/N^2)$ critical exponent calculation but these do not check $O(N)$ terms of anomalous dimension
- Still non-zero at $N = \frac{1}{2}$

- Explicitly

$$\tilde{\gamma}_m(g)\Big|_{N=\frac{1}{2}} \;=\; [3\zeta(3) - 4]\,\frac{g^4}{64\pi^4} \;+\; O(g^5)$$

- True anomalous dimension, $\gamma_m(g)$, emerges via Rossi et al's projection technique

$$\gamma_m(g) \;=\; \tilde{\gamma}_m(g) \;+\; \sum_{k=3}^{\infty} \rho_m^{(k)}(g)\beta_k(g)$$

  where k ranges over evanescent operators only

- Projection formula is

$$\int d^dx\,\mathcal{N}[\mathcal{O}_k]\Big|_{\substack{g_{(i)}=0\\ d=2}} \;=\; \int d^dx\,\Big(\rho^{(k)}(g)\mathcal{N}[i\bar{\psi}\partial\!\!\!/\psi \;-\; m\bar{\psi}\psi \;+\; 2g\mathcal{O}_0] \;-\; \rho_m^{(k)}(g)\,\mathcal{N}[m\bar{\psi}\psi] \;+\; C^{(k)}(g)\mathcal{N}[\mathcal{O}_0]\Big)\Big|_{\substack{g_{(i)}=0\\ d=2}}$$

- Inserted into a 2 or 4-point function in d-dimensions and then renormalized; projection functions essentially defined by the finite parts

- For four loops the projection term has been deduced

$$\rho_m(g) \;=\; -\,\frac{g}{\pi} \;+\; O(g^2)$$

- Final mass anomalous dimension in two dimensions is

$$\begin{aligned}\gamma_m(g) \quad &= \quad -\,(2N-1)\frac{g}{2\pi} \;+\; (2N-1)\frac{g^2}{8\pi^2} \;+\; (4N-3)(2N-1)\frac{g^3}{32\pi^3} \\ &\quad + \left[12(2N-13)(N-1)\zeta(3) - 20N^2 - 46N + 57\right]\frac{(2N-1)g^4}{384\pi^4} \\ &\quad + O(g^5)\end{aligned}$$

- Vanishes at $N = \frac{1}{2}$

Conclusions

- Consistent value for the four loop $\overline{\mathrm{MS}}$ mass anomalous dimension in the Gross-Neveu model
- Resolves discrepancy in previous computations of evanescent operator coupling constant renormalization
- Opens up the possibility of computing mass dimension in non-abelian Thirring model to examine Casimir structure
- Useful interplay of vacuum bubble integrals in different dimensions