NLO corrections with the OPP method

Costas G. Papadopoulos

NCSR "Demokritos", Athens

Loops and Legs 2008, Sonderhausen, 20-25 April 2008

Outline

1. Introduction: Wishlists and Troubles
2. OPP Reduction
   - Rational terms
3. Numerical Tests
   - 4-photon amplitudes
   - 6-photon amplitudes
   - VVV production

Introduction: LHC needs NLO

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)

Introduction: LHC needs NLO

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)
- In the last years we have seen a remarkable progress in the theoretical description of multi-particle processes at tree-order, thanks to very efficient recursive algorithms

Introduction: LHC needs NLO

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)
- In the last years we have seen a remarkable progress in the theoretical description of multi-particle processes at tree-order, thanks to very efficient recursive algorithms
- The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO)

Introduction: LHC needs NLO

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)
- In the last years we have seen a remarkable progress in the theoretical description of multi-particle processes at tree-order, thanks to very efficient recursive algorithms
- The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO)
- As a result, a big effort has been devoted by several groups to the problem of an efficient computation of one-loop corrections for multi-particle processes!

NLO Wishlist Les Houches

[from G. Heinrich's Summary talk]

Wishlist Les Houches 2007
1. $pp \to V\,V + \text{jet}$
2. $pp \to t\bar{t}\,b\bar{b}$
3. $pp \to t\bar{t} + 2$ jets
4. $pp \to W\,W\,W$
5. $pp \to V\,V\,b\bar{b}$
6. $pp \to V\,V + 2$ jets
7. $pp \to V + 3$ jets
8. $pp \to t\bar{t}\,b\bar{b}$
9. $pp \to 4$ jets

Processes for which a NLO calculation is both desired and feasible

Will we "finish" in time for LHC?

What has been done? (2005-2007)

Some recent results → Cross Sections available

- $pp \to ZZZ$ $pp \to t\bar{t}Z$ [Lazopoulos, Melnikov, Petriello]
- $pp \to H + 2$ jets [Campbell, et al., J. R. Andersen, et al.]
- $pp \to VV + 2$ jets via VBF [Bozzi, Jäger, Oleari, Zeppenfeld]
- $pp \to VV + 1$ jet [S. Dittmaier, S. Kallweit and P. Uwer]
- $pp \to t\bar{t} + 1$ jet [S. Dittmaier, P. Uwer and S. Weinzierl]

Mostly $2 \to 3$, very few $2 \to 4$ complete calculations.

- $e^+ e^- \to 4$ fermions [Denner, Dittmaier, Roth]
- $e^+ e^- \to H H \nu \bar{\nu}$ [GRACE group (Boudjema et al.)]

This is NOT a complete list

(A lot of work has been done at NLO → calculations & new methods)

NLO TROUBLES

Problems arising in NLO calculations

- Large Number of Feynman diagrams
- Reduction to Scalar Integrals (or sets of known integrals)
- Numerical Instabilities (inverse Gram determinants, spurious phase-space singularities)
- Extraction of soft and collinear singularities (we need virtual and real corrections)

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
  - general applicability major achievements
  - but major problem: not designed @ amplitude level

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
- Semi-Numerical Approach (Algebraic/Partly Numerical – Improved traditional) $\rightarrow$ Reduction to set of well-known integrals
- Numerical Approach (Numerical/Partly Algebraic) $\rightarrow$ Compute tensor integrals numerically
  - Ellis, Giele, Glover, Zanderighi;
  - Binoth, Guillet, Heinrich, Schubert;
  - Denner, Dittmaier; Del Aguila, Pittau;
  - Ferroglia, Passera, Passarino, Uccirati;
  - Nagy, Soper; van Hameren, Vollinga, Weinzierl;

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
- Semi-Numerical Approach (Algebraic/Partly Numerical – Improved traditional) $\longrightarrow$ Reduction to set of well-known integrals
- Numerical Approach (Numerical/Partly Algebraic) $\longrightarrow$ Compute tensor integrals numerically
- Analytic Approach (Twistor-inspired)
  $\longrightarrow$ extract information from lower-loop, lower-point amplitudes
  $\longrightarrow$ determine scattering amplitudes by their poles and cuts
  - ⋆ major advantage: designed to work @ amplitude level
  - ⋆ **quadruple and triple cuts major simplifications**
  - Bern, Dixon, Dunbar, Kosower, Berger, Forde;
  - Anastasiou, Britto, Cachazo, Feng, Kunszt, Mastrolia;

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
- Semi-Numerical Approach (Algebraic/Partly Numerical – Improved traditional) $\rightarrow$ Reduction to set of well-known integrals
- Numerical Approach (Numerical/Partly Algebraic) $\rightarrow$ Compute tensor integrals numerically
- Analytic Approach (Twistor-inspired)
  $\rightarrow$ extract information from lower-loop, lower-point amplitudes
  $\rightarrow$ determine scattering amplitudes by their poles and cuts

⋆ OPP Integrand-level reduction
combine: reduction@integrand + n-particle cuts

OPP Reduction - Intro

G. Ossola., C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007) – arXiv:hep-ph/0609007

and JHEP **0707** (2007) 085 – arXiv:0704.1271 [hep-ph]

R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP **0803**, 003 (2008)

Any m-point one-loop amplitude can be written, before integration, as

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

External momenta p_i are 4-dimensional objects

The old "master" formula

$$
\begin{aligned}
\int A &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
&+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
&+ \text{rational terms}
\end{aligned}
$$

OPP "master" formula - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP "MASTER" FORMULA - II

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) \qquad \right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c(i_0 i_1 i_2) \qquad] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i$$
$$+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) \qquad \right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} [a(i_0) \qquad] \prod_{i\neq i_0}^{m-1} D_i$$

- The quantities $d(i_0 i_1 i_2 i_3)$ are the coefficients of 4-point functions with denominators labeled by i_0, i_1, i_2, and i_3.
- $c(i_0 i_1 i_2)$, $b(i_0 i_1)$, $a(i_0)$ are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

The quantities $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are the "spurious" terms

- They still depend on q (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?

Spurious Terms - I

Following F. del Aguila and R. Pittau, arXiv:hep-ph/0404120

- Express any q in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^4 G_i \, \ell_i^\mu \; , \; {\ell_i}^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2 \, , \quad k_2 = \ell_2 + \alpha_2 \ell_1 \, , \; k_i = p_i - p_0$$
$${\ell_3}^\mu =< \ell_1|\gamma^\mu|\ell_2] \, , \; {\ell_4}^\mu =< \ell_2|\gamma^\mu|\ell_1]$$

- The coefficients G_i either reconstruct denominators D_i

$\rightarrow$ They give rise to d, c, b, a coefficients

Spurious Terms - I

Following F. del Aguila and R. Pittau, arXiv:hep-ph/0404120

- Express any q in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^{4} G_i \, \ell_i^\mu \; , \; {\ell_i}^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2 \, , \quad k_2 = \ell_2 + \alpha_2 \ell_1 \, , \; k_i = p_i - p_0$$
$$\ell_3{}^\mu = < \ell_1|\gamma^\mu|\ell_2] \, , \; \ell_4{}^\mu = < \ell_2|\gamma^\mu|\ell_1]$$

- The coefficients G_i either reconstruct denominators D_i or vanish upon integration

→ They give rise to d, c, b, a coefficients
→ They form the spurious $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ coefficients

Spurious Terms - II

- $\tilde{d}(\mathrm{q})$ term (only 1)

$$\tilde{d}(q) = \tilde{d}\, T(q)\,,$$

where $\tilde{d}$ is a constant (does not depend on q)

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5]$$

Spurious Terms - II

- $\tilde{d}(q)$ term (only 1)

$$\tilde{d}(q) = \tilde{d}\, T(q)\,,$$

where $\tilde{d}$ is a constant (does not depend on q)

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5]$$

- $\tilde{c}(q)$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}[(q + p_0)\cdot \ell_3]^j + \tilde{c}_{2j}[(q + p_0)\cdot \ell_4]^j \right\}$$

In the renormalizable gauge, $j_{max} = 3$

Spurious Terms - II

- $\tilde{d}(q)$ term (only 1)

$$\tilde{d}(q) = \tilde{d}\, T(q)\,,$$

where $\tilde{d}$ is a constant (does not depend on q)

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5]$$

- $\tilde{c}(q)$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}[(q+p_0)\cdot \ell_3]^j + \tilde{c}_{2j}[(q+p_0)\cdot \ell_4]^j \right\}$$

In the renormalizable gauge, $j_{max} = 3$

- $\tilde{b}(q)$ and $\tilde{a}(q)$ give rise to 8 and 4 terms, respectively

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\frac{1}{D_{i_4}(q^+)} + \frac{1}{D_{i_4}(q^-)} \right)$$

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\frac{1}{D_{i_4}(q^+)} + \frac{1}{D_{i_4}(q^-)} \right)$$

- Melrose, Nuovo Cim. 40 (1965) 181
- G. Källén, J.Toll, J. Math. Phys. 6, 299 (1965)

More on historical roots

W. L. van Neerven and J. A. M. Vermaseren, “Large Loop Integrals,” Phys. Lett. B **137**, 241 (1984)

More on historical roots

W. L. van Neerven and J. A. M. Vermaseren, "Large Loop Integrals," Phys. Lett. B **137**, 241 (1984)

The derivation of the reduction formula starts as in ref. [1] with the Schouten identity which is a relation between five Levi-Civita tensors:

$$\epsilon^{p_1 p_2 p_3 p_4} Q_\mu = \epsilon^{\mu p_2 p_3 p_4} Q\cdot p_1 + \epsilon^{p_1 \mu p_3 p_4} Q\cdot p_2 + \epsilon^{p_1 p_2 \mu p_4} Q\cdot p_3 + \epsilon^{p_1 p_2 p_3 \mu} Q\cdot p_4 \,. \tag{6}$$

W. L. van Neerven and J. A. M. Vermaseren, "Large Loop Integrals," Phys. Lett. B **137**, 241 (1984)

The derivation of the reduction formula starts as in ref. [1] with the Schouten identity which is a relation between five Levi-Civita tensors:

$$\epsilon^{p_1 p_2 p_3 p_4} Q_\mu = \epsilon^{\mu p_2 p_3 p_4} Q\cdot p_1 + \epsilon^{p_1 \mu p_3 p_4} Q\cdot p_2 + \epsilon^{p_1 p_2 \mu p_4} Q\cdot p_3 + \epsilon^{p_1 p_2 p_3 \mu} Q\cdot p_4 \,. \tag{6}$$

which yields the final formula for the scalar one-loop five-point function:

$$E_{01234}(w^2 - 4\Delta_4 m_0^2) = D_{1234}\,[2\Delta_4 - w\cdot(v_1 + v_2 + v_3 + v_4)]$$
$$+ D_{0234}\, v_1\cdot w + D_{0134}\, v_2\cdot w + D_{0124}\, v_3\cdot w + D_{0123}\, v_4\cdot w \,. \tag{19}$$

More on historical roots

W. L. van Neerven and J. A. M. Vermaseren, "Large Loop Integrals," Phys. Lett. B **137**, 241 (1984)

The derivation of the reduction formula starts as in ref. [1] with the Schouten identity which is a relation between five Levi-Civita tensors:

$$\epsilon^{p_1 p_2 p_3 p_4} Q_\mu = \epsilon^{\mu p_2 p_3 p_4} Q\cdot p_1 + \epsilon^{p_1 \mu p_3 p_4} Q\cdot p_2 + \epsilon^{p_1 p_2 \mu p_4} Q\cdot p_3 + \epsilon^{p_1 p_2 p_3 \mu} Q\cdot p_4 \,. \quad (6)$$

which yields the final formula for the scalar one-loop five-point function:

$$E_{01234}(w^2 - 4\Delta_4 m_0^2) = D_{1234}\,[2\Delta_4 - w\cdot(v_1 + v_2 + v_3 + v_4)]$$

$$+ D_{0234}\, v_1\cdot w + D_{0134}\, v_2\cdot w + D_{0124}\, v_3\cdot w + D_{0123}\, v_4\cdot w \,. \quad (19)$$

This method is completely different from the one used in ref. [3].

More on historical roots

W. L. van Neerven and J. A. M. Vermaseren, "Large Loop Integrals," Phys. Lett. B **137**, 241 (1984)

The derivation of the reduction formula starts as in ref. [1] with the Schouten identity which is a relation between five Levi-Civita tensors:

$$\epsilon^{p_1 p_2 p_3 p_4} Q_\mu = \epsilon^{\mu p_2 p_3 p_4} Q\cdot p_1 + \epsilon^{p_1 \mu p_3 p_4} Q\cdot p_2 + \epsilon^{p_1 p_2 \mu p_4} Q\cdot p_3 + \epsilon^{p_1 p_2 p_3 \mu} Q\cdot p_4\,. \quad (6)$$

which yields the final formula for the scalar one-loop five-point function:

$$E_{01234}(w^2 - 4\Delta_4 m_0^2) = D_{1234}\,[2\Delta_4 - w\cdot(v_1 + v_2 + v_3 + v_4)]$$
$$+ D_{0234}\, v_1\cdot w + D_{0134}\, v_2\cdot w + D_{0124}\, v_3\cdot w + D_{0123}\, v_4\cdot w\,. \quad (19)$$

This method is completely different from the one used in ref. [3].

References

[1] G. 't Hooft and M. Veltman, Nucl. Phys. B153 (1979) 365.
[2] J.A.M. Vermaseren, Nucl. Phys. B229 (1983) 347.
[3] G. Passarino and M. Veltman, Nucl. Phys. B160 (1979) 151. ←

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating N(q) for a set of values of the integration momentum q

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating N(q) for a set of values of the integration momentum q

There is a very good set of such points: Use values of q for which a set of denominators D_i vanish → The system becomes "triangular": solve first for 4-point functions, then 3-point functions and so on

Example

$$N(q) = d + \tilde{d}(q) + \sum_{i=0}^{3} \left[c(i) + \tilde{c}(q;i) \right] D_i + \sum_{i_0 < i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] D_{i_0} D_{i_1}$$

$$+ \sum_{i_0=0}^{3} \left[a(i_0) + \tilde{a}(q; i_0) \right] D_{i \neq i_0} D_{j \neq i_0} D_{k \neq i_0}$$

We look for a q of the form $q^\mu = -p_0^\mu + x_i \ell_i^\mu$ such that

$$D_0 = D_1 = D_2 = D_3 = 0$$

$\rightarrow$ we get a system of equations in x_i that has two solutions $q_0^\pm$

Example

$$N(q) = d + \tilde{d}(q)$$

Our "master formula" for $q = q_0^{\pm}$ is:

$$N(q_0^{\pm}) = [d + \tilde{d}\; T(q_0^{\pm})]$$

$\longrightarrow$ solve to extract the coefficients d and $\tilde{d}$

Example

$$N(q) - d - \tilde{d}(q) = \sum_{i=0}^{3} [c(i) + \tilde{c}(q;i)] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] D_{i_0} D_{i_1}$$

$$+ \sum_{i_0=0}^{3} [a(i_0) + \tilde{a}(q; i_0)] D_{i \neq i_0} D_{j \neq i_0} D_{k \neq i_0}$$

Then we can move to the extraction of c coefficients using

$$N'(q) = N(q) - d - \tilde{d}\,T(q)$$

and setting to zero three denominators (ex: $D_1 = 0$, $D_2 = 0$, $D_3 = 0$)

Example

$$N(q) - d - \tilde{d}(q) \quad = \quad [c(0) + \tilde{c}(q;0)]\, D_0$$

We have infinite values of q for which

$$D_1 = D_2 = D_3 = 0 \quad \text{and} \quad D_0 \neq 0$$

$\longrightarrow$ Here we need 7 of them to determine $c(0)$ and $\tilde{c}(q;0)$

Rational Terms - I

- Let's go back to the integrand

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

- Insert the expression for $N(q) \to$ we know all the coefficients

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d + \tilde{d}(q) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c + \tilde{c}(q)] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i + \cdots$$

- Finally rewrite all denominators using

$$\frac{D_i}{\bar{D}_i} = \bar{Z}_i \,, \qquad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i} \right)$$

Rational Terms - I

$$
\begin{aligned}
A(\bar{q}) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i
\end{aligned}
$$

The rational part is produced, after integrating over $d^n q$, by the $\tilde{q}^2$ dependence in $\bar{Z}_i$

$$
\bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)
$$

Rational Terms - I

The "Extra Integrals" are of the form

$$I_{s;\mu_1\cdots\mu_r}^{(n;2\ell)} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)}\,,$$

where

$$\bar{D}(k_i) \equiv (\bar{q}+k_i)^2 - m_i^2\,, k_i = p_i - p_0$$

These integrals:

- **have dimensionality** $\mathcal{D} = 2(1+\ell-s)+r$
- **contribute only when** $\mathcal{D} \geq 0$**, otherwise are of** $\mathcal{O}(\epsilon)$

Rational Terms - II

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

Rational Terms - II

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

Rational Terms - II

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$
m_i^2 \rightarrow m_i^2 - \tilde{q}^2
$$

Rational Terms - II

Polynomial dependence on $\tilde{q}^2$

$$b(ij;\tilde{q}^2) = b(ij) + \tilde{q}^2 b^{(2)}(ij)\,, \quad c(ijk;\tilde{q}^2) = c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,.$$

Rational Terms - II

Polynomial dependence on $\tilde{q}^2$

$$b(ij;\tilde{q}^2) \;=\; b(ij) + \tilde{q}^2 b^{(2)}(ij)\,, \quad c(ijk;\tilde{q}^2) \;=\; c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,.$$

$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} \;=\; -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3}\right] + \mathcal{O}(\epsilon)\,,$$

$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} \;=\; -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,, \quad \int d^n\bar{q}\,\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} \;=\; -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon)\,.$$

Furthermore, by defining

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) \equiv \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3;\tilde{q}^2) + \tilde{d}(q;i_0 i_1 i_2 i_3;\tilde{q}^2) \right] \prod_{i \neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \,,$$

the following expansion holds

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q) \,,$$

where the last coefficient is independent on q

$$d^{(2m-4)}(q) = d^{(2m-4)} \,.$$

Rational Terms - II

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$\begin{aligned} \mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\ &- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) . \end{aligned}$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

Rational Terms - R_2

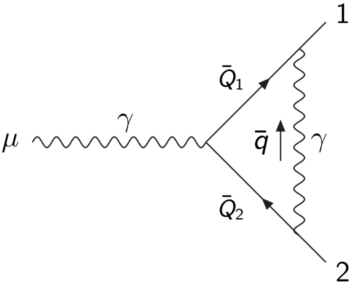
A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\, \bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\, \bar{q}\, \mathcal{R}_2$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

New vertices/particles or GKM-approach



$$\bar{Q}_1 = \bar{q} + p_1 = Q_1 + \tilde{q}$$
$$\bar{Q}_2 = \bar{q} + p_2 = Q_2 + \tilde{q}$$

$$\bar{D}_0 = \bar{q}^2$$

$$\bar{D}_1 = (\bar{q} + p_1)^2$$

$$\bar{D}_2 = (\bar{q} + p_2)^2$$

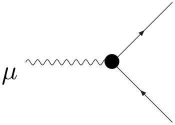
$$\begin{aligned}
\bar{N}(\bar{q}) &\equiv e^3 \left\{ \bar{\gamma}_{\bar{\beta}} \, (\not{\bar{Q}}_1 + m_e) \, \gamma_\mu \, (\not{\bar{Q}}_2 + m_e) \, \bar{\gamma}^{\bar{\beta}} \right\} \\
&= e^3 \, \{ \gamma_\beta (\not{Q}_1 + m_e) \gamma_\mu (\not{Q}_2 + m_e) \gamma^\beta \\
&- \epsilon \, (\not{Q}_1 - m_e) \gamma_\mu (\not{Q}_2 - m_e) + \epsilon \tilde{q}^2 \, \gamma_\mu - \tilde{q}^2 \, \gamma_\beta \gamma_\mu \gamma^\beta \} \; ,
\end{aligned}$$

Rational Terms - R_2

$$\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_0\bar{D}_1\bar{D}_2} = -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,,$$

$$\int d^n\bar{q}\frac{q_\mu q_\nu}{\bar{D}_0\bar{D}_1\bar{D}_2} = -\frac{i\pi^2}{2\epsilon} g_{\mu\nu} + \mathcal{O}(1)\,,$$

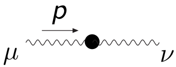
$$\mathrm{R}_2 = -\frac{ie^3}{8\pi^2}\gamma_\mu + \mathcal{O}(\epsilon)\,,$$



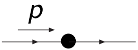
$$= -\frac{ie^3}{8\pi^2}\gamma_\mu$$

Rational Terms - R_2

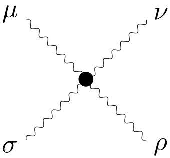
Rational counterterms



$$= -\frac{ie^2}{8\pi^2}\, g_{\mu\nu}\left(2m_e^2 - p^2/3\right)$$



$$= \frac{ie^2}{16\pi^2}\left(-\not{p} + 2m_e\right)$$



$$= \frac{ie^4}{12\pi^2}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)$$

Summary

Calculate $N(q)$

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly
- Calculate $N(q)$ numerically via recursion relations

SUMMARY

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly
- Calculate $N(q)$ numerically via recursion relations
- Just specify external momenta, polarization vectors and masses and proceed with the reduction!

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly
- Calculate $N(q)$ numerically via recursion relations
- Just specify external momenta, polarization vectors and masses and proceed with the reduction!

Compute all coefficients

Summary

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly
- Calculate $N(q)$ numerically via recursion relations
- Just specify external momenta, polarization vectors and masses and proceed with the reduction!

Compute all coefficients

- by evaluating N(q) at certain values of integration momentum

Summary

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly
- Calculate $N(q)$ numerically via recursion relations
- Just specify external momenta, polarization vectors and masses and proceed with the reduction!

Compute all coefficients

- by evaluating N(q) at certain values of integration momentum

Evaluate scalar integrals

Summary

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly
- Calculate $N(q)$ numerically via recursion relations
- Just specify external momenta, polarization vectors and masses and proceed with the reduction!

Compute all coefficients

- by evaluating N(q) at certain values of integration momentum

Evaluate scalar integrals

- massive integrals $\longrightarrow$ FF [G. J. van Oldenborgh]
- massless+massive integrals $\longrightarrow$ OneLOop [A. van Hameren]

What we gain

- PV:

- Unitarity Methods:

What we gain

- PV:
  - $N(q)$ or $A(q)$ hasn't to be known analytically
- Unitarity Methods:

What we gain

- PV:
  - $N(q)$ or $A(q)$ hasn't to be known analytically
  - No computer algebra

- Unitarity Methods:

What we gain

- PV:
  - $N(q)$ or $A(q)$ hasn't to be known analytically
  - No computer algebra
  - Mathematica $\rightarrow$ Numerica
- Unitarity Methods:

What we gain

- PV:
  - $N(q)$ or $A(q)$ hasn't to be known analytically
  - No computer algebra
  - Mathematica $\rightarrow$ Numerica
- Unitarity Methods:
  - A more transparent algebraic method

What we gain

- PV:
  - $N(q)$ or $A(q)$ hasn't to be known analytically
  - No computer algebra
  - Mathematica $\rightarrow$ Numerica
- Unitarity Methods:
  - A more transparent algebraic method
  - A solid way to get all rational terms

What we gain

- PV:
  - $N(q)$ or $A(q)$ hasn't to be known analytically
  - No computer algebra
  - Mathematica $\rightarrow$ Numerica
- Unitarity Methods:
  - A more transparent algebraic method
  - A solid way to get all rational terms

Cuttools

G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **0803**, 042 (2008) [arXiv:0711.3596 [hep-ph]]

Properties of the master equation

The Master Equation

Properties of the master equation

- Polynomial equation in q

The Master Equation

Properties of the master equation

- Polynomial equation in q
- Highly redundant: the a-terms have a degree of $m^2 - 2$ compared to m as a function of q

Properties of the master equation

- Polynomial equation in q
- Highly redundant: the a-terms have a degree of $m^2 - 2$ compared to m as a function of q
- Zeros of (a tower of) polynomial equations

The Master Equation

Properties of the master equation

- Polynomial equation in q
- Highly redundant: the a-terms have a degree of $m^2 - 2$ compared to m as a function of q
- Zeros of (a tower of) polynomial equations
- Different ways of solving it, besides 'unitarity method'

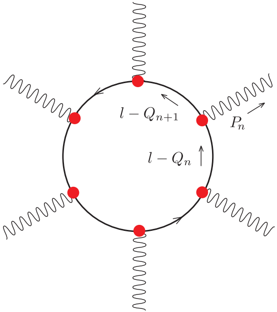
Properties of the master equation

- Polynomial equation in q
- Highly redundant: the a-terms have a degree of $m^2 - 2$ compared to m as a function of q
- Zeros of (a tower of) polynomial equations
- Different ways of solving it, besides 'unitarity method'

The $N \equiv N$ test

A tool to efficiently treat phase-space points with numerical instabilities

As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass m_f)

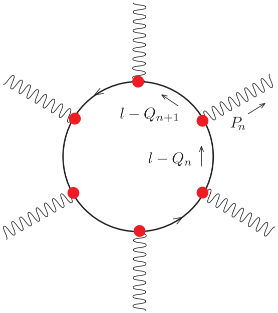


Input parameters for the reduction:

- External momenta p_i
- Masses of propagators in the loop
- Polarization vectors

4-PHOTON AND 6-PHOTON AMPLITUDES

As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass m_f)



Input parameters for the reduction:

- External momenta p_i $\longrightarrow$ in this example massless, i.e. $p_i^2 = 0$
- Masses of propagators in the loop $\longrightarrow$ all equal to m_f
- Polarization vectors $\longrightarrow$ various helicity configurations

Four Photons – Comparison with *Gounaris et al.*

$$\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8$$

Rational Part

Four Photons – Comparison with *Gounaris et al.*

$$
\begin{aligned}
\frac{F^{f}_{++++}}{\alpha^2 Q_f^4} &= -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right) B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right) B_0(\hat{t}) \\
&\quad - 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\right)\left[\hat{t}\, C_0(\hat{t}) + \hat{u}\, C_0(\hat{u})\right] \\
&\quad - 4\left[\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\right] D_0(\hat{t},\hat{u})
\end{aligned}
$$

Massless four-photon amplitudes

$$
\begin{aligned}
\frac{F^f_{++++}}{\alpha^2 Q_f^4} &= -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right)B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right)B_0(\hat{t}) \\
&- 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}}\right)[\hat{t}C_0(\hat{t}) + \hat{u}C_0(\hat{u})] \\
&- 4\left[4m_f^4 - (2\hat{s}m_f^2 + \hat{t}\hat{u})\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} + \frac{4m_f^2\hat{t}\hat{u}}{\hat{s}}\right]D_0(\hat{t},\hat{u}) \\
&+ 8m_f^2(\hat{s} - 2m_f^2)[D_0(\hat{s},\hat{t}) + D_0(\hat{s},\hat{u})]
\end{aligned}
$$

Massive four-photon amplitudes

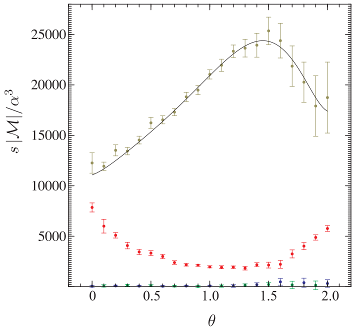
Four Photons – Comparison with *Gounaris et al.*

$$\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right)B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right)B_0(\hat{t})$$
$$-8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}-\frac{4m_f^2}{\hat{s}}\right)[\hat{t}C_0(\hat{t})+\hat{u}C_0(\hat{u})]$$
$$-4\left[4m_f^4-(2\hat{s}m_f^2+\hat{t}\hat{u})\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}+\frac{4m_f^2\hat{t}\hat{u}}{\hat{s}}\right]D_0(\hat{t},\hat{u})$$
$$+8m_f^2(\hat{s}-2m_f^2)[D_0(\hat{s},\hat{t})+D_0(\hat{s},\hat{u})]$$

Massive four-photon amplitudes

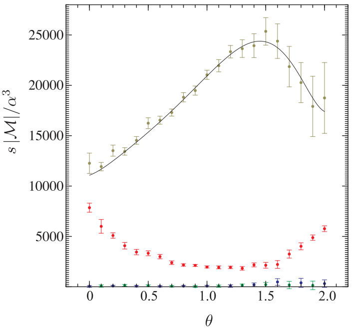
Results also checked for F^f_{+++-} and F^f_{++--}

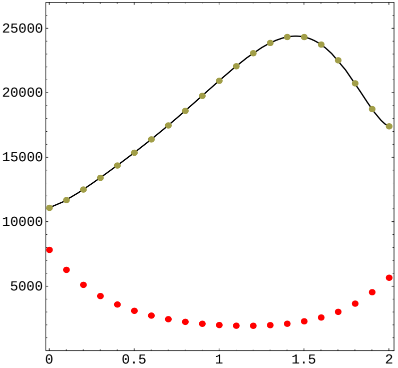
Massless case: $[++----]$ and $[+--++-]$



Plot presented by Nagy and Soper hep-ph/0610028
(also Binoth et al., hep-ph/0703311)

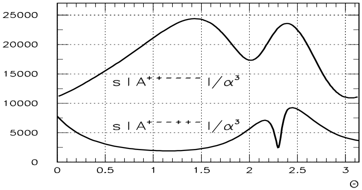
Massless case: $[+ + - - - -]$ and $[+ - - + + -]$

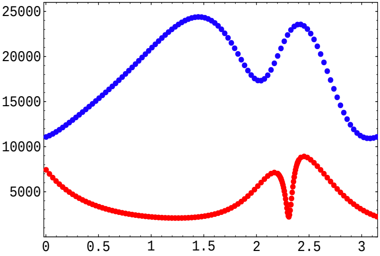




Analogous plot produced with OPP reduction

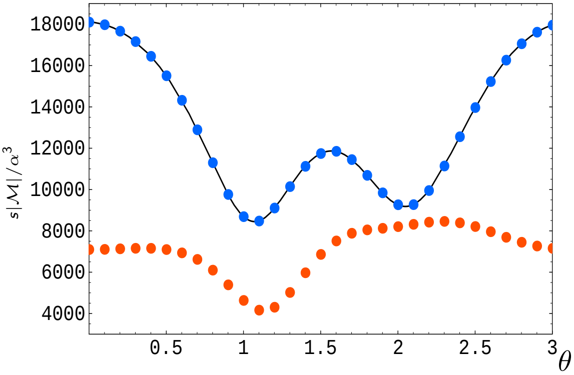
Massless case: $[++----]$ and $[++--+-]$





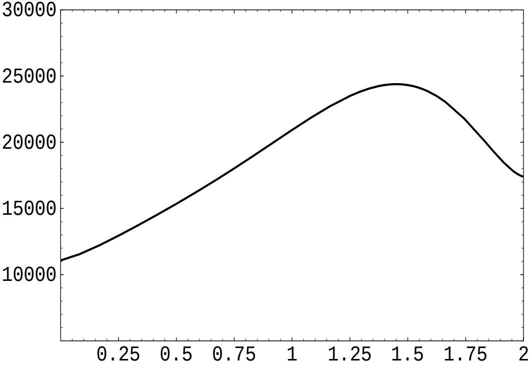
Same plot as before for a wider range of θ

Massless case: $[++----]$ and $[++--+-]$



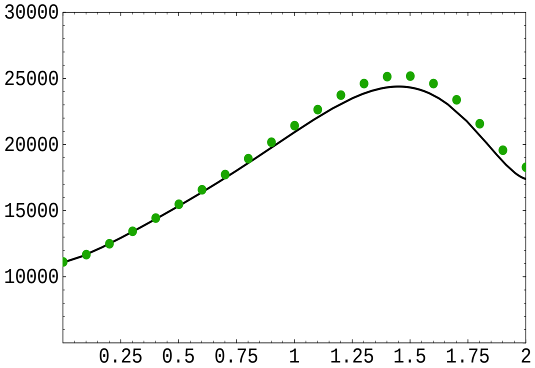
Same idea for a different set of external momenta

Six Photons with Massive Fermions



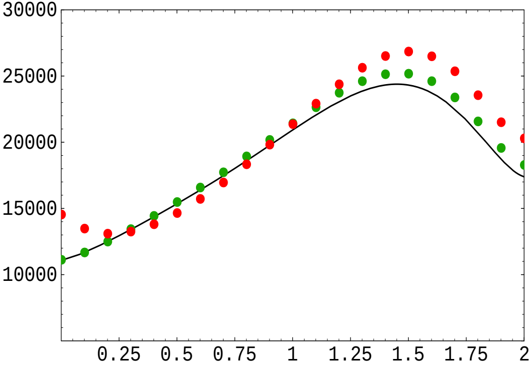
- Massless result [Mahlon]

Six Photons with Massive Fermions



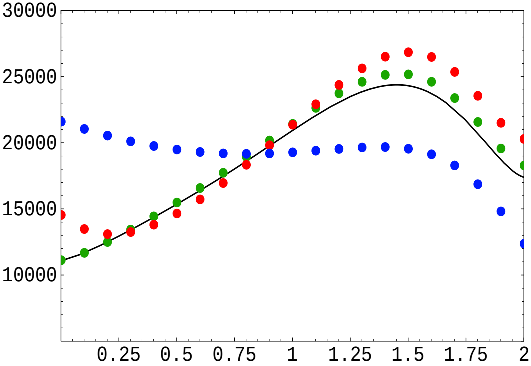
- Massless result [Mahlon]
- $m = 0.5$ GeV

Six Photons with Massive Fermions

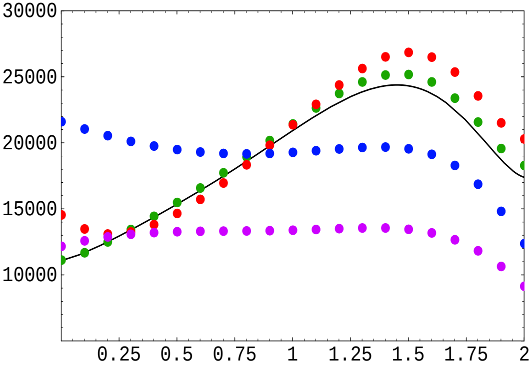


- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV

Six Photons with Massive Fermions



- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV
- $m = 12.0$ GeV



- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV
- $m = 12.0$ GeV
- $m = 20.0$ GeV

$pp \to VVV$ NLO

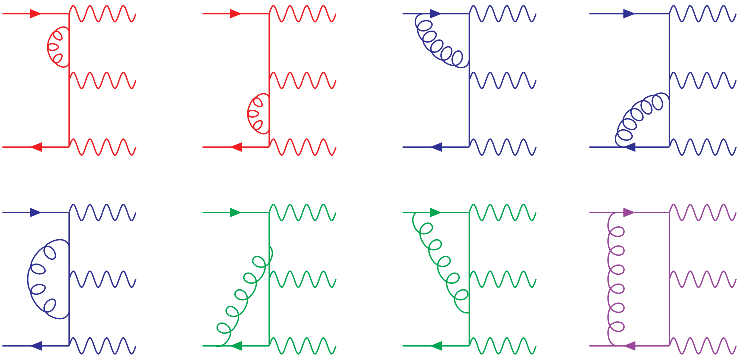
NLO corrections to tri-boson production

- $pp \to ZZZ$
- $pp \to W^+ZZ$
- $pp \to W^+W^-Z$
- $pp \to W^+W^-W^+$

T. Binoth, G. Ossola, C. G. Papadopoulos and R. Pittau, arXiv:0804.0350 [hep-ph]

$pp \to ZZZ$ VIRTUAL CORRECTIONS

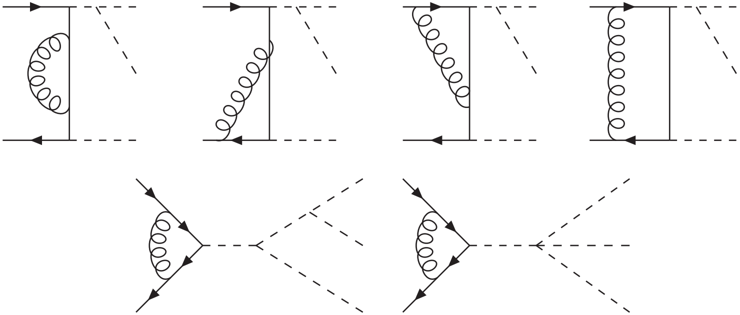
A. Lazopoulos, K. Melnikov and F. Petriello, [arXiv:hep-ph/0703273]



POLES $1/\epsilon^2$ AND $1/\epsilon$

$$\sigma^{\mathrm{NLO,virt}}|_{\mathrm{div}} = -C_F \frac{\alpha_s}{\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} (s_{12})^{-\epsilon} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \sigma^{\mathrm{LO}}$$

$pp \to WWZ$ VIRTUAL CORRECTIONS



Hankele and Zeppenfeld arXiv:0712.3544 [hep-ph]

$pp \to VVV$ VIRTUAL CORRECTIONS

A still naive implementation

A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !

A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !
- Calculate 4d and rational R_1 terms by `CutTools`

$pp \longrightarrow VVV$ VIRTUAL CORRECTIONS

A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !
- Calculate 4d and rational R_1 terms by `CutTools`
- R_2 terms added by hand

$pp \to VVV$ VIRTUAL CORRECTIONS

A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !
- Calculate 4d and rational R_1 terms by `CutTools`
- R_2 terms added by hand

Comparison with LMP

$pp \to VVV$ VIRTUAL CORRECTIONS

A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !
- Calculate 4d and rational R_1 terms by `CutTools`
- R_2 terms added by hand

Comparison with LMP

- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms

$pp \to VVV$ VIRTUAL CORRECTIONS

A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !
- Calculate 4d and rational R_1 terms by `CutTools`
- R_2 terms added by hand

Comparison with LMP

- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms
- An 'easy' agreement for all graphs with up to 4-point loop integrals

$pp \to VVV$ VIRTUAL CORRECTIONS

A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !
- Calculate 4d and rational R_1 terms by `CutTools`
- R_2 terms added by hand

Comparison with LMP

- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms
- An 'easy' agreement for all graphs with up to 4-point loop integrals
- A bit more work to uncover the differences in scalar function normalization that happen to show to order ϵ^2 thus influence only 5-point loop integrals.

$pp \to VVV$ VIRTUAL CORRECTIONS

Typical precision:

$pp \rightarrow VVV$ VIRTUAL CORRECTIONS

Typical precision:

- LMP: 9.573(66) about 1% error

$pp \to VVV$ VIRTUAL CORRECTIONS

Typical precision:

- LMP: 9.573(66) about 1% error
- OPP:

$$\begin{cases} -26.4570674281\,5552 \\ -26.4570674281\,65503661018557937723426 \end{cases}$$

$pp \to VVV$ virtual corrections

Typical precision:

- LMP: 9.573(66) about 1% error
- OPP:

$$\begin{cases} -26.45706742815552 \\ -26.4570674281655036610185579377234 26 \end{cases}$$

Typical time: 10^4 times faster (for non-singular PS-points)

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{q\bar{q}}^{NLO} = \int_{VVVg} \left[d\sigma_{q\bar{q}}^{R} - d\sigma_{q\bar{q}}^{A} \right] + \int_{VVV} \left[d\sigma_{q\bar{q}}^{B} + d\sigma_{q\bar{q}}^{V} + \int_{g} d\sigma_{q\bar{q}}^{A} + d\sigma_{q\bar{q}}^{C} \right]$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{q\bar{q}}^{NLO} = \int_{VVVg} \left[d\sigma_{q\bar{q}}^{R} - d\sigma_{q\bar{q}}^{A} \right] + \int_{VVV} \left[d\sigma_{q\bar{q}}^{B} + d\sigma_{q\bar{q}}^{V} + \int_{g} d\sigma_{q\bar{q}}^{A} + d\sigma_{q\bar{q}}^{C} \right]$$

$$\mathcal{D}^{q_1 g_6, \bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x}\, p_1 \cdot p_6} \left(\frac{1+\tilde{x}^2}{1-\tilde{x}} \right) \left| \mathcal{M}_{q\bar{q}}^{B}(\tilde{p}_{16}, p_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5) \right|^2$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{q\bar{q}}^{NLO} = \int_{VVVg} \left[d\sigma_{q\bar{q}}^{R} - d\sigma_{q\bar{q}}^{A} \right] + \int_{VVV} \left[d\sigma_{q\bar{q}}^{B} + d\sigma_{q\bar{q}}^{V} + \int_{g} d\sigma_{q\bar{q}}^{A} + d\sigma_{q\bar{q}}^{C} \right]$$

$$\mathcal{D}^{q_1 g_6, \bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x}\, p_1 \cdot p_6} \left(\frac{1+\tilde{x}^2}{1-\tilde{x}} \right) \left| \mathcal{M}_{q\bar{q}}^{B}(\tilde{p}_{16}, p_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5) \right|^2$$

$$\tilde{x} = \frac{p_1 \cdot p_2 - p_2 \cdot p_6 - p_1 \cdot p_6}{p_1 \cdot p_2}$$

$$\tilde{p}_{16} = \tilde{x}\, p_1 \quad , \quad K = p_1 + p_2 - p_6 \quad , \quad \tilde{K} = \tilde{p}_{16} + p_2$$

$$\Lambda^{\mu\nu} = g^{\mu\nu} - \frac{2(K^\mu + \tilde{K}^\mu)(K^\nu + \tilde{K}^\nu)}{(K+\tilde{K})^2} + \frac{2\tilde{K}^\mu K^\nu}{K^2}$$

$$\tilde{p}_j = \Lambda\, p_j$$

$pp \to VVV$ REAL CORRECTIONS

$$d\sigma^R_{q\bar{q}} - d\sigma^A_{q\bar{q}} = \frac{C_S}{N}\frac{1}{2s_{12}}\left[C_F\,|\mathcal{M}^R_{q\bar{q}}(\{p_j\}')|^2 - \mathcal{D}^{q_1 g_6,\bar{q}_2} - \mathcal{D}^{\bar{q}_2 g_6,q_1}\right]d\Phi_{VVVg}$$

$$d\sigma^R_{q\bar{q}} - d\sigma^A_{q\bar{q}} = \frac{C_S}{N}\frac{1}{2s_{12}}\left[C_F\,|\mathcal{M}^R_{q\bar{q}}(\{p_j\}')|^2 - \mathcal{D}^{q_1 g_6,\bar{q}_2} - \mathcal{D}^{\bar{q}_2 g_6, q_1}\right] d\Phi_{VVVg}$$

$$d\sigma^C_{q\bar{q}} + \int_g d\sigma^A_{q\bar{q}} = \frac{\alpha_s C_F}{2\pi}\frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}-\frac{2\pi^2}{3}\right] d\sigma^B_{q\bar{q}}$$

$$+\frac{\alpha_s C_F}{2\pi}\int_0^1 dx\,\mathcal{K}^{q,q}(x)\,d\sigma^B_{q\bar{q}}(xp_1,p_2) + \frac{\alpha_s C_F}{2\pi}\int_0^1 dx\,\mathcal{K}^{\bar{q},\bar{q}}(x)\,d\sigma^B_{q\bar{q}}(p_1,xp_2)$$

$$\mathcal{K}^{q,q}(x) = \mathcal{K}^{\bar{q},\bar{q}}(x) = \left(\frac{1+x^2}{1-x}\right)_+ \log\left(\frac{s_{12}}{\mu_F^2}\right) + \left(\frac{4\log(1-x)}{1-x}\right)_+ + (1-x) - 2(1+x)\log(1-x)$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int\limits_{VVV} \left[\int\limits_{q} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int\limits_{VVVq} \left[d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int_{VVV} \left[\int_{q} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVq} \left[d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]$$

$$d\sigma_{gq}^{R} - d\sigma_{gq}^{A} = \frac{C_S}{N} \frac{1}{2s_{12}} \left[T_R |\mathcal{M}_{gq}^{R}|^2 - \mathcal{D}^{g_1 q_6, q_2} \right] d\Phi_{VVVq}$$

$pp \longrightarrow VVV$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int_{VVV} \left[\int_q d\sigma_{gq}^A + d\sigma_{gq}^C \right] + \int_{VVVq} \left[d\sigma_{gq}^R - d\sigma_{gq}^A \right]$$

$$d\sigma_{gq}^R - d\sigma_{gq}^A = \frac{C_S}{N} \frac{1}{2s_{12}} \left[T_R |\mathcal{M}_{gq}^R|^2 - \mathcal{D}^{g_1 q_6, q_2} \right] d\Phi_{VVVq}$$

$$\mathcal{D}^{g_1 q_6, q_2} = \frac{8\pi\alpha_s \, T_R}{\tilde{x} \, 2 \, p_1 \cdot p_6} [1 - 2\,\tilde{x}\,(1 - \tilde{x})] |\mathcal{M}_{q\bar{q}}^B(\tilde{p}_j)|^2$$

$pp \to VVV$ REAL CORRECTIONS

$$d\sigma^{C}_{gq} + \int_{q} d\sigma^{A}_{gq} = \frac{\alpha_s T_R}{2\pi} \int_{0}^{1} dx\, \mathcal{K}^{g,q}(x)\, d\sigma^{B}_{q\bar{q}}(xp_1, p_2)$$

$$\mathcal{K}^{g,q}(x) = [x^2 + (1-x)^2] \log\left(\frac{s_{12}}{\mu_F^2}\right) + 2x(1-x) + 2[x^2 + (1-x)^2] \log(1-x)$$

$pp \longrightarrow VVV$ REAL CORRECTIONS

$$d\sigma^C_{gq} + \int_q d\sigma^A_{gq} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx\, \mathcal{K}^{g,q}(x)\, d\sigma^B_{q\bar{q}}(xp_1, p_2)$$

$$\mathcal{K}^{g,q}(x) = [x^2 + (1-x)^2] \log\left(\frac{s_{12}}{\mu_F^2}\right) + 2x(1-x) + 2[x^2 + (1-x)^2] \log(1-x)$$

$$d\sigma(P_1, P_2) = \sum_{ab} \int dz_1 dz_2 f_a(z_1, \mu_F) f_b(z_2, \mu_F) d\sigma_{ab}(z_1 P_1, z_2 P_2)$$

$q\bar{q}$, $\bar{q}q$, gq, qg, $g\bar{q}$, $\bar{q}g$

$$d\sigma^C_{gq} + \int_q d\sigma^A_{gq} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx\, \mathcal{K}^{g,q}(x)\, d\sigma^B_{q\bar{q}}(xp_1, p_2)$$

$$\mathcal{K}^{g,q}(x) = [x^2 + (1-x)^2] \log\left(\frac{s_{12}}{\mu_F^2}\right) + 2x(1-x) + 2[x^2 + (1-x)^2] \log(1-x)$$

$$d\sigma(P_1, P_2) = \sum_{ab} \int dz_1 dz_2 f_a(z_1, \mu_F) f_b(z_2, \mu_F) d\sigma_{ab}(z_1 P_1, z_2 P_2)$$

$q\bar{q}$, $\bar{q}q$, gq, qg, $g\bar{q}$, $\bar{q}g$

- check also with phase-space slicing method

$pp \to VVV$ NLO

- Virtual contributions obtained with `Cuttools`
- $O(100ms)$ per "event" $\to$ factor $O(10-10^2)$

$pp \to VVV$ NLO

- Virtual contributions obtained with `Cuttools`
- $O(100ms)$ per "event" $\to$ factor $O(10 - 10^2)$

- Real contributions obtained with `Helac`
- Positive/negative (un)weighted events

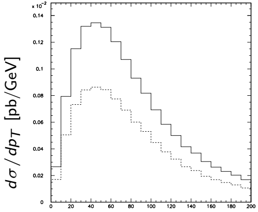
$pp \to VVV$ NLO

- Virtual contributions obtained with `Cuttools`
- $O(100ms)$ per "event" $\to$ factor $O(10 - 10^2)$

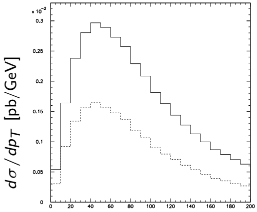
- Real contributions obtained with `Helac`
- Positive/negative (un)weighted events

Process	scale μ	Born cross section [fb]	NLO cross section [fb]
ZZZ	$3M_Z$	9.7(1)	15.3(1)
WZZ	$2M_Z + M_W$	20.2(1)	40.4(2)
WWZ	$M_Z + 2M_W$	96.8(6)	181.7(8)
WWW	$3M_W$	82.5(5)	146.2(6)

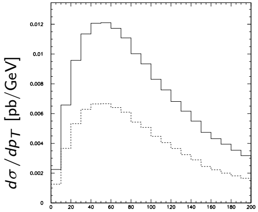
$pp \to VVV$ NLO



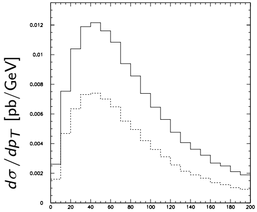
ZZZ



W^+ZZ

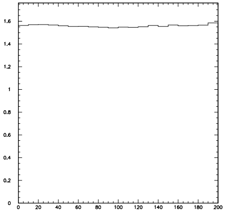


W^+W^-Z

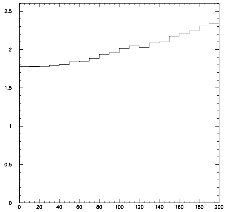


$W^+W^-W^+$

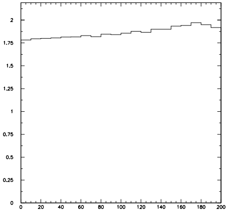
$pp \to VVV$ NLO



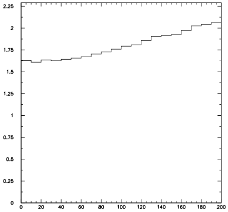
ZZZ



W^+ZZ

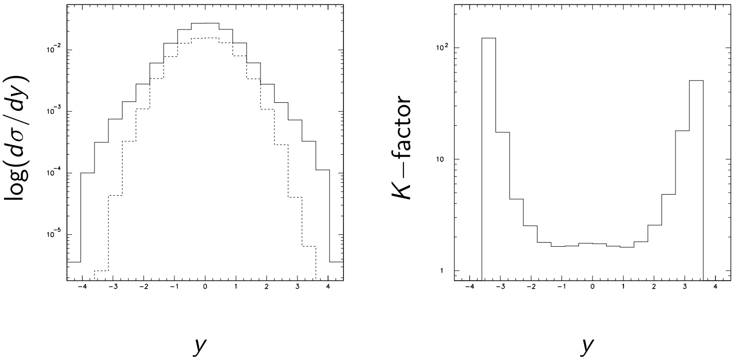


W^+W^-Z



$W^+W^-W^+$

$pp \to VVV$ NLO



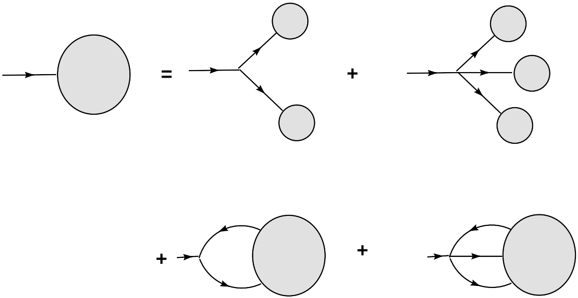
$\log(d\sigma/dy)$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
−4 −3 −2 −1 0 1 2 3 4
y
K−factor
10^{2}
10
1
−4 −3 −2 −1 0 1 2 3 4
y

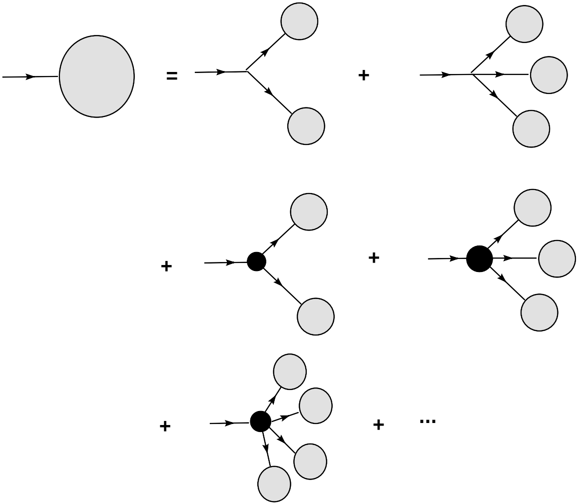
$pp \to VVV$ NLO

scale	σ_B	σ_{NLO}	K
$\mu = M/2$	82.7(5)	153.2(6)	1.85
$\mu = M$	81.4(5)	144.5(6)	1.77
$\mu = 2M$	81.8(5)	139.1(6)	1.70

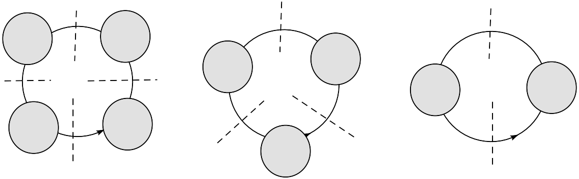
scale	σ_B	σ_{NLO}	K
$\mu = M/2$	20.2(1)	43.0(2)	2.12
$\mu = M$	20.0(1)	39.7(2)	1.99
$\mu = 2M$	19.7(1)	37.8(2)	1.91

Amplitude calculation-I





Amplitude calculation-III



Outlook

Reduction at the integrand level

Outlook

Reduction at the integrand level

- changes the computational approach at one loop

Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Current

Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Current

- Understand potential stability problems

Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Current

- Understand potential stability problems
- Combine with the real corrections

Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Current

- Understand potential stability problems
- Combine with the real corrections

Future

Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Current

- Understand potential stability problems
- Combine with the real corrections

Future

- Automatize through Dyson-Schwinger equations

Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Current

- Understand potential stability problems
- Combine with the real corrections

Future

- Automatize through Dyson-Schwinger equations

A generic NLO calculator seems feasible