On the Cuts of Scattering Amplitudes

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@: `arXiv:0710.5559 [hep-ph], arXiv:0803.1989 [hep-ph], arXiv:0803.3964 [hep-ph]`

Loops & Legs 2008

Outline

- Generalised Unitarity *vs* Complex Analysis

- Analytical Tool: Integration with Spinor-Variables
  - Contour Integrals of Rational Functions

- Semi-Numerical Tool: Optimizing the Integral-Reduction
  - Discrete Fourier Transform & Polynomials's coefficients

One Loop Amplitudes

- Reduction in D-shifted Basis

Passarino-Veltman; Tarasov;
Bern, Chalmers, Dixon, Dunbar, Kosower, Morgan;
Binoth, Guillet, Heinrich;
Giele, Kunszt, Melnikov

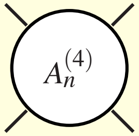
$$A_n^{(D)} = \sum_{r=0} g_{(r)} I_n^{(D+2r)} + \sum_{r=0} f_{(r)} I_{n-1}^{(D+2r)} + \ldots + \sum_{r=0} e_{(r)} I_5^{(D+2r)} + \sum_{r=0} d_{(r)} I_4^{(D+2r)} + \sum_{r=0} c_{(r)} I_3^{(D+2r)} + \sum_{r=0} b_{(r)} I_2^{(D+2r)} + \sum_{r=0} a_{(r)} I_1^{(D+2r)}$$

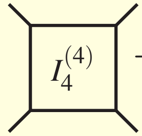
$$\stackrel{\varepsilon \to 0}{=} \text{PolyLogarithms} + \text{Rational}$$

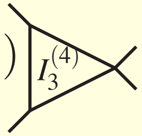
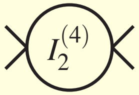
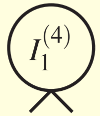
- $a, b, c, d, e, \ldots, f, g$ are the unknowns: they are known to be rational functions of kinematic invariants, and D-independent in this basis.

- Loop Splitting: $L_{(D)} = \ell_{(4)} + \mu_{(-2\epsilon)} \qquad \Rightarrow \qquad \int d^{4-2\epsilon} L = \int d^{-2\epsilon}\mu \int d^4 \ell$

- 4-dim Kernel



$= \Big(g(\mu^2)\, \pi_n(\mu^2) + f(\mu^2)\, \pi_{n-1}(\mu^2) + \ldots + e(\mu^2)\, \pi_5(\mu^2) + d(\mu^2) \Big)$  $+$

$+\, c(\mu^2)$  $+\, b(\mu^2)$  $+\, a(\mu^2)$ 

where:

▷ $g(\mu^2), \ldots, e(\mu^2), d(\mu^2), \ldots, a(\mu^2)$ are polynomials of in μ^2, ex: $\qquad d(\mu^2) = \sum_r d_{(r)}\, (\mu^2)^r$

▷ $\pi_n(\mu^2) =$ coefficient of the reduction of the I_n, $(n > 4)$ to I_4 at $D = 4$ (non-polynomial).

- The polynomial structure of $g(\mu^2), \ldots, d(\mu^2), c(\mu^2)$, and $b(\mu^2)$ is responsible for the *D*-shifted integrals:

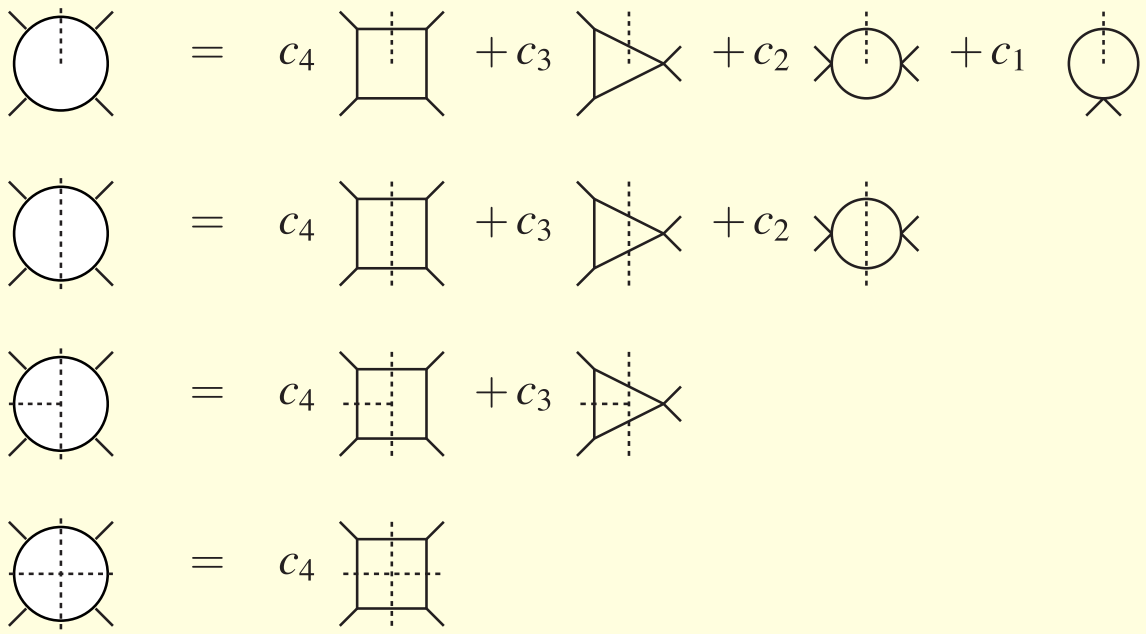
$$\int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} (\mu^2)^r f(\mu^2) \quad = \quad -\epsilon(1-\epsilon)(2-\epsilon)\cdots(r-1-\epsilon)(4\pi)^r \int \frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} f(\mu^2)$$

▷ The reconstruction of the 4-dim kernel of any one-loop amplitude contains all the information for the complete reconstruction of the amplitude in *D*-dimensions.

Generalised Unitarity

- coefficients show up entangled in a given cut: how do we disentangle them?

The polylogarithmic structure of the 4-D master integrals is different. Therefore their multiple cuts have specific signature which enable us to distinguish unequivocally among them.



On-Shell Complex Momenta enable the *fulfillment* of the cut-constraints!

Analytical Implementation of Cut-Conditions

Double-Cut Phase Space Measure

- 4-dim LIPS Cacahazo, Svrček & Witten

$$\int d^4\Phi = \int d^4\ell_0 \, \delta^{(+)}(\ell_0^2) \, \delta^{(+)}((\ell_0 - K)^2) = \int \frac{\langle \ell \, d\ell \rangle [\ell \, d\ell]}{\langle \ell | K | \ell]} \int t \, dt \, \delta^{(+)}\left(t - \frac{K^2}{\langle \ell | K | \ell]}\right)$$

$$\Leftarrow \qquad \ell_0^2 = 0 \, , \quad \ell_0 = |\ell_0\rangle [\ell_0| \equiv t |\ell\rangle [\ell|$$

- D-dim LIPS Anastasiou, Britto, Feng, Kunszt, & P.M.; Britto, Feng; Britto, Feng, & P.M.

$$\int d^{4-2\varepsilon}\Phi \quad = \quad \Omega(\varepsilon) \int d\mu^{-2\varepsilon} \int d^4\Phi(\mu^2) \, ,$$

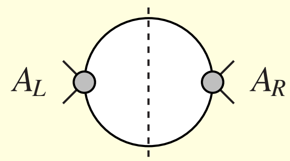
$$\begin{aligned}
\int d^4\Phi(\mu^2) \quad &= \quad \int d^4L \, \delta^{(+)}(L^2 - M_1^2 - \mu^2) \, \delta^{(+)}((L-K)^2 - M_2^2 - \mu^2) \\
&= \quad \int \frac{\langle \ell \, d\ell \rangle [\ell \, d\ell]}{\langle \ell | K | \ell]} \int t \, dt \, \delta^{(+)}\left(t - \frac{(1 - 2z_0) K^2}{\langle \ell | K | \ell]}\right)
\end{aligned}$$

$$\Leftarrow \qquad L = \ell_0 + z_0 K \, , \qquad \text{with } \ell_0^2 = 0 \, , \qquad \ell_0 \equiv t |\ell\rangle [\ell| \qquad z_0 = \frac{K^2 + M_1^2 - M_2^2 - \sqrt{\lambda[K^2, M_1^2, M_2^2] - 4\mu^2}}{2K^2} \, ,$$

Double-Cut ⊕ Spinor-Integration

Britto, Buchbinder, Cachazo & Feng; Britto, Feng & P.M.

Anastasiou, Britto, Feng, Kunszt & P.M.



$$M = \Omega(\varepsilon) \int d\mu^{-2\varepsilon} \, \Delta \, , \qquad \Delta = \int d^4\Phi(\mu^2) \; A_L^{\text{tree}}(t, |\ell\rangle, |\ell]) \; \times \; A_R^{\text{tree}}(t, |\ell\rangle, |\ell])$$

- trivial t-integration ⊕ Schouten identity: factorizing the dependence on $|\ell\rangle$ and $|\ell]$

$$\Delta = \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \; G\Big(|\ell\rangle\Big) \frac{[\eta \, \ell]^n}{\langle \ell | P_1 | \ell]^{n+1} \langle \ell | P_2 | \ell]}$$

- Feynman Parametrization:

$$\Delta = \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \; (n+1) \int dx \, (1-x)^n \; G\Big(|\ell\rangle\Big) \frac{[\eta \, \ell]^n}{\langle \ell | R | \ell]^{n+2}} \, , \qquad \not{R} = x \not{P}_1 + (1-x) \not{P}_2$$

The parametric integration is responsible for the logarithmic-term on the double-cut !

(back of a envelop) Spinor-Integration

- Change of Variables:

$$\forall p,q : q^2 = p^2 = 0 \quad \Rightarrow \quad |\ell\rangle \equiv |p\rangle + z|q\rangle \quad \& \quad |\ell] \equiv |p] + \bar{z}|q] \quad \Rightarrow \quad \langle \ell\, d\ell\rangle[\ell\, d\ell] = -\langle q|p|q]\, dz\, d\bar{z}\,,$$

$$\Delta = (n+1)\int dx\,(1-x)^n \langle q|p|q] \int dz\, d\bar{z}\, G(z) \frac{([\eta\, p] + \bar{z}[\eta\, q])^n}{(\langle p|R|p] + z\langle q|R|p] + \bar{z}\langle p|R|q] + z\bar{z}\langle q|R|q])^{n+2}}$$

- Primitive in $\bar{z}$

$$\frac{([\eta\, p] + \bar{z}[\eta\, q])^n}{(\langle p|R|p] + z\langle q|R|p] + \bar{z}\langle p|R|q] + z\bar{z}\langle q|R|q])^{n+2}} = \frac{(\alpha+\beta\bar{z})^n}{(\gamma+\delta\bar{z})^{n+2}} = \frac{d}{d\bar{z}}\Pi(z,\bar{z})\,, \qquad \Pi(z,\bar{z}) = \frac{1}{(n+1)} \frac{(\alpha+\beta\bar{z})^{n+1}}{(\beta\delta-\alpha\gamma)(\gamma+\delta\bar{z})^{n+1}}$$

$$\Delta = (n+1)\int dx\,(1-x)^n \langle q|p|q] \int dz\, d\bar{z}\, \frac{d}{d\bar{z}}\Big(G(z)\Pi(z,\bar{z})\Big)$$

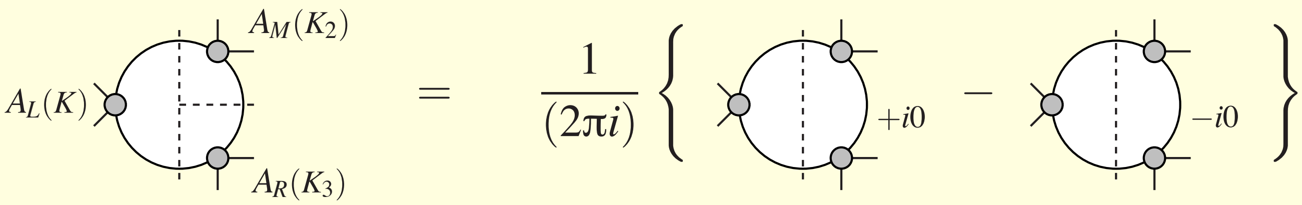
- Cauchy's Residue Theorem in z

$$\int dz\, d\bar{z}\, \frac{d}{d\bar{z}}\Big(G(z)\Pi(z,\bar{z})\Big) = \sum_{\text{res } @z-\text{poles}} G(z)\Pi(z,z^*)$$

where: *(i)* G and Π are rational functions in z and $\bar{z}$; and *(ii)* $z - \text{poles} \in \{G(z), \delta(z)\}$.

Triple-Cut ⊕ Spinor-Integration

P.M. (2007)



The integration over the Feynman parameter is frozen.

▷ *Cuts* in Feynman Parameters

$$\frac{1}{(ax+b)+i0} \quad \to \quad K_1(x) = \frac{1}{a}\delta(x-x_0)$$

$$\frac{1}{(ax^2+bx+c)+i0} \quad \to \quad K_2(x) = \frac{1}{a\,|x_1-x_2|}\Big(\delta(x-x_1)+\delta(x-x_2)\Big)$$

where $x_{0,1,2}$ are the *zeroes* of the corresponding denominators.

- alternatively: $|\ell\rangle = |p\rangle + z|q\rangle \ , \qquad |\ell] = |p] + \bar{z}|q]$

Forde; Britto, Feng;

Bjerrum-Bohr, Dunbar, Ita, Perkins

Polar coordinates: $z \equiv r\,\tau \ \oplus \ \tau \equiv e^{i\theta} \quad \Rightarrow \quad dz d\bar{z} = r dr \times (-i) d\tau/\tau$

▷ r-integration frozen by the $3^{\rm rd}$-cut; ▷ Cauchy's residue in τ.

Coefficients in Closed Form

Britto, Feng (2007)

Britto, Feng & Yang (2008)

Britto, Feng & P.M. (2008)

- Generic Massive Double-cut in the K^2-channel:

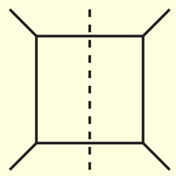
$$\int d^4L\, \delta^{(+)}\left(L^2 - M_1^2 - \mu^2\right)\, \delta^{(+)}\left((L-K)^2 - M_2^2 - \mu^2\right) \frac{\prod_j \langle a_j|L|b_j]}{\prod_i \left((L-K_i)^2 - m_i^2 - \mu^2\right)}$$

$$= \int \langle \ell\, d\ell\rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) \frac{(K^2)^{n+1}}{\langle \ell|K|\ell]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle \ell|R_j|\ell]}{\prod_{i=1}^{k} \langle \ell|Q_i|\ell]}$$

where

$$\begin{aligned} P_j &= |a_j\rangle [b_j| \\ R_j &= -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) P_j + \frac{-z(2P_j \cdot K)}{K^2} K, \\ Q_j &= -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) K_i + \frac{K_i^2 + M_1^2 - m_i^2 - 2zK \cdot K_i}{K^2} K \end{aligned}$$

• I_4-coefficient



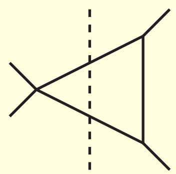
$$C[Q_r, Q_s, K] = \frac{(K^2)^{2+n}}{2} \left(\frac{\prod_{j=1}^{k+n} \langle P_{sr,1} | R_j | P_{sr,2}]}{\langle P_{sr,1} | K | P_{sr,2}]^{n+2} \prod_{t=1, t \neq i,j}^{k} \langle P_{sr,1} | Q_t | P_{sr,2}]} + \{ P_{sr,1} \leftrightarrow P_{sr,2} \} \right).$$

$$\Delta_{sr} = (2Q_s \cdot Q_r)^2 - 4Q_s^2 Q_r^2$$

$$P_{sr,1} = Q_s + \left(\frac{-2Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r$$

$$P_{sr,2} = Q_s + \left(\frac{-2Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r$$

• I_3-coefficient



$$C[Q_s,K] \quad = \quad \frac{(K^2)^{1+n}}{2}\frac{1}{(\sqrt{\Delta_s})^{n+1}}\frac{1}{(n+1)!\langle P_{s,1}\,P_{s,2}\rangle^{n+1}}\,\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{\prod_{j=1}^{k+n}\langle\gamma_{s,1}|R_j\,Q_s|\gamma_{s,1}\rangle}{\prod_{t=1,t\neq s}^{k}\langle\gamma_{s,1}|Q_t\,Q_s|\gamma_{s,1}\rangle}+\{P_{s,1}\leftrightarrow P_{s,2}\}\right)\Bigg|_{\tau=0}$$

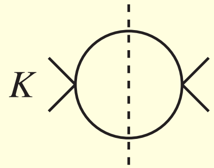
$$|\gamma_{s,1}\rangle = |P_{s,1}\rangle - \tau|P_{s,2}\rangle$$

$$\Delta_s = (2Q_s\cdot K)^2 - 4Q_s^2K^2$$

$$P_{s,1} = Q_s + \left(\frac{-2Q_s\cdot K+\sqrt{\Delta_s}}{2K^2}\right)K$$

$$P_{s,2} = Q_s + \left(\frac{-2Q_s\cdot K-\sqrt{\Delta_s}}{2K^2}\right)K$$

• I_2-coefficient



$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{1}{q!} \frac{d^q}{ds^q} \left(B^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(B^{(r;a-q;1)}_{n,n-a}(s) - B^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0} ,$$

$$B^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n! [\eta|\phi\, K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \gamma_0 | R_j\ (K - s\eta) | \gamma_0 \rangle}{\langle \gamma_0\, \eta \rangle^{n+1} \prod_{p=1}^{k} \langle \gamma_0 | Q_p\ (K - s\eta) | \gamma_0 \rangle} \right) \Bigg|_{\tau=0} ,$$

$$B^{(r;b;1)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\langle \gamma_{r,1} | \eta | P_{r,1}]^{t+1}}{\langle \gamma_{r,1} | K | P_{r,1}]^{t+1}} \frac{\langle \gamma_{r,1} | Q_r\ \eta | \gamma_{r,1} \rangle^b \prod_{j=1}^{n+k} \langle \gamma_{r,1} | R_j\ (K - s\eta) | \gamma_{r,1} \rangle}{\langle \gamma_{r,1} | \eta\ K | \gamma_{r,1} \rangle^{n+1} \prod_{p=1, p \neq r}^{k} \langle \gamma_{r,1} | Q_p\ (K - s\eta) | \gamma_{r,1} \rangle} \right) \Bigg|_{\tau=0} ,$$

$$B^{(r;b;2)}_{n,t}(s) \equiv (-1)^{2b+1} B^{(r;b;1)}_{n,t}(s) \ \Bigg|_{P_{r,1} \leftrightarrow P_{r,2}}$$

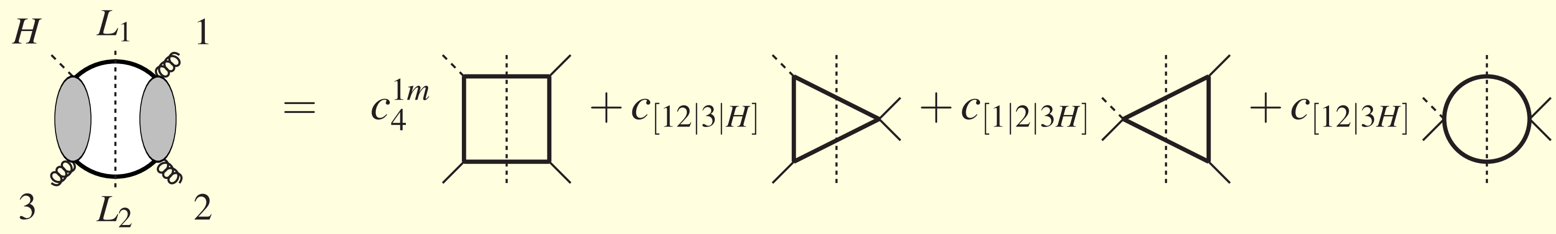
η and ϕ, massless reference momenta

$$|\gamma_0\rangle = (\not{K} - \tau\phi)|\eta] , \qquad |\gamma_{r,1}\rangle = |P_{r,1}\rangle - \tau |P_{r,2}\rangle , \qquad |\gamma_{r,2}\rangle = |P_{r,2}\rangle - \tau |P_{r,1}\rangle ,$$

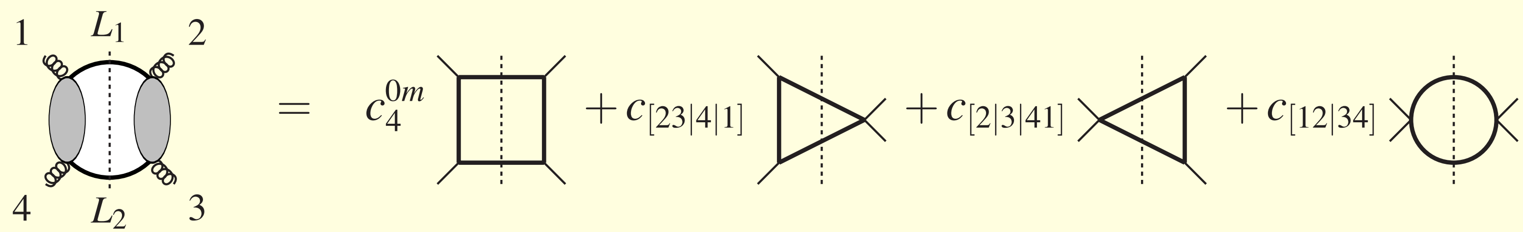
Integral Reduction by Pattern-matching

- Successfull Tests

Rozowsky (1997)



Bern-Morgan (1994)



- Implemented in S@M (Spinors @ MATHEMATICA) Maître & P.M. (2007)

Semi-Numerical Implementation of Cut-Conditions

OPP Integral-Reduction (in a nutshell)

Ossola, Papadopoulos, Pittau

Ellis, Giele, Kunszt

Giele, Kunszt, Melnikov

- OPP-decomposition

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

Top-Down Polynomial Structures

- **Quadruple-cut.** When q is solution of $D_0 = D_1 = D_2 = D_3 = 0$

$$R(q) \quad \equiv \quad [d(0123) + \tilde{d}(q;0123)]$$

is a polynomial with 2 terms.

- **Triple-cut.** When q is solution of $D_0 = D_1 = D_2 = 0 \quad \text{and} \quad D_i \neq 0 \ \ \forall i \neq 0,1,2$

$$R'(q) \quad \equiv \quad [c(012) + \tilde{c}(q;012)]$$

is a polynomial with 7 terms.

- **Double-cut.** When q is solution of $D_0 = D_1 = 0 \quad \text{and} \quad D_i \neq 0 \ \ \forall i \neq 0,1$

$$R''(q) \quad \equiv \quad [b(01) + \tilde{b}(q;01)]$$

is a polynomial with 9 terms.

- **Single-cut.** when q is solution of $D_0 = 0 \quad \text{and} \quad D_i \neq 0 \ \ \forall i \neq 0\,,$

$$R'''(q) \quad \equiv \quad [a(0) + \tilde{a}(q;0)]$$

is a polynomial with 5 terms (only $a(0)$ is relevant).

`Pb:` How extracting their coefficients efficiently?

OPPtiMizing the Integral Reduction

Ossola, Papadopoulos, Pittau, & P.M.

- Discrete Fourier Transform

Hp: F, known only numerically in N points, F_n $(n=0,...,N-1)$. Each of this values admits a DFT, defined as,

$$F_n \equiv \sum_{k=0}^{N-1} f_k \, e^{-2\pi i \frac{k}{N} \, n} \, .$$

Th: by using the orthogonality relation,

$$\sum_{n=0}^{N-1} e^{2\pi i \frac{k}{N} \, n} \, e^{-2\pi i \frac{k'}{N} \, n} = N \, \delta_{kk'} \, .$$

obtain

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n \, e^{2\pi i \frac{k}{N} \, n} \, .$$

- Polynomial coefficients by Projections

Britto, Feng & P.M. (2008)

Ossola, Papadopoulos, Pittau, & P.M. (2008)

BlackHat-collab'n (2008)

$$P_n(x) = \sum_{\ell=0}^{n} c_\ell \, x^\ell \, ,$$

1. Generate the set of discrete values $P_{n,k}$ $(k = 0, \ldots, n)$,

$$P_{n,k} \equiv P_n(x_k) = \sum_{\ell=0}^{n} c_\ell \, \rho^\ell \, e^{-2\pi i \frac{k}{(n+1)} \ell} \, ,$$

   by sampling $P_n(x)$ at the points (equidistant on the ρ-circle)

$$x_k = \rho \, e^{-2\pi i \frac{k}{(n+1)}} \, .$$

2. Using the orthogonality relations for the wave planes, one can obtain the coefficient c_ℓ simply by,

$$c_\ell = \frac{\rho^{-\ell}}{n+1} \sum_{k=0}^{n} P_{n,k} \, e^{2\pi i \frac{k}{(n+1)} \ell}$$

- **3ple-cut** $q = x_1 \ell_1 + x_2 \ell_2 + x_3 \ell_3 + x_4 \ell_4 \,, \qquad x_3\, x_4 = C$

$$R'(q) = P(x_3, x_4) \quad = \quad \underbrace{c_{00} + c_{10}\, x_3 + c_{20}\, x_3^2 + c_{30}\, x_3^3 + c_{01}\, x_3 + c_{02}\, x_4^2 + c_{03}\, x_4^3}_{7 \text{ coefficients}}$$

▷ DFT:

$$P_k^{(1)} = P(C\, e^{-2\pi i\, k/4},\, e^{+2\pi i\, k/4}) \quad (k = 0,1,2,3)$$
$$P_k^{(2)} = P(e^{-2\pi i\, k/3},\, C\, e^{+2\pi i\, k/3}) \quad (k = 0,1,2)$$

- **2ple-cut** $q = y k_1 + y_v v + y_7 \ell_7 + x_8 \ell_8 \,, \qquad y_7\, y_8 = F_y$

$$R''(q) = P(y, y_7, y_8) \quad = \quad \underbrace{b_{000} + b_{100}\, y + b_{200}\, y^2 + b_{010}\, y_7 + b_{020}\, y_7^2 + b_{001}\, y_8 + b_{002}\, y_8^2 + b_{110}\, y\, y_7 + b_{101}\, y\, y_8}_{9 \text{ coefficients}}$$

▷ DFT:

$$P_k^{(1)} = P(0, F_0\, e^{-2\pi i\, k/3},\, e^{+2\pi i\, k/3}) \quad (k = 0,1,2)$$
$$P_k^{(2)} = P(0, e^{-2\pi i\, k/2},\, F_0\, e^{+2\pi i\, k/2}) \quad (k = 0,1)$$
$$P_k^{(3)} = P(-1, F_{-1}\, e^{-2\pi i\, k/2},\, e^{+2\pi i\, k/2}) \quad (k = 0,1)$$
$$P_k^{(4)} = P(-1, e^{-2\pi i\, k/1},\, F_{-1}\, e^{+2\pi i\, k/1}) \quad (k = 0)$$
$$P_k^{(5)} = P(1, F_1\, e^{-2\pi i\, k/1},\, e^{+2\pi i\, k/1}) \quad (k = 0)$$

in conclusion

- Generalised Unitarity $\Leftrightarrow$ Complex Analysis
- Analytical Tool: Integration with Spinor-Variables
  - Contour Integrals of Rational Functions $\sim$ Integrals by *partial fractioning*
    - + No standard PV-Tensor reduction
    - + No need for “spurious” terms/poles subtraction
    - + No assumption on the integrand
    - + Closed Forms for Generating Masters’ coefficients for Massive Amplitudes
    - – Tadpoles’ are not (yet) detected
- Semi-Numerical Tool: Optimizing the Integral-Reduction
  - Discrete Fourier Transform & Polynomials on the Unit-Circle
    - + No system-inversion
    - + High-flexibility for cut-solutions
    - + Stability and reduction of Computing-Time
    - + Wide applicability to problems involving polynomial structures