Automating One-Loop Amplitudes

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for the BlackHat collaboration

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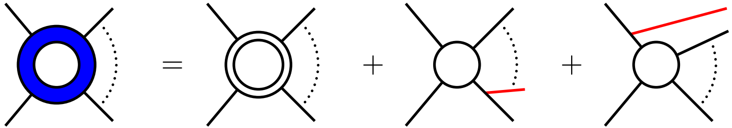
NLO corrections

NLO corrections are very important

- NLO corrections are large in particular in QCD
- NLO corrections affect the shape of distributions
- New production channels may open at NLO, affecting both cross sections and distributions
- Reduce the scale uncertainty of tree level cross sections
- The signals can closely resemble the background
- A precise understanding of the background is then mandatory
- Precise studies at the LHC will require NLO corrections for many high multiplicity QCD background processes

Ingredients of NLO computation

The NLO correction has two parts



- Real radiation part
  - Challenge is systematic extraction of the singularities
  - Different methods
    - Subtraction method (dipole, antenna)
    - Sector decomposition
    - Phase space slicing
  - Automation possible [Gleisberg,Krauss;Seymour,Tevlin]
- Virtual part
  - Current bottleneck for the automation of the NLO corrections

Feynman diagrams

- Traditional approach using Feynman diagram has a rapid growth in complexity with the number of external states
- Very large number of terms makes numerical evaluation very challenging
- $2 \to 3, 4$ processes at the LHC
  - $pp \to WW + j$ [Campbell,Ellis,Zanderighi;Dittmaier,Kallweit,Uwer]
  - $gg \to gggg$ [Ellis,Giele,Zanderighi]
  - $pp \to H + 2j$ (via gluon fusion) [Campbell,Ellis,Zanderighi]
  - $pp \to H + 2j$ (via weak interactions) [Ciccolini,Denner,Dittmaier]
  - $pp \to W/Zb\bar{b}$ [Febres Cordero, Reina, Wackeroth]
  - $pp \to WWZ$ [Hankele,Zeppenfeld]
  - $pp \to H + t\bar{t}$ [Beenakker, Dittmaier, Krämer, Plümper, Spira, Zerwas; Dawson, Jackson, Reina, Wackeroth]
  - $pp \to H + 3j$ (leading contribution) [Figy,Hankele,Zeppenfeld]

One loop corrections

- Numerical methods based on sector decomposition
  $pp \to ZZZ$ [Lazopoulos,Melnikov,Petriello]
  $pp \to t\bar{t}Z$ [Lazopoulos,McElmurry,Melnikov,Petriello]
- Unitarity method
  - Many new analytic results
  - (Few) numerical results using unitarity techniques:
    $e^+e^- \to Z \to 4partons$ [Bern,Dixon,Kosower,Weinzierl]
    included in MCFM [Campbell,Ellis]
    ZZZ,WZZ,WWZ,ZZZ [Binoth,Ossola,Papadopoulos,Pittau]
  - Tools: CutTools [Ossola,Papadopoulos,Pittau]
  - Numerical implementations [Ellis,Giele,Kunszt,Melnikov; Mastrolia,Ossola,Papadopoulos,Pittau]

The BlackHat Project

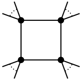
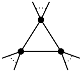
Goal: Automating the computation of one-loop amplitudes using on-shell methods

- C++ code
- To compute one-loop virtual amplitudes we use
  - Unitarity bootstrap
    - generalized unitarity + recursion relations
  - Spinor-helicity formalism
  - Complex momenta

One-Loop Decomposition

A one-loop amplitude can be decomposed into a sum of coefficients multiplying scalar integrals and rational terms.

$$A = R + C$$

$C = \sum_i b_i$  $+ \sum_i c_i$  $+ \sum_i d_i$ 

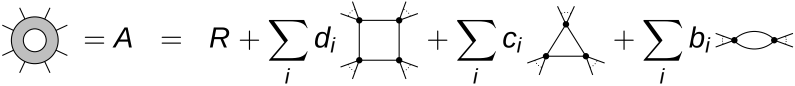
- The task is reduced to determining the coefficients
- The coefficients b_i, c_i, d_i can be computed using generalized unitarity techniques in $d = 4$ dimensions
- The rational part has to be computed separately.
  $\Rightarrow$ use recursion relations

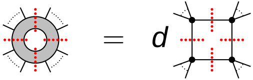
Generalized Unitarity

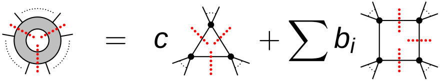
A unitarity cut is the replacement

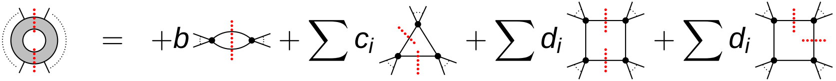
$$\frac{i}{p^2 - m^2 + i\epsilon} \to 2\pi\delta(p^2 - m^2)$$

Cut can be used to isolate integral coefficient

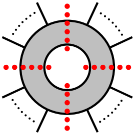








Quadruple cut



$$= \int d^4 l\, \delta(l_1^2)\delta(l_2^2)\delta(l_3^2)\delta(l_4^4)\mathcal{A}$$

$$= A_1^{\text{tree}}(l^\pm)A_2^{\text{tree}}(l^\pm)A_3^{\text{tree}}(l^\pm)A_4^{\text{tree}}(l^\pm)$$

With one massless leg in one corner:

$$l_1^{\pm\mu} = \frac{\langle 1^\mp|\not{K}_2\not{K}_3\not{K}_4\gamma^\mu|1^\pm\rangle}{2\langle 1^\mp|\not{K}_2\not{K}_4|1^\pm\rangle}, \quad l_2^{\pm\mu} = -\frac{\langle 1^\mp|\gamma^\mu\not{K}_2\not{K}_3\not{K}_4|1^\pm\rangle}{2\langle 1^\mp|\not{K}_2\not{K}_4|1^\pm\rangle},$$

$$l_3^{\pm\mu} = \frac{\langle 1^\mp|\not{K}_2\gamma^\mu\not{K}_3\not{K}_4|1^\pm\rangle}{2\langle 1^\mp|\not{K}_2\not{K}_4|1\rangle}, \quad l_4^{\pm\mu} = -\frac{\langle 1^\mp|\not{K}_2\not{K}_3\gamma^\mu\not{K}_4|1^\pm\rangle}{2\langle 1^\mp|\not{K}_2\not{K}_4|1^\pm\rangle}.$$

Gram determinant for $K_1^2 = 0$

$$\Delta_4 = -2\langle 1^-|\not{K}_2\not{K}_4|1^+\rangle\langle 1^+|\not{K}_2\not{K}_4|1^-\rangle$$

Three-particle cut

Momentum parametrization with two massless four vectors
[del Aguila,Ossola,Papadopoulos,Pittau;Forde]

$$l^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{\alpha_3}{2} \langle k_1|\gamma^\mu|k_2] + \frac{\alpha_4}{2} \langle k_2|\gamma^\mu|k_1]$$

The three delta functions fix three of the coefficients α_i

$$l^\mu = \tilde{K}_1^\mu + \tilde{K}_2^\mu + \frac{t}{2} \left\langle \tilde{K}_1|\gamma^\mu|\tilde{K}_2\right] + \frac{1}{2t} \left\langle \tilde{K}_2|\gamma^\mu|\tilde{K}_1\right]$$

Triple cut is a function of t

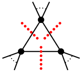
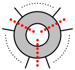
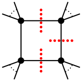
$$C(t) = \frac{c_{-3}}{t^3} + \frac{c_{-2}}{t^2} + \frac{c_{-1}}{t} + c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \sum_{poles} \frac{d_i}{\xi_i (t - t_i)}$$

Poles in t originate from additional propagators going on-shell

$$(l_i - K)^2 \to (t - t_i)\xi_i$$

Three-particle cut

$$T(t) \equiv C(t) - \sum \frac{d_i}{\xi_i(t - t_i)}$$

c  $=$  $- \sum d_i$ 

Subtracted triple cut is a function of t [OPP]

$$T(t) = \frac{c_{-3}}{t^3} + \frac{c_{-2}}{t^2} + \frac{c_{-1}}{t} + c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

$$c_0 = \frac{1}{2p+1} \sum_{j=-p}^{p} T_3(t_0 e^{2\pi i j/(2p+1)})$$

- Discrete Fourier projection avoids numerically unstable matrix invertions.
- Ensures good numerical accuracy

Two-particle cut

Momentum parametrization with two massless four vectors

$$l^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{\alpha_3}{2}\langle k_1|\gamma^\mu|k_2] + \frac{\alpha_4}{2}\langle k_2|\gamma^\mu|k_1]$$

The two delta functions leave two free parameters α_i

$$l^\mu = y\tilde{K}_1^\mu + (1-y)\tilde{\chi}^\mu + \frac{t}{2}\left\langle \tilde{K}_1|\gamma^\mu|\chi\right] + \frac{y(1-y)}{2t}\left\langle \chi|\gamma^\mu|\tilde{K}_1\right]$$

Double cut integrand is a function of t and y

$$\begin{aligned} C_2(y,t) &= A_1(t,y)A_2(t,y) \\ B_2(y,t) &\equiv C_2(y,t) - \sum box - \sum triangle \end{aligned}$$

$$b_0 = \frac{1}{20}\sum_{j=0}^{4}\left[B_2\left(0, t_0\, e^{2\pi i j/5}\right) + 3B_2\left(2/3, t_0\, e^{2\pi i j/5}\right)\right].$$

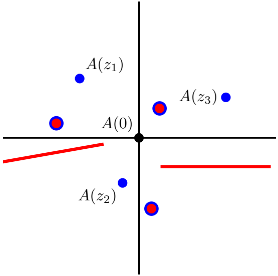
Recursion for Rational Terms

Use the analytic properties of the one-loop amplitude to construct the rational term

- Complex shift $p_1 \rightarrow p_1(z)$, $p_2 \rightarrow p_2(z)$

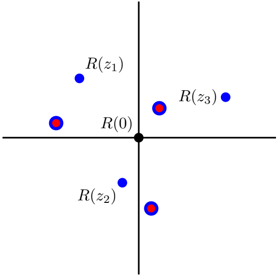
$$A(z) = R(z) + C(z)$$

$$Res_{z_s} \frac{R_S(z_s)}{z_s} = -Res_{z_s} \frac{C(z_s)}{z_s}$$



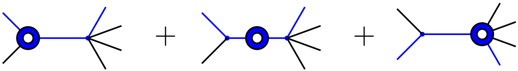
- Spurious poles z_s appear in $C(z)$ and $R(z)$ due to Gram determinants
- The residues of $R(z)/z$ and $C(z)/z$ at the unphysical poles have to cancel since $A(z)$ has no spurious poles.

$$\frac{R(z)}{z}$$



- $R(z)$ factorizes at the physical pole locations, so that we can use recursion relations. [Bern,Dixon,Kosower]

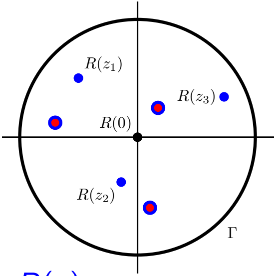
$$\mathrm{Res}_{z_p} \frac{R(z_p)}{z_p} =$$



$$R_D = -\sum_{z_p} \mathrm{Res}_{z_p} \frac{R(z_p)}{z_p}$$

Rational term

$$\frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z} = R_\infty$$



$$\begin{aligned} R(0) &= R_\infty - \sum_{\text{poles}\,\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R(z)}{z} \\ &= R_\infty - \sum_{\text{spur}} \operatorname*{Res}_{z_s} \frac{R(z)}{z} - \sum_{\text{phys}} \operatorname*{Res}_{z_p} \frac{R(z)}{z} \\ &= R_\infty + \sum_{\text{spur}} \operatorname*{Res}_{z_s} \frac{C(z_s)}{z_s} + R_D \end{aligned}$$

The value R_∞ of the contour integral at ∞ can be constructed using an auxiliary recursion.

Numerical extraction of the spurious poles

Numerical spurious extraction is tricky, but possible because

- Precise cut part input
- Location of the spurious poles is known a priori
- Only need to evaluate a small part of $C(z)$ around the pole.
- Only need rational part of the expansion of the integral functions around vanishing Gram determinant

$$I_3^{3\mathrm{m}}(s_1,s_2,s_3) \to \ +\frac{1}{6}\frac{\Delta_3}{s_1 s_2 s_3} - \frac{s_1+s_2+s_3}{120}\left(\frac{\Delta_3}{s_1 s_2 s_3}\right)^2$$

$$-\frac{1}{2}\sum_{i=1}^{3}\ln(-s_i)\frac{s_i - s_{i+1} - s_{i-1}}{s_{i+1}s_{i-1}}\left[1 - \frac{1}{6}\frac{\Delta_3}{s_{i+1}s_{i-1}} + \frac{1}{30}\left(\frac{\Delta_3}{s_{i+1}s_{i-1}}\right)^2\right] + \cdots$$

- We only need the pure rational pieces.

Challenges

- Evaluation time
- Control of numerical precision for exceptional points
  - Gram determinants
  - Large cancellations (between rational- and cut terms)
  - Using increased precision (32 digits, 64 digits) when needed for some pieces
  - Automatic diagnosis

Precision diagnosis

The precision of the computed amplitude can be assessed

- Cut part

$$A_n^{\rm oneloop}|_{1/\epsilon,\,{\rm non-log}} = \frac{1}{\epsilon}\sum_k b_k = -\left[\frac{1}{\epsilon}\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N_c}\right)\right]A_n^{\rm tree}\,,$$

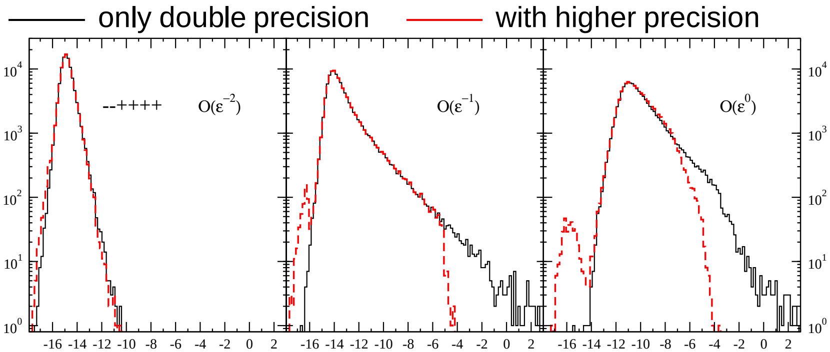
- Spurious poles

$$A_n^{\rm oneloop}(z_s)|_{1/\epsilon,\,{\rm non-log}} = \frac{1}{\epsilon}\sum_k b_k(z_s) = 0\,,$$

- Big cut/rational cancellations

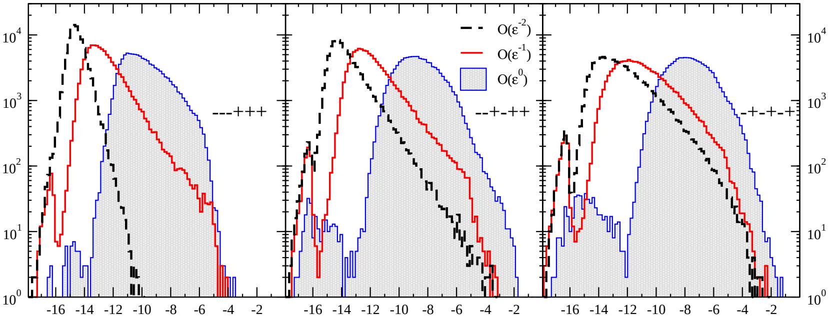
Split MHV results

- Accuracy: $\log_{10}\left(\frac{|A^{\rm num}-A^{\rm ref}|}{|A^{\rm ref}|}\right)$

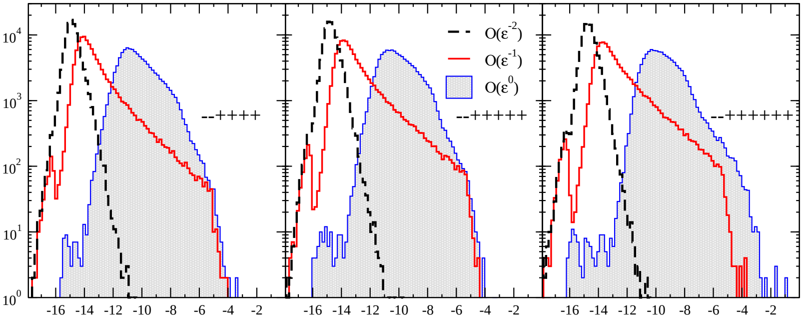


- 100’000 points with cuts
  - $E_T > 0.01\sqrt{s}$
  - Pseudo-rapidity $\eta < 3$
  - $\Delta_R > 0.4$ $\quad \Delta_R = \sqrt{\Delta_\eta^2 + \Delta_\phi^2}$

NMHV amplitudes



MHV amplitudes



Timing

- 2.33 GHz Xeon processor

helicity	cut part only	double prec. only	multi-prec.
$--++++$	2.4 ms	6.8 ms	8.3 ms
$--+++++$	4.2 ms	10.5 ms	14 ms
$--++++++$	6.1 ms	28 ms	43 ms
$-+-+++$	3.1 ms	17.3 ms	24 ms
$-++-++$	3.3 ms	60 ms	76 ms
$---+++$	4.4 ms	12 ms	16 ms
$--+-++$	5.9 ms	42 ms	48 ms
$-+-+-+$	6.9 ms	62 ms	80 ms

- Bottleneck is the spurious pole evaluation
- The effect of higher precision on the evaluation time is noticeable but not dominant

Outlook

First tests passed

- Numerical stability is under control
- Evaluation time is reasonable

But still a long way to go...

- Sums (color orderings+ external helicities)
- Fermions (internal+external)
- Vector bosons (external)
- Masses (internal+external)
- Speed and precision improvements
- Combine with real part