Generalised Unitarity for Massive One Loop Amplitudes

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Loops and Legs 2008

24^{th} April 2008

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Work in Collaboration with David Kosower

Introduction

- NLO QCD corrections are essential ingredients for LHC physics
  - Accurate predictions of QCD backgrounds
    $\Rightarrow$ Observation of deviations from the SM
- Aim: Provide efficient one loop matrix element generator
- Traditional Feynman techniques are extremely inefficient
- On-shell techniques exploit gauge invariance to reduce complexity
- Inclusion of arbitrary masses for one loop processes:
  - Spinor/helicity formalism for massive particles
  - Generalised Unitarity and integral coefficients
  - Application to top production through gluon fusion, $gg \to t\bar{t}$

See talks by:

Britto,Boels,Dunbar,Kosower,Mastrolia,Maitre,Rodrigo

Overview Of On-Shell Techniques

- Recursion relations for tree level processes [Britto,Cachazo,Feng,Witten]
  Massive Particles [SB,Glover,Khoze,Svrček]
- Recursion relations for 1-loop rational terms
  Gluon amplitudes [Berger,Bern,Dixon,Forde,Kosower]
  Higgs amplitudes [Berger,Del Duca,Dixon]
  [SB,Glover,Risager]
- D-dimensional unitarity cuts
  Massive cases, top loops : gggg,Hggg [Anastasiou,Britto,Feng,Kunszt,Mastrolia]
  6-point amplitudes [Ellis,Giele,Kunszt,Melnikov]
- Automated for massless gluon amplitudes
  8-point amplitudes `[BlackHat]`

Spinor-Helicity Basis

- Write all massless momenta and external wavefunctions as 2-component spinors

$$2p^\mu = \langle p|\gamma^\mu|p] \qquad \varepsilon^\mu_\pm(p,\xi) = \pm\frac{\langle p\pm|\gamma^\mu|\xi\pm\rangle}{\sqrt{2}\langle p\mp|\xi\pm\rangle}$$

- Can also apply also to massive momenta: [Kleiss,Stirling]

$$P^\mu = \alpha p^{\flat,\mu} + \beta\eta^\mu, \qquad \alpha\beta = \frac{m_P^2}{\langle\eta|P|\eta]}$$

- Fermion wavefunctions [Rodrigo;Schwinn,Weinzierl]

$$u_\pm(Q,m) = \frac{(Q+m)|\eta\mp\rangle}{\sqrt{\alpha}\langle q^\flat\pm|\eta\mp\rangle} \qquad v_\pm(Q,m) = \frac{(Q-m)|\eta\mp\rangle}{\sqrt{\alpha}\langle q^\flat\pm|\eta\mp\rangle}$$

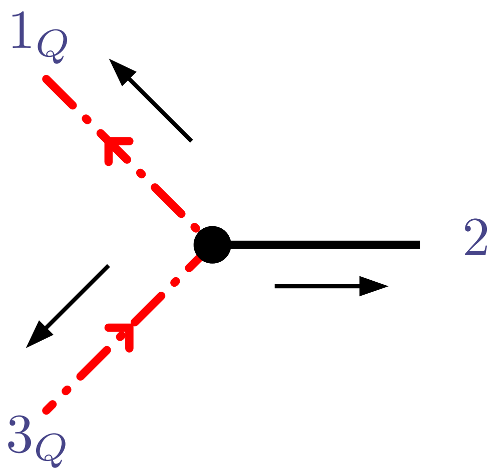
- Set $\alpha = 1$ but keep η free
- Build all amplitudes from on-shell 3-point vertices

Three-Point QQg Vertices

$$A_3(1_Q^+,2^+,3_Q^-) = \frac{\langle \eta_1|1|\eta_3]\langle\xi|3|2]}{\langle\eta_1 1^\flat\rangle[\eta_3 3^\flat]\langle 2\xi\rangle} \qquad A_3(1_Q^+,2^+,3_Q^+) = m\frac{\langle\eta_1\eta_3\rangle\langle\xi|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_3 3^\flat\rangle\langle 2\xi\rangle}$$

$$A_3(1_Q^+,2^-,3_Q^-) = \frac{\langle \eta_3|2|\eta_1]\langle 2|1|\xi]}{\langle\eta_1 1^\flat\rangle\langle\eta_3 3^\flat\rangle[2\xi]} \qquad A_3(1_Q^+,2^-,3_Q^+) = m\frac{\langle\eta_1\eta_3\rangle\langle 2|1|\xi] - \langle\eta_1 2\rangle\langle\eta_3 2\rangle[\xi 2]}{\langle\eta_1 1^\flat\rangle\langle\eta_3 3^\flat\rangle[\xi 2]}$$

- Independent of reference, ξ, for polarisation vector
- Non-trivial dependence on η_i
- Convenient massless limit

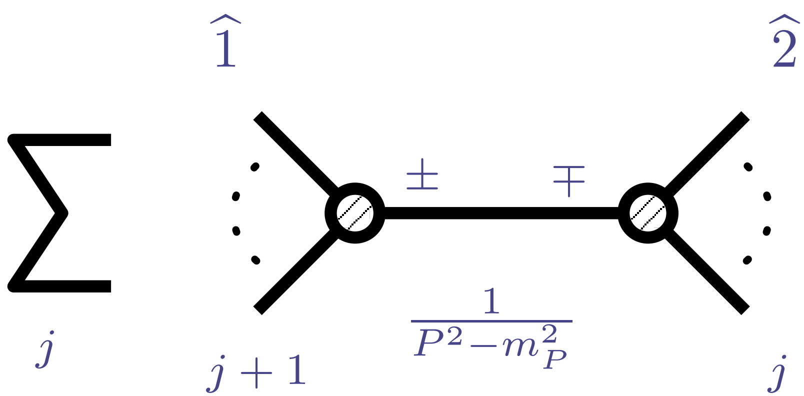


Tree-Level Recursion

[Britto,Cachazo,Feng,Witten]

- Form higher point amplitudes from 3-point amplitudes
- Complex momentum shift

$$\widehat{1} = 1 - z|1\rangle[2| \qquad \widehat{2} = 2 + z|1\rangle[2|$$



- Compact analytic representations

One Loop Amplitudes

- General integral basis:

$$A_n^{(1)} = \sum_k C_4^k I_4^k + \sum_k C_3^k I_3^k + \sum_k C_2^k I_2^k + C_1 I_1 + R_n$$

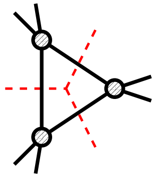
- The basis of functions is known for arbitrary internal and external masses

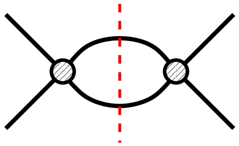
`qcdloop.fnal.gov` [Ellis et al.]

[Denner,Dittmaier,Scharf]

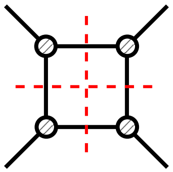
- Reconstruct coefficients from unitarity cuts [Bern,Dixon,Dunbar,Kosower]

$$\frac{i}{p^2 - m^2} \rightarrow (2\pi)\delta(p^2 - m^2)$$





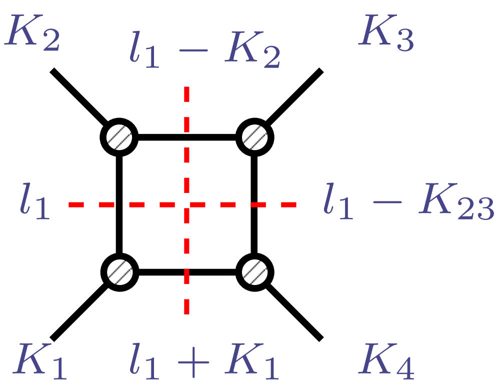
- Cutting four propagators freezes 4-dimensional loop integral



Complex momenta $\Rightarrow$ 3-point vertices non zero on-shell

[Britto,Cachazo,Feng]

Coefficient Extraction From Generalised Cuts



Parametrise the loop momentum to find on-shell conditions [Ossala,Papadopolous,Pittau]

$$l_1 = aK_1^\flat + bK_2^\flat + c|K_1^\flat\rangle[K_2^\flat| + d|K_2^\flat\rangle[K_1^\flat|$$

$$K_1^\flat = K_1 - \frac{S_1}{\gamma}K_2 \qquad K_2^\flat = K_2 - \frac{S_2}{\gamma}K_1$$

$$S_i = K_i^2 \qquad \gamma = K_1 \cdot K_2 \pm \sqrt{(K_1 \cdot K_2)^2 - S_1 S_2}$$

- Solving constraints fixes all four coefficients $\rightarrow$ 2 solutions

$$\{l_1^2 = m_1^2, (l_1 - K_2)^2 = m_2^2, (l_1 - K_{23})^2 = m_3^2, (l_1 + K_1)^2 = m_4^2\}$$

- Massive internal propagators: same loop basis [Kilgore]
- Box coefficient given as sum over both solutions [Britto,Cachazo,Feng]

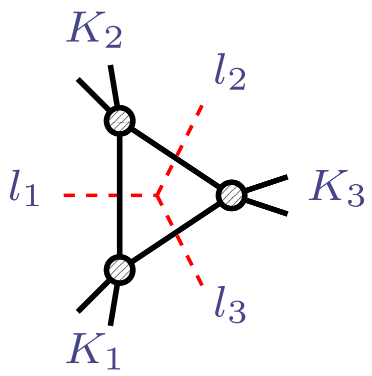
$$C_4 = \frac{i}{2}\sum_{c=c_\pm} A_1 A_2 A_3 A_4(l_1(c))$$

Direct Extraction Of Triangle Coefficients

- Examine analytic behaviour of triple cut

[Ossala,Papadopolous,Pittau]

[Forde]



$$l_1 = aK_1^\flat + bK_2^\flat + t|K_1^\flat\rangle[K_2^\flat| + \frac{d}{t}|K_1^\flat\rangle[K_2^\flat|$$

$$\Rightarrow \int d^4 l \Pi \delta(l_i^2) A_1 A_2 A_3 \rightarrow \int dt J_t A_1 A_2 A_3$$

- Consider t as a complex variable then one can see that

$$\int dt J_t A_1 A_2 A_3 = \int dt J_t \underbrace{\mathrm{Inf}_t}[A_1 A_2 A_3] + \frac{\mathrm{Res}_{t=t_i}(A_1 A_2 A_3)}{(t - t_i)}$$

Laurent series around $t = \infty$
$\mathrm{Inf}_t[X(t)] = x_0 + x_1 t + x_2 t^2 + x_3 t^3$

- Integrals over non-zero powers of t vanish

$$C_3 = -\sum_\gamma \mathrm{Inf}_t[A_1 A_2 A_3]|_{t^0}$$

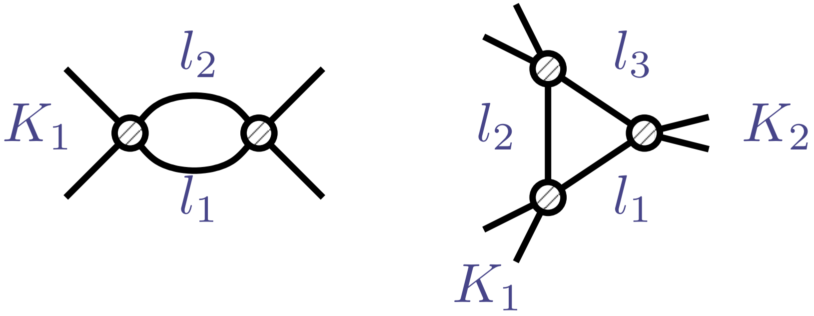
Bubble Coefficients

- Here the two unfixed integrals can be parametrised by,

$$l = yK_1^\flat + a(1-y)K_2^\flat + t|K_1^\flat\rangle[K_2^\flat| + by(1-y)|K_2^\flat\rangle[K_1^\flat|$$

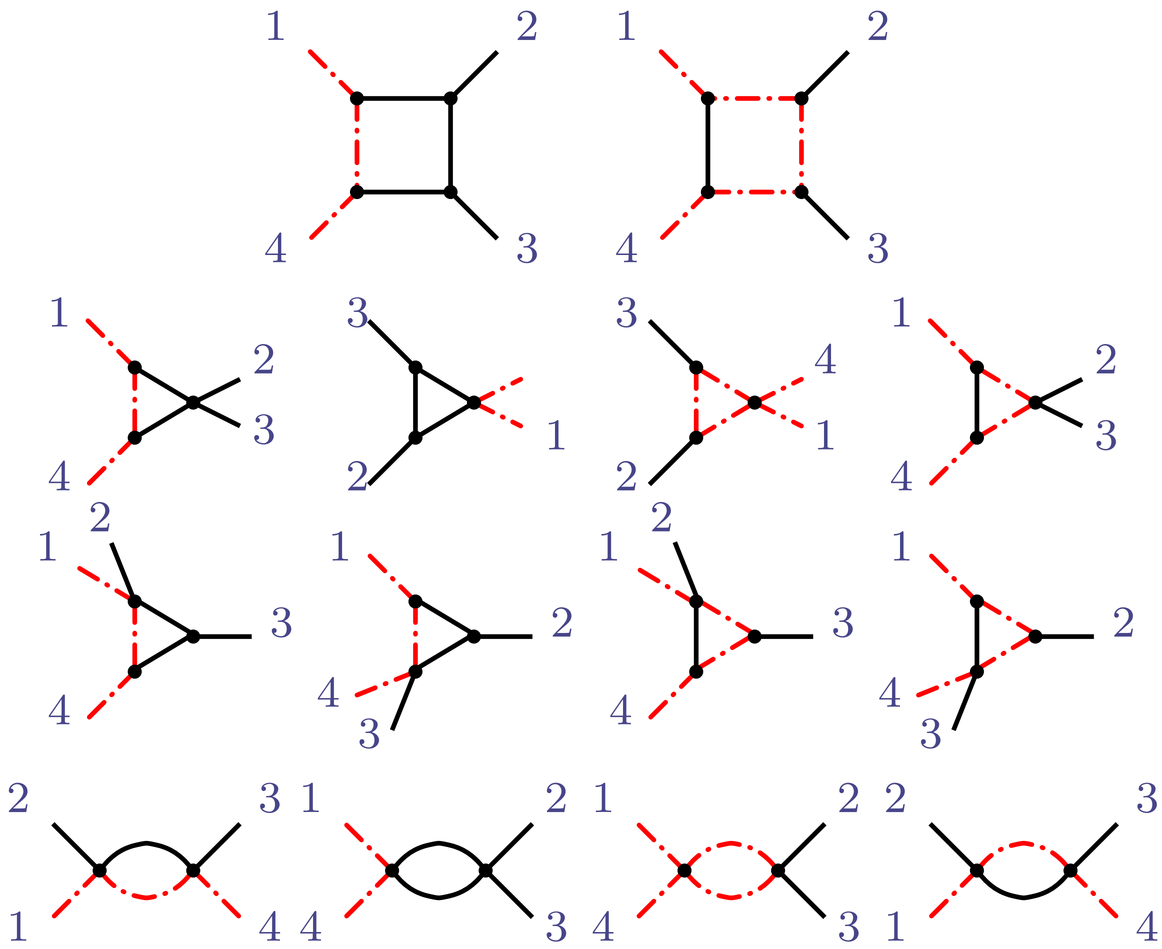
- Decomposition and complex integration similar - non-vanishing integrals
- Integrals over y and t also have mass dependence
- Final Coefficient

$$C_2 = -i\mathrm{Inf}_t[\mathrm{Inf}_y[A_1A_2]]|_{t^0, y^i \to Y_i} - \frac{1}{2}\sum_{\{K_2\}}\sum_{y_\pm}\mathrm{Inf}_t[A_1A_2A_3(K_2)]|_{t^i \to T_i}$$

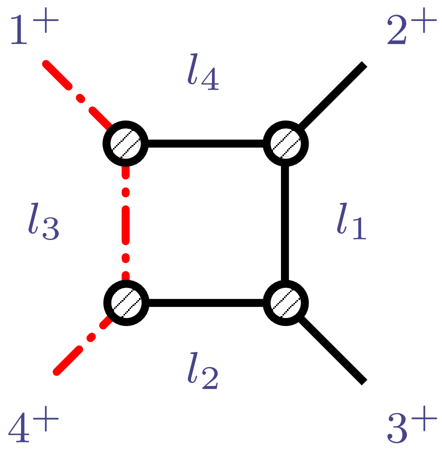


The One Loop $t\bar{t}gg$ Amplitude

$$A_n^{(1)}(1_Q, 2, 3, 4_Q) = N_c A_n^{[L]} + \frac{1}{N_c} A_n^{[R]} + n_f A_n^{[f]} + A_n^{[t]}$$



An Example Coefficient



$$l_1 = c|2\rangle[3| \qquad c = -\frac{(1+2)^2 - m^2}{\langle 2|1|3]}$$

$$l_1' = d|3\rangle[2| \qquad d = -\frac{(1+2)^2 - m^2}{\langle 3|1|2]}$$

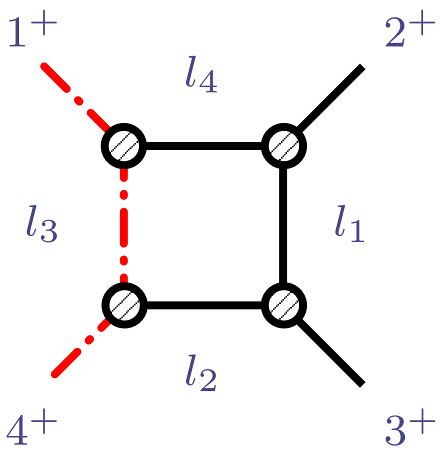
- Sum over both internal helicity configurations
- Use $\tilde{\eta} = \eta_{1,4}$ reduce to 2 contributions

$$C_4^{000m}(1_Q^+, 2^+, 3^+, 4_Q^+) = \frac{1}{2}\sum_{l_1, l_1'}$$

$$A_3(1_Q^+, l_4^+, -l_{3,Q}^-)A_3(-l_4^-, 2^+, l_1^+)A_3(-l_1^-, 3^+, l_2^+)A_3(l_{3,Q}^+, -l_2^-, 4_Q^+)$$

$$+A_3(1_Q^+, l_4^-, -l_{3,Q}^+)A_3(-l_4^+, 2^+, l_1^-)A_3(-l_1^+, 3^+, l_2^-)A_3(l_{3,Q}^-, -l_2^+, 4_Q^+)$$

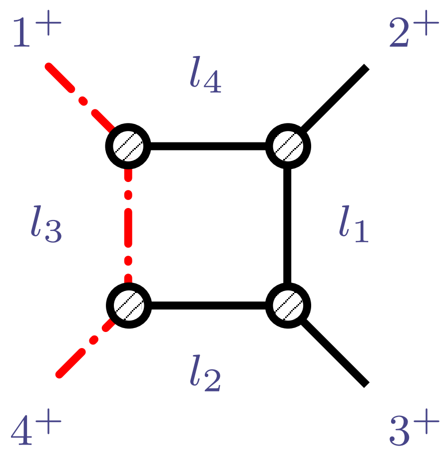
An Example Coefficient



$$C_4^{000m}(1_Q^+, 2^+, 3^+, 4_Q^+) =$$

$$-2m^3 \frac{\langle \eta_1 \eta_4 \rangle [23]^2}{\langle \eta_1 1^\flat \rangle \langle \eta_4 4^\flat \rangle}$$

An Example Coefficient



$$C_4^{000m}(1_Q^+, 2^+, 3^+, 4_Q^+) = -2m^3 \frac{\langle\eta_1\eta_4\rangle[23]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle}$$

$$\begin{aligned}
C_4^{000m}(1_Q^+2^+3^+4_Q^+) =\; & 2\frac{s_{23}m\langle\eta_4 2\rangle[23]\langle\eta_1|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle} - 2\frac{s_{23}m\langle\eta_4 2\rangle[23]\langle\eta_1|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]} \\
& - 2\frac{m\langle\eta_1 2\rangle[23]\langle\eta_4|4|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]} + \frac{m\langle\eta_4 2\rangle s_{23}^2[23]\langle\eta_1|1|2]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]} - 2\frac{m\langle\eta_4 2\rangle[23]\langle\eta_1|1|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]} \\
& - \frac{m\langle\eta_1|1|2]s_{23}^2\langle\eta_4|4|3]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]} + \frac{m^3\langle\eta_1|1|2]s_{23}^2\langle\eta_4|4|2]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]^2} - \frac{s_{12}s_{23}^2m\langle\eta_1|1|2]\langle\eta_4|4|2]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]^2} \\
& + \frac{s_{23}m\langle\eta_1|1|2]\langle\eta_4|4|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]^2} + \frac{s_{23}m\langle\eta_1|1|2]\langle\eta_4|4|2]\langle 3|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]\langle 3|1|2]^2} + \frac{1}{2}\frac{m^3\langle\eta_1\eta_4\rangle s_{23}^2[23]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]^2\langle 3|1|2]^2} \\
& - \frac{s_{23}m\langle\eta_1 2\rangle[23]\langle\eta_4|4|2]\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]^2\langle 3|1|2]^2} - \frac{s_{23}m\langle\eta_4 2\rangle[23]\langle\eta_1|1|2]\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]^2\langle 3|1|2]^2} - \frac{1}{2}\frac{m\langle\eta_1 2\rangle s_{23}^2[23]\langle\eta_4|4|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]^2\langle 3|1|2]^2} \\
& - \frac{1}{2}\frac{m\langle\eta_4 2\rangle s_{23}^2[23]\langle\eta_1|1|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]^2\langle 3|1|2]^2} - \frac{1}{2}\frac{m[23]s_{23}^2\langle 2|1|2]AA(\eta_1 14\eta_4)\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]^2\langle 3|1|2]^2} + \frac{s_{23}m^3\langle\eta_1\eta_4\rangle\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} \\
& - 2\frac{s_{23}m\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 3|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} - 2\frac{s_{23}m\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 3|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} - \frac{s_{23}m\langle 2|1|2]AA(\eta_1 14\eta_4)}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} \\
& + 2\frac{m\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 2|1|2]\langle 3|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} + 2\frac{m\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 2|1|2]\langle 3|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} + \frac{m^3\langle\eta_1 2\rangle s_{23}^2\langle\eta_4|4|2]\langle 2|1|2]^2\langle 3|1|3]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} \\
& + \frac{m^3\langle\eta_4 2\rangle s_{23}^2\langle\eta_1|1|2]\langle 2|1|2]^2\langle 3|1|3]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} - \frac{m^3\langle 2|1|2]^2 s_{23}^2\langle 3|1|3]AA(\eta_1 14\eta_4)}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} + \frac{m^5\langle\eta_1\eta_4\rangle s_{23}^2\langle 3|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} \\
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& + 2\frac{m^5\langle\eta_4 2\rangle s_{23}^2\langle\eta_1|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} + \frac{s_{23}m^3\langle\eta_1\eta_4\rangle\langle 2|1|2]\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} + 2\frac{s_{23}m^3\langle\eta_1\eta_4\rangle\langle 2|1|2]^4\langle 3|1|3]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} \\
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& - 2\frac{s_{23}m\langle 3|1|3]AA(\eta_1 14\eta_4)\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]^2\langle 3|1|2]^2} + 2\frac{s_{23}m^3\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 23\rangle^2} + 2\frac{s_{23}m^3\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 23\rangle^2} \\
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& - \frac{m\langle\eta_4 2\rangle s_{23}^2\langle\eta_1|1|3]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 23\rangle^2} + \frac{m^3\langle\eta_1 3\rangle s_{23}^2\langle\eta_4 2\rangle[23]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} - \frac{s_{12}s_{23}m^3\langle\eta_1\eta_4\rangle\langle 2|1|2]\langle 3|1|3]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} \\
& + \frac{s_{12}s_{23}m\langle 2|1|2]\langle 3|1|3]AA(\eta_1 14\eta_4)}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} - \frac{s_{23}m^3\langle\eta_1\eta_4\rangle\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} + \frac{s_{23}m^3AA(\eta_1 14\eta_4)\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} \\
& - \frac{s_{23}m^5\langle\eta_1\eta_4\rangle\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} - \frac{s_{23}m\langle\eta_1 2\rangle\langle\eta_4|4|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} - \frac{s_{23}m\langle\eta_4 2\rangle\langle\eta_1|1|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} \\
& + \frac{s_{23}m\langle 2|1|2]AA(\eta_1 14\eta_4)\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]\langle 23\rangle^2} + 2\frac{s_{23}m^3\langle\eta_1 3\rangle\langle\eta_4|4|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} + 2\frac{s_{23}m^3\langle\eta_4 3\rangle\langle\eta_1|1|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} \\
& - \frac{3}{2}\frac{s_{23}m\langle\eta_1 3\rangle\langle\eta_4|4|2]\langle 2|1|2]\langle 3|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} + \frac{1}{2}\frac{s_{23}m\langle\eta_1 3\rangle\langle\eta_4|4|2]\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} - \frac{3}{2}\frac{s_{23}m\langle\eta_4 3\rangle\langle\eta_1|1|2]\langle 2|1|2]\langle 3|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} \\
& + \frac{1}{2}\frac{s_{23}m\langle\eta_4 3\rangle\langle\eta_1|1|2]\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} - 2\frac{m\langle\eta_1 3\rangle\langle\eta_4|4|2]\langle 2|1|2]^4\langle 3|1|3]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} - 2\frac{m\langle\eta_4 3\rangle\langle\eta_1|1|2]\langle 2|1|2]^4\langle 3|1|3]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 2|1|3]\langle 3|1|2]^2\langle 23\rangle^2} \\
& + 2\frac{m\langle\eta_1 3\rangle\langle\eta_4|4|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2} + 2\frac{m\langle\eta_4 3\rangle\langle\eta_1|1|2]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2} - \frac{m^5\langle\eta_1 2\rangle s_{23}^2\langle\eta_4|2|3]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} \\
& + \frac{s_{12}s_{23}^2m^3\langle\eta_1 2\rangle\langle\eta_4|2|3]\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} - \frac{s_{12}s_{23}^2m\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} - \frac{s_{12}s_{23}^2m\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} \\
& - 2\frac{s_{23}m^3\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} - \frac{s_{23}m^3\langle\eta_1 2\rangle\langle\eta_4|2|3]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} - \frac{s_{23}m^3\langle\eta_1 2\rangle\langle\eta_4|2|3]\langle 3|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} \\
& - 2\frac{s_{23}m^3\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 2|1|2]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} + \frac{1}{2}\frac{s_{23}m\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 2|1|2]\langle 3|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} - \frac{1}{2}\frac{s_{23}m\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} \\
& + \frac{1}{2}\frac{s_{23}m\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 2|1|2]\langle 3|1|3]\langle 2|1|2]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} - \frac{1}{2}\frac{s_{23}m\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 2|1|2]^4}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} - 2\frac{m\langle\eta_1 2\rangle\langle\eta_4|4|3]\langle 2|1|2]\langle 2|1|2]^2\langle 3|1|3]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2} \\
& - 2\frac{m\langle\eta_4 2\rangle\langle\eta_1|1|3]\langle 2|1|2]\langle 2|1|2]^2\langle 3|1|3]^2}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 3|1|2]\langle 23\rangle^2\langle 2|1|3]^2}
\end{aligned}$$

(Matrix elements)^2 [Beenaaker et al.(1989)]

Analytic results [Körner,Merebashvilli (2002)]

Complete Cut-Constructible Parts

- Automated extraction for all 8 helicity configurations
- Compact analytic expressions
- Numerical comparison with Feynman results
- Numerical tests for independence of loop bases and reference vectors
- Checks for leading order poles:

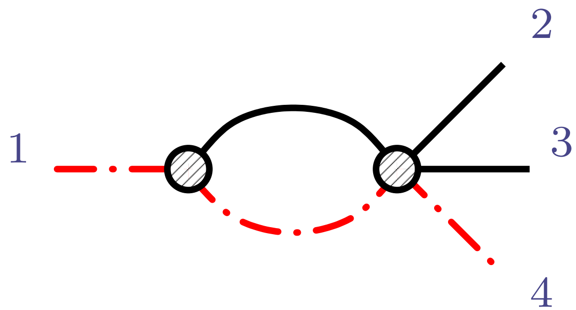
$$A_4^{(1)} = -\frac{A_4^{(0)}}{\epsilon^2} + \mathcal{O}(\epsilon^{-1})$$

$$\Rightarrow C_4^{000m} + s_{23}(C_{3;12}^{00m} + C_{3;34}^{00m}) + \langle 2|1|2] C_3^{000} = -s_{23}\langle 2|1|2] A_4^{(0)}$$

- Check for smooth massless limit $\rightarrow$ need to fix $\log(\mu^2/m^2)$ terms

Fixing The I_1 Coefficient

- 4-dimensional unitarity cuts are cumbersome when considering wave-function renormalisation terms



$$l_1 \cdot l_2 = 0$$

$$A_5(l_{1,Q}, l_2, 2, 3, 4_Q) \sim \frac{1}{l_1 \cdot l_2}$$

- Universal IR and UV behaviour is now well understood [Moch,Mitov]

$$A_n^{(1),CC} = (\text{boxes,triangles,bubbles}) + \frac{C_1}{\epsilon}\left(\frac{\mu^2}{m^2}\right)^{\epsilon}$$

- Fixing remaining $1/\epsilon$ discrepancy gives us C_1 directly

$$C_1 = \frac{5C_f}{2} A_4^{(0)} - N_c(C_{2;12}^{0m} + C_{2;23}^{00}) - \frac{1}{N_c}(C_{2;12}^{m0} + \tilde{C}_{2;23}^{mm})$$

Summary

- On-shell methods with complex momenta are an efficient way to calculate scattering amplitudes - even including massive particles.
- Re-computation of the cut-constructible contributions to $gg \to t\bar{t}$
  - Compact analytic results for all helicity configurations
- Easily automated for both numerical and analytic results
- Further study to evaluate rational pieces
  - D-dimensional cuts vs. On-shell recursion

$$l_{[D]}^2 = l_{[4]}^2 - \mu^2 = 0$$

- Future applications to higher multiplicity amplitudes: $pp \to t\bar{t} +$ jets