On-Shell Methods in Gauge Theories

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Calculations in Gauge Theories

- Explicit calculations are the only way to really learn something from experiment — or from theory
- Want NLO calculations of differential cross sections to understand LHC backgrounds quantitatively
- One-loop calculations of amplitudes in QCD have been the bottleneck to obtaining NLO predictions
- Higher-loop calculations desirable for precision observables and as error estimates
- Higher-loop calculations are playing an important role in developing our understanding of new structures in gauge theories

Why Feynman Diagrams Won't Get You There

- Huge number of diagrams in calculations of interest — factorial growth
- $2 \to 6$ jets: 34300 tree diagrams, $\sim 2.5 \cdot 10^7$ terms

  $\sim 2.9 \cdot 10^6$ 1-loop diagrams, $\sim 1.9 \cdot 10^{10}$ terms

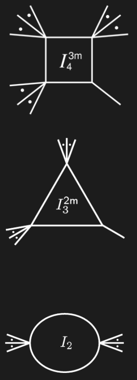
- But answers often turn out to be very simple
- Vertices and propagators involve gauge-variant off-shell states
- Each diagram is not gauge invariant — huge cancellations of gauge-noninvariant, redundant, parts in the sum over diagrams

New Technologies: On-Shell Methods

- Use only information from physical states
- Use properties of amplitudes as calculational tools
  - Factorization → on-shell recursion relations
  - Unitarity → unitarity method
  - Underlying field theory → integral basis

- Formalism

Known integral basis:



$$\mathrm{Ampl} = \sum_{j \in \mathrm{Basis}} c_j \, \mathrm{Int}_j + \mathrm{Rational}$$

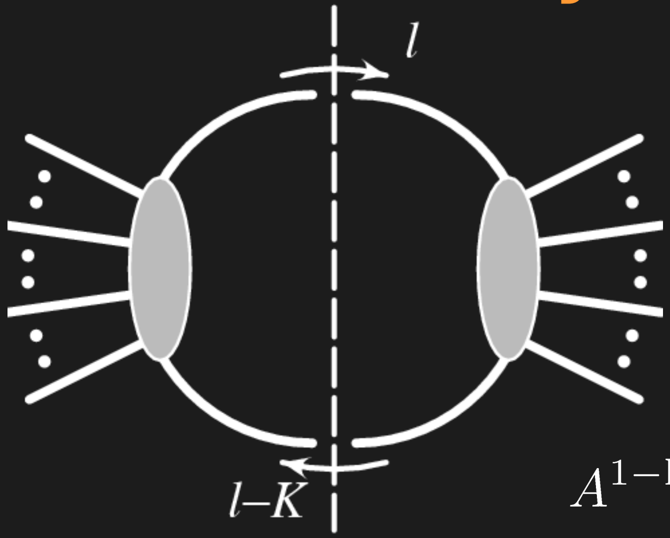
Unitarity

On-shell Recursion

Unitarity: Prehistory

- General property of scattering amplitudes in field theories $\mathrm{Disc}\, T = T^\dagger T$
- Understood in '60s at the level of single diagrams in terms of Cutkosky rules
  - obtain absorptive part of a one-loop diagram by integrating tree diagrams over phase space
  - obtain dispersive part by doing a dispersion integral
- In principle, could be used as a tool for computing 2 → 2 processes
- No understanding
  - of how to do processes with more channels
  - of how to handle massless particles
  - of how to combine it with field theory: false gods of *S*-matrix theory

Unitarity as a Practical Tool



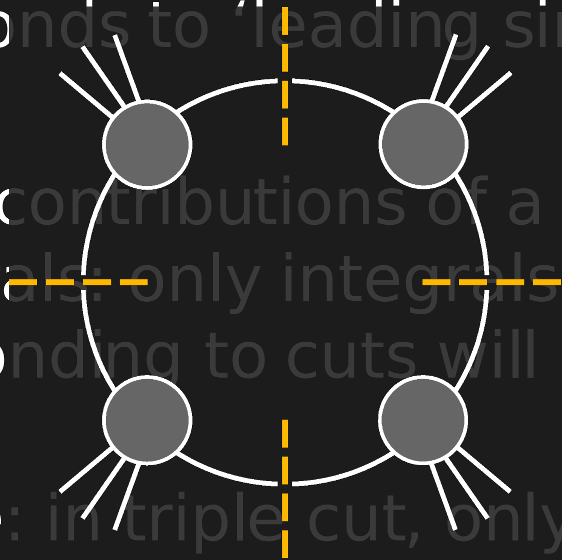
Bern, Dixon, Dunbar, & DAK (1994)

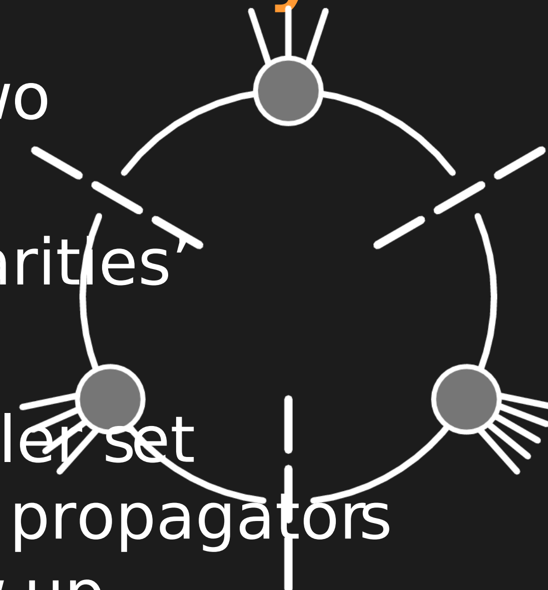
$$A^{1-\text{loop}} = \text{“}\sum_{\text{cuts } K^2}\text{”} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \, \frac{i}{\ell^2} A_L^{\text{tree}} \frac{i}{(\ell-K)^2} A_R^{\text{tree}}$$

- Compute cuts in a set of channels
- Compute required tree amplitudes
- Reconstruct corresponding Feynman integrals
- Perform algebra to identify coefficients of master integrals
- Assemble the answer, merging results from different channels

Generalized Unitarity

- Can sew together more than two tree amplitudes
- Corresponds to ‘leading singularities’





- Isolates contributions of a smaller set of integrals: only integrals with propagators corresponding to cuts will show up

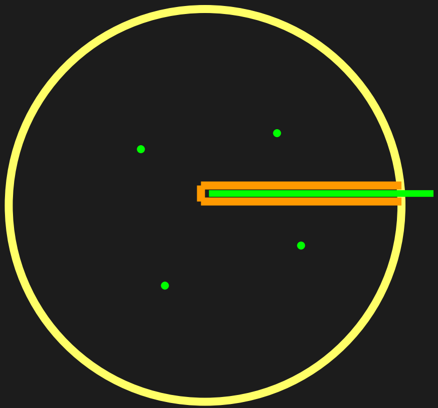
Bern, Dixon, DAK (1997)

- Example: in triple cut, only boxes and triangles will contribute

(in $\mathcal{N} = 4$, a smaller set of boxes)

- Combine with use of complex momenta to determine box coeffs directly in terms of tree amplitudes

Rational Terms



- Consider contour integral $\frac{1}{2\pi i}\oint_C \frac{dz}{z}\, A(z)$

$$\Rightarrow \quad A(0) = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z} + \int_{\text{Branch}} \text{Cut}\, \frac{dz}{z}\, \text{Disc}_B A(z)$$

Spurious singularities $\frac{\ln(s_1/s_2)}{(s_1-s_2)^2} + \frac{1}{s_1-s_2}$

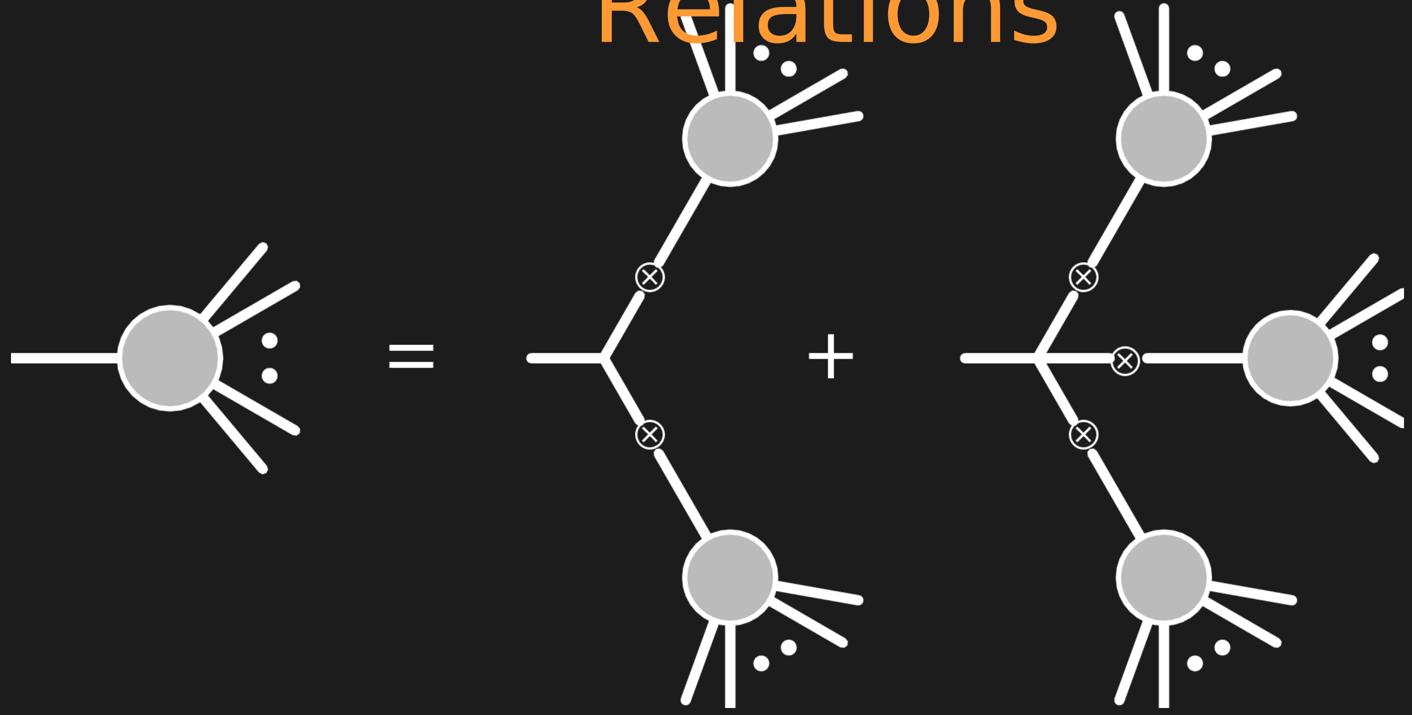
$$A(0) = -\sum_{\substack{\text{spurious}\\ \text{poles }\alpha}} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Cut}(z)}{z} - \sum_{\substack{\text{physical}\\ \text{poles }\alpha}} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Rat}(z)}{z} + \text{Cut}$$

On-shell recursion

Computational Complexity

- Basic object: color-ordered helicity amplitude
- Differential cross section needs sum over color orderings and helicities
  - Use phase-space symmetry to reduce sum over color orderings to polynomial number (or do sum by Monte Carlo)
  - Sum over helicities by Monte Carlo
- What is the complexity of a helicity amplitude? $C \exp n$ vs $C n^p$
  - Asymptotically, only overall behavior really matters
  - We're interested in moderate n
  - Brute-force calculations have huge prefactors as well as exp behavior
- Want polynomial behavior
  - Purely analytic answers for generic helicities are exponential
- Require a (partly) numerical approach for maximal

Berends–Giele Recursion Relations



J_5 appearing inside J_{10} is identical to J_5 appearing inside J_{17}

Compute once numerically $\Rightarrow$ maximal reuse

$\Rightarrow$ Polynomial complexity per helicity

Intelligent Automation

- Only the unitarity method combined with a numerically recursive approach for the cut trees can yield a polynomial-complexity calculation at loop level
- To date: bespoke calculations

- Need industrialization $\Rightarrow$ automation
- Numerical approach overall: worry about numerical stability

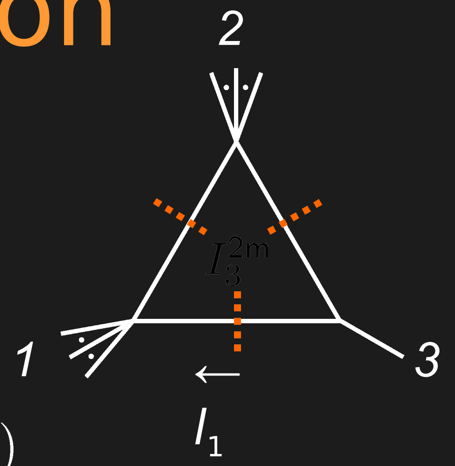
- Do analysis analytically
- Do algebra numerically

What’s New?

- Two years to the day since last talk at Loops & Legs Eisenach
- Completed groundwork for practical technology
- Completed analytic calculations

- Many new ideas, complementary lines of attack within the unitarity framework:
  Ossola, Papadopoulos, Pittau
  Ellis, Giele, Kunszt, Melnikov
  Anastasiou, Britto, Feng, Mastrolia
- Internal masses: Britto, Feng, Mastrolia; Badger $\Rightarrow$ talk
- Framework for automated numerical calculations

Analytic Isolation



- Define massless momenta

$$\tilde{K}_1^\mu = \xi\alpha(\gamma K_1^\mu + K_1^2 K_3^\mu)$$
$$\tilde{K}_3^\mu = -\xi\alpha'(\gamma K_3^\mu + K_3^2 K_1^\mu)$$

where $\alpha = \dfrac{K_3^2(K_1^2-\gamma)}{K_1^2 K_3^2 - \gamma^2}\,, \quad \alpha' = \dfrac{K_1^2(K_3^2-\gamma)}{K_1^2 K_3^2 - \gamma^2}$

$$\gamma = \gamma_\pm = -K_1\cdot K_3 \pm \sqrt{\Delta}\,, \quad \xi = \frac{\gamma}{\gamma^2 - K_1^2 K_3^2}\,,$$

$$\Delta = -\det(K_i\cdot K_j) = (K_1\cdot K_3)^2 - K_1^2 K_3^2$$

Parametrize

$$l_1^\mu(t) = \tilde{K}_1^\mu + \tilde{K}_3^\mu + \frac{t}{2}\langle \tilde{K}_1^-|\gamma^\mu|\tilde{K}_3^-\rangle + \frac{1}{2t}\langle \tilde{K}_3^-|\gamma^\mu|\tilde{K}_1^-\rangle$$

del Aguila, Ossola, Papadopoulos, Pittau (2006); Forde (2007)

Forde (2007)

- t is remaining unfrozen degree of freedom
- Each box corresponds to a pole in t
- Triangle has no poles at finite t, only higher-order poles at 0 and ∞
- Integrals of powers of t vanish ('total derivatives')

$$\int \frac{d^4 l}{(2\pi)^4} \, \langle \tilde{K}_3^- | l | \tilde{K}_1^- \rangle^n = 0 \quad \Longrightarrow \quad \int dt \, \mathrm{Jac}_t \, t^n = 0$$

- Constant term gives triangle coefficient: take large-t expansion of product of trees, then set $t = 0$

$$\mathrm{coeff} = -[\mathrm{Inf}_t \, A_1 A_2 A_3](t)\Big|_{t=0}$$

BlackHat

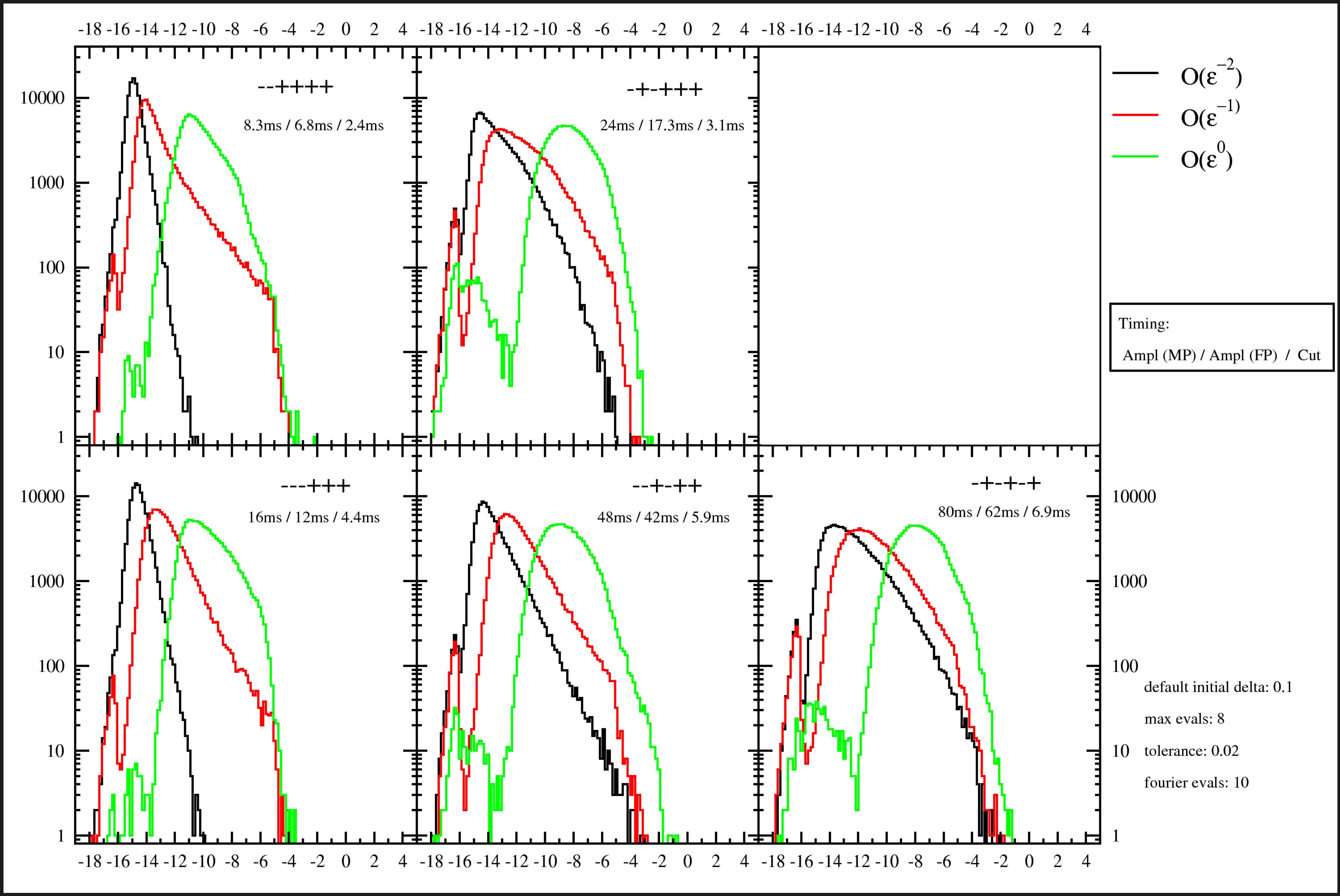
$\Rightarrow$ Daniel Maître's talk

Carola Berger, Z. Bern, L. Dixon, Fernando Febres Cordero,
Darren Forde, Harald Ita, DAK, Daniel Maître

- Written in C++
- Framework for automated one-loop calculations
  - Organization in terms of integral basis (boxes, triangles, bubbles)
  - Assembly of different contributions
- Library of functions (spinor products, integrals, residue extraction)
- Tree amplitudes (ingredients)
- Caching

Outline of a Calculation

- Unitarity freezes all propagators in box integrals: coefficients are given by a product of tree amplitudes with complex momenta (Britto, Cachazo, Feng) — compute them numerically
- Unitarity leaves one degree of freedom in triangle integrals: coefficients are the residues at ∞ (Forde): compute these numerically using discrete Fourier projection after subtracting boxes *à la* Ossola, Papadopoulos, Pittau
- Two degrees of freedom in bubbles: two-dimensional discrete Fourier projection, after subtractions
- Physical poles: on-shell recursion numerically (complex momenta)
- Spurious poles (‘cut completion’): locations numerically, functional form from each integral



--++++
8.3ms / 6.8ms / 2.4ms
-+-+++
24ms / 17.3ms / 3.1ms
---+++
16ms / 12ms / 4.4ms
--+-++
48ms / 42ms / 5.9ms
-+-+-+
80ms / 62ms / 6.9ms
$O(\varepsilon^{-2})$
$O(\varepsilon^{-1})$
$O(\varepsilon^{0})$
Timing:
Ampl (MP) / Ampl (FP) / Cut
default initial delta: 0.1
max evals: 8
tolerance: 0.02
fourier evals: 10

The Party's Not Over

- General automated packages
- Refinement of techniques within the unitarity method
- Exploiting analytic structure of integrands:
  - Ossola–Papadopoulos–Pittau subtraction of higher-point integrands
  - Forde: use analytic behavior (pole at infinity) to extract coefficient
- Refinement of techniques for computing rational terms
  - On-shell recursion
  - Giele–Kunszt–Melnikov use of different integer dimensions to effect *D*-dimensional unitarity

Challenges at Higher Loops

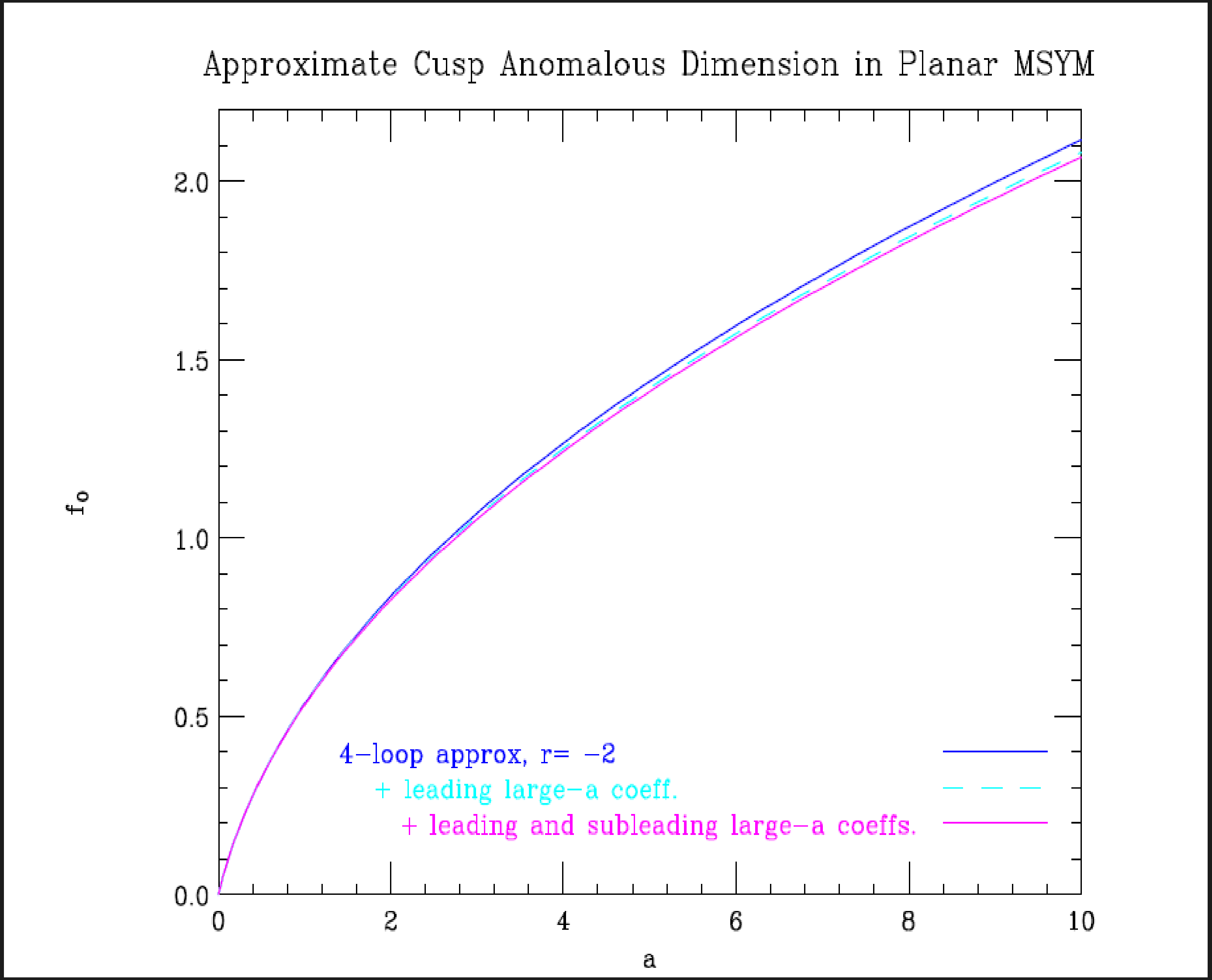
- Construct basis of master integrals
- Analyze analytic structure of integrals and integrands to obtain algorithm for extracting coefficients
- Understand factorization and extraction of rational parts

Development Laboratory

- Need a simpler theory
- Field theorists' gut prejudice: scalar field theory is simplest, because Lagrangian is simplest
- Or perhaps QED is "easiest"
- But these theories have the *least* symmetry
- The simplest theory is the one with the *most* symmetry
- N = 4 supersymmetric gauge theory

- Confirmed by simplicity of its loop amplitudes
- Fewer integrals, simpler by power counting, “dual conformal invariance”
- Amplitudes satisfy “maximum transcendentality”: all contributions have polylog weight = 2 L

- Important to probing the AdS/CFT duality


Approximate Cusp Anomalous Dimension in Planar MSYM
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10
a
f_o
4−loop approx, r= −2
+ leading large−a coeff.
+ leading and subleading large−a coeffs.

Iteration Relation

Anastasiou, Bern, Dixon, DAK (2003); Bern, Dixon, Smirnov (2005)

$$\ln \mathcal{M}_n^{\rm MHV} = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon) \right)$$

$$a \equiv (4\pi e^{-\gamma})^{\epsilon} \left(\frac{\lambda}{8\pi^2} = \frac{g^2 N_c}{8\pi^2} = \frac{N_c \alpha_s}{2\pi} \right)$$

Evidence: two-loop four- and five-point amplitudes; three-loop four-point amplitude computed using the unitarity method

It Extends to Strong Coupling!

Alday & Maldacena (2007)

- Use AdS/CFT to perform a string calculation: fixed-angle high-momentum

$$s_{ij}^{\rm Gauge} = ({\rm Warp})\, s_{ij}^{\rm String}$$

- Use saddle-point (classical solution) approach

Gross & Mende (1988)

- Equivalent to Wilson loop calculation

$$\ln \mathcal{M}_4 = -\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda_4 \mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda_4 \mu^{2\epsilon}}{s^\epsilon}\right) + (s \to t)$$
$$+ \frac{f(\lambda)}{8}\left[\ln^2 s/t + 4\pi^2/3\right] + C(\lambda)$$

- Also works for five-point amplitude
- Wilson loops at weak coupling: at one loop, M is equal to the Wilson loop for any number of legs (up to an addititve constant)

Drummond, Korchemsky, Sokatchev (2007)

Brandhuber, Heslop, & Travaglini (2007)

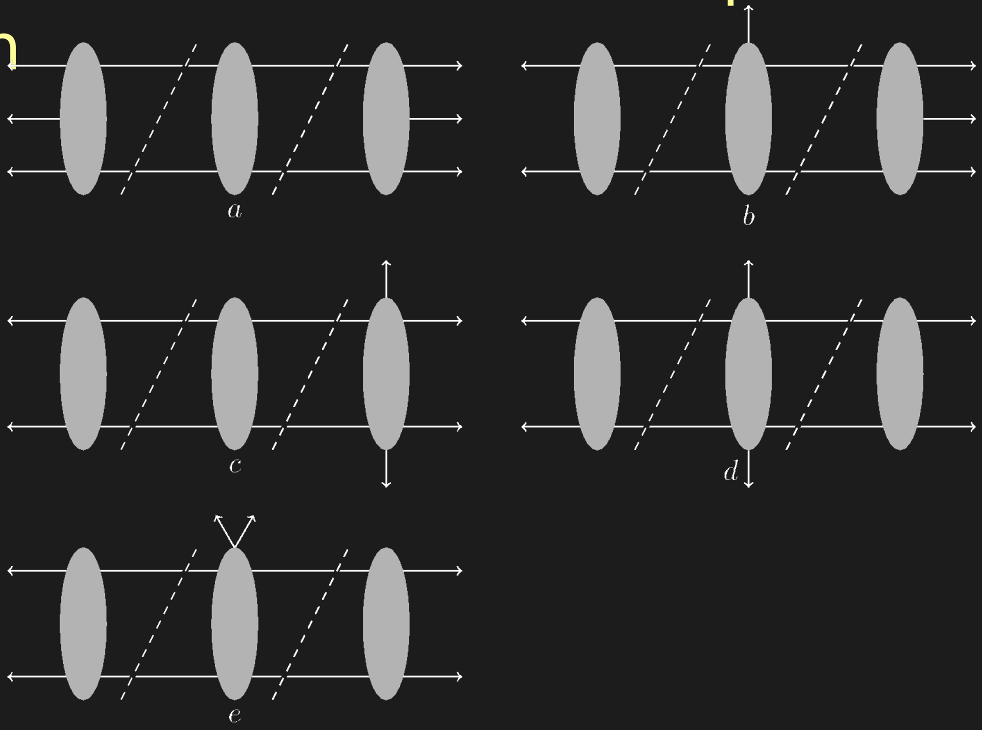
- Wilson loop = Amplitude holds for four- and five-point amplitudes at two loops

Drummond, Henn, Korchemsky, Sokatchev (2007–8)

- Explained by 'dual conformal invariance'

- Simple iteration relation breaks down for six-point two-loop amplitude

Z. Bern, L. Dixon, R. Roiban, M. Spradlin, C. Vergu, & A. Volovich



- But equality with Wilson loop holds!

Comparison with Drummond, Henn, Korchemsky, & Sokatchev

$\Rightarrow$New structure?

Summary

- On-shell methods are method of choice for QCD calculations for colliders
- First light for automated one-loop calculations
- On-shell methods are also playing an important role in uncovering new structures in gauge theories, and in understanding the AdS/CFT duality