It’s Simpler to be Singular

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Based on work done in collaboration with Stefano Goria
Discussions with Ansgar Denner and Stefan Dittmaier are acknowledged.

Outlines

(1, 2,)

1. *From the analytical structure of Feynman diagrams*
2. *to their numerical evaluation*

what else, but the inevitable!

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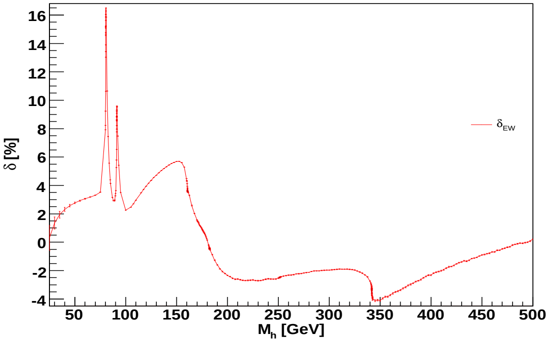
Part I

Intermezzo

A complete two-loop calculation

Oooops $\ldots\ H \to \gamma\gamma, gg \rightsquigarrow$

This is what I should have been talking about
S. Actis, C. Sturm, S. Uccirati and myself ($\approx$ 10 kilohour)



Part II

Sonata form

A celebrated result with too many fathers

Theorem

$$\sum \left\{ \textit{1-loop n-legs Feynman diagrams} \right\} = \sum_{\mathcal{D}} B_{\mathcal{D}}\, D_0\left(P_1^{\mathcal{D}}, \ldots, P_4^{\mathcal{D}}\right) + \cdots$$

$\mathcal{D}$ *partition of* $\{1 \ldots n\}$ *into 4 non-empty sets*
$P_i^{\mathcal{D}}$ *sum of momenta in* $i \in \mathcal{D}$

Bases are bases, and troubles are troubles

Scalar one-loop integrals

form a basis. Thus, coefficients are uniquely determined, although some method can be more efficient than others in their determination. However, troublesome points will always be there (Denner-Dittmaier anathema). What to do?

- Change (adapt) bases?
- Avoid bases (expansion)?
- Rethinking necessary.

Part III

Factorization of Feynman amplitudes

Factorization

Any Feynman diagram

is particularly simple when evaluated around its anomalous threshold.

Kershaw theorem (1972)

The singular part of a scattering amplitude around its leading Landau singularity may be written as an algebraic product of the scattering amplitudes for each vertex of the corresponding Landau graph times a certain explicitly determined singularity factor which depends only on the type of singularity (triangle graph, box graph, etc.) and on the masses and spins of the internal particles.

One-loop, multi-legs

Define

scalar one-loop N -leg integral in n -dimensions as

$$S_{n\,;\,N} = \frac{\mu^{\epsilon}}{i\,\pi^2}\int d^n q\,\frac{1}{\prod_{i=0,N-1}(i)},$$
$$(i) = (q+k_0+\cdots+k_i)^2+m_i^2,$$

Use N -simplex

$$\int dS_N = \prod_{i=1}^{N}\int_0^{x_{i-1}} dx_i\,, \qquad x_0 = 1.$$

One-loop, multi-legs II

In parametric space we get

$$S_{n\,N} \;=\; \left(\frac{\mu^2}{\pi}\right)^{2-n/2} \Gamma\left(N-\frac{n}{2}\right)\,[N]_n.$$

Example

$$[N]_n \;=\; \int dS_{N-1}\, V_N^{n/2-N},$$

with

$$V_N \;=\; x^t\, H_N\, x + 2\, K_N^t\, x + L_N, \quad X_N = -K_N^t\, H_N^{-1}.$$

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One-loop, multi-legs III

Useful jargon (used by addicts)

BST factor

$$B_N \quad = \quad L_N - K_N^t \, H_N^{-1} \, K_N$$

Caley (determinant)

$$M = \begin{pmatrix} H_N & K_N \\ K_N^t & L_N \end{pmatrix}$$

Gram (determinant)

$$H_{ij} = - k_i \cdot k_j \qquad\qquad G = \det H$$

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One-loop, multi-legs IV

It follows

$B = C/G$, where $C = \mathrm{det}M$ is the so-called modified Cayley determinant of the diagram.

LS as pinches (masses $\&$ invariants $\in R$)

$V_N = (x - X_N)^t \; H \; (x - X_N) + B_N$

No discussion of the complex part of singular surface

$B_N = 0$ induces a pinch on the integration contour at the point of coordinates $x = X_N$; therefore, if the conditions,

$$B_N = 0, \qquad 0 < X_{N,N-1} < \; \ldots \; < X_{N,1} < 1,$$

are satisfied we will have the leading singularity of the diagram.

Why to avoid Gram^{-1}?

A common wisdom, but?

- The vanishing of the Gram determinant is the condition for the occurence of non-Landau singularities, connected with the distorsion of the integration contour to infinity;
- furthermore, for complicated diagrams, there may be pinching of Landau ($C = 0$) and non-Landau singularities ($G = 0$), giving rise to a non-Landau singularity whose position depends upon the internal masses (so-called D^2 wild points).

AT and factorization

It follows:

- Given the above properties the factorization of Kershaw theorem follows.
- The beauty of being at the anomalous threshold is that everything is frozen and the amplitude factorizes.
- But, what to do with a point?
- It looks perfect for boundary conditions, as long as we can reach it. Alternative: expand & match residues at a given AT (Cachazo 2008).

Standard reduction vs modern techniques

Example

$$\frac{\mu^{\epsilon}}{i\,\pi^2}\int d^n q\,\frac{q\cdot p_1}{\prod_{i=0,3}(i)} \;=\; \sum_{i=1}^{3} D_{1i}\,p_1\cdot p_i = -\sum_{i=1}^{3} D_{1i}\,H_{1i}\,.$$

carefull application of the method

$$D_{1i} \;=\; -\frac{1}{2}H^{-1}_{ij}\,d_j, \quad d_i = D_0^{(i+1)} - D_0^{(i)} - 2\,K_i\,D_0,$$

where $D_0^{(i)}$ is the scalar triangle obtained by removing propagator i from the box.

Standard reduction vs modern techniques II

Therefore we obtain

$$\frac{\mu^\epsilon}{i\,\pi^2}\int d^nq\,\frac{q\cdot p_1}{\prod_{i=0,3}(i)} \;=\; \frac{1}{2}\sum_{i,j=1}^{3} H^{-1}_{ij}\,H_{1i}\,d_j = \frac{1}{2}\,d_1,$$

(no G_3). Furthermore, the coefficient of D_0 in the reduction is

$$\frac{1}{2}\left(m_0^2 - m_1^2 - p_1^2\right)$$

(General feature of tensor-N $\rightarrow$ scalar-N)

Standard reduction vs modern techniques III

Theorem

At the leading Landau singularity of the box we must have

$$q^2 + m_0^2 \;=\; 0, \qquad (q+p_1)^2 + m_1^2 = 0, \qquad \textit{etc.}$$

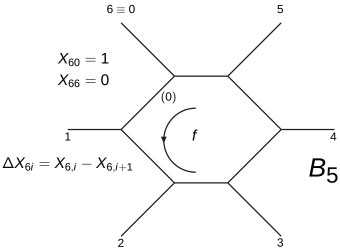
Therefore

the coefficient of D_0 is fixed by

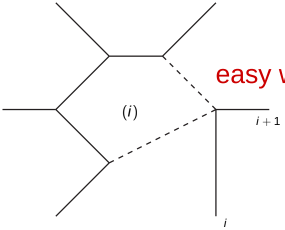
$$2\,q \cdot p_1\Big|_{AT} \;=\; m_0^2 - m_1^2 - p_1^2,$$

which is what a careful application of SR gives. Note that one gets the coeff. without having to require a physical singularity.

From hexagons up: factorization at SubLeadingLandau . . . Landshoff



$$B_5(i) \to 0$$



easy with BST

$$F\left(\{n\}_5\right) \quad \sim \quad \frac{1}{6}\,\frac{\Delta X_{6i}}{B_6}\, X_{51}^{n_1}(i)\,\ldots\, X_{5i}^{n_i+n_{i+1}}(i)\,\ldots\, X_{54}^{n_5}(i)\, E_0^{\text{sing}}(i)$$

$$\text{or} \quad \frac{1}{6}\,\frac{\Delta X_{65}}{B_6}\, X_{51}^{n_1}(5)\,\ldots\, X_{54}^{n_4}(5)\, E_0^{\text{sing}}(5)\, \delta_{n_5,0} \quad i = 5$$

Sunny-side up of factorization

Progress

- At least in one *point* we can avoid reduction, all integrals are scalar;
- but, do we need to have the AT inside the physical region R_{phys} (support of $\Delta^{\pm}$ in R)?

Problems

- Since this is a rare event (see later) we must have a generalization:
- prove that the AT, even with invariants $\notin R_{\mathrm{phys}}$, implies a frozen q.

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Generalized factorization I

Define

if $\frac{1}{i\,\pi^2}\int d^nq\,\frac{1}{\prod_{i=0,N-1}(i)}$ is singular at $x = X \in R$

Then (example)

$$\frac{1}{i\,\pi^2}\int d^nq\,\frac{q\cdot p_l}{\prod_{i=0,N-1}(i)} = -\sum_{i=1}^{N}[N]_n(i)\,p_l\cdot p_i$$

$$\sum_{i=1}^{N}[N]_n(i)\,H_{li} \;\underset{AT}{\sim}\; \sum_{i=1}^{N}[N]_n(1)\,H_{li}\,X_i = -K_l\,[N]_n.$$

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Generalized factorization II

Where

$$X_i = -K_j H_{ji}^{-1}, \qquad H X = -K$$

$\rightsquigarrow$ Factorization

At the AT all scalar products $\rightarrow$ solution of

$$(q + \cdots + p_i)^2 + m_i^2, \qquad i = 0, \ldots, N-1.$$

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Part IV

More on the AT

How frequent is AT in your calculation?

For $N = 4$ there are 14 branches in p-(real) space,

$p_i^0 > 0, p_k^0 < 0$

$$M_i^2 < (m_i + m_l)^2\,,\ M_j^2 > \left(m_i - m_j\right)^2\,,\ M_k^2 < \left(m_j + m_k\right)^2\,,\ M_l^2 < (m_k - m_l)^2\,,$$

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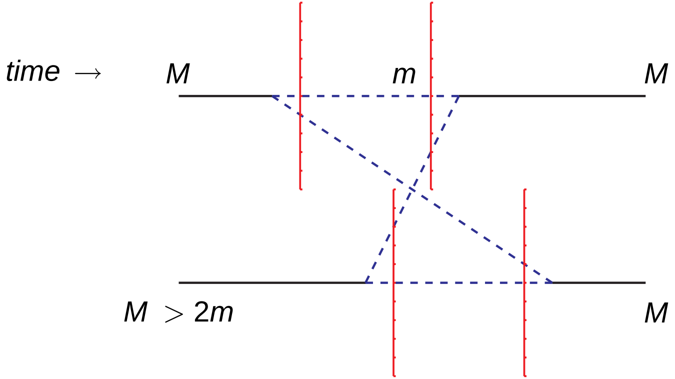
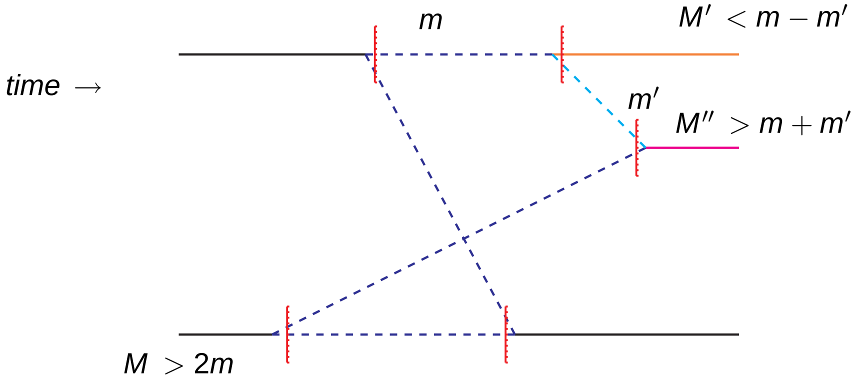
It's easier with Coleman - Norton



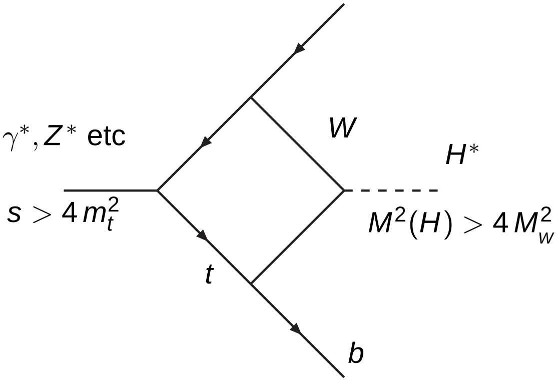
In $2 \rightarrow 2$ two unstable particles $\in\ |\mathrm{in}>$ are needed!

Example for pentagon



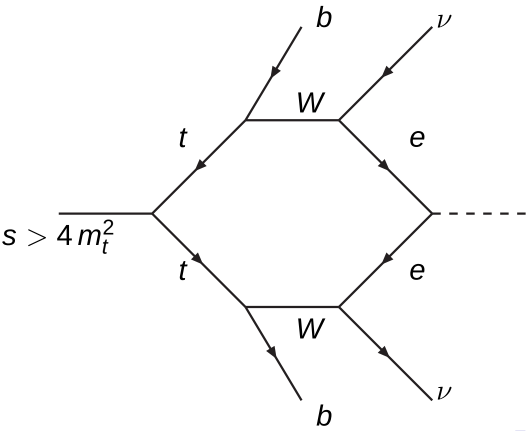
AT watch (ain't a tornado but)

For those who don't want an AT in their MC, beware of



AT watch II (Denner's devil)

Hexagons don't count but pentagons $\leftarrow$ hexagons do!



Expansion around AT Eden . . . Melrose

Expansion around AT

of Feynman integrals is easy to derive analytically

Requires

- Mellin-Barnes
- Sector decomposition

Leading behavior†

- $C_0 \sim \ln B_3$;
- $D_0 \sim B_4^{-1/2}$;
- $E_0 \sim B_5^{-1}$;
- F_0 none in 4 d.

e.g. $\mathrm{Im}\, C_0$ has a log singularity, $\mathrm{Re}\, C_0$ has a discontinuity
†) NO IR/coll configuration, otherwise enhancement of singular behavior (in the residues of IR/coll poles).

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Non integrable pentagon singularity?

Problem

pentagon $\longrightarrow$ non-integrable pole

Solutions?

1. spin + gauge cancellations
2. unstable particles $\longrightarrow$ complex masses

Preliminar

1. simple examples $\longrightarrow$ not the case
2. unitarity?
3. for integ. sing. average over a Breit-Wigner of the invariant mass of unstable ext particles

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Part V

Differential equations

Differential equations, Regge ... Kotikov ... Remiddi

Everything is suggesting DE with boundary conditions at the AT

But we want

- ODE for the amplitude;
- real momenta †;
- one boundary condition.

Advantages

- no reduction;
- extedibility to higher loops.

Requires

- the right variable

$\dagger$) $p \in C$ means $\mathrm{SL}(2, C) \otimes \mathrm{SL}(2, C) \rightarrow$ double cover of $\mathrm{SO}(3,1)$

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ODE vs PDE

The case

- non-homogeneous systems of ODE are easy to obtain with IBP but the non-homogeneous part requires (a lot) of additional work;
- PDE are notoriously much more difficult!

However

homogeneous (compatible) systems of nth-order PDE are easy to derive, a fact that has to do with the hypergeometric character of one-loop diagrams.

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For the fun of it

Use

- Kershaw expansion around pseudo-threshold and
- generalization of Horn-Birkeland-Ore theory (see Bateman bible)

to write one-loop diagrams as

$$F(z_1, \ldots, z_m) = \sum_{\{n_i\}} A(n_1, \ldots, n_m) \prod_i \frac{z_i^{n_i}}{n_i\,!}$$

Since

$$\frac{A(\ldots, n_i + 1, \ldots)}{A(\ldots, n_i, \ldots)} = \frac{P_i(\{n_i\})}{Q_i(\{n_i\})} = \frac{\text{fin. pol.}}{\text{fin. pol.}}$$

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Hypergeometry of Feynman integrals

Then

$$\left[Q_i\left(\left\{z_i\,\frac{\partial}{\partial z_i}\right\}\right)z_i^{-1} - P_i\left(\left\{z_i\,\frac{\partial}{\partial z_i}\right\}\right)\right]F \;=\; 0.$$

With, e.g. for $N = 4$ ($N = 5$ P, Q are of third order)

$$\begin{aligned}
s_{ij} &= -\left(p_i + \ldots + p_{j-1}\right)^2 \quad z_{ij} = \frac{s_{ij} - (m_i - m_j)^2}{4\,m_i m_j} \\
P_{ij} &= (n_i + 1)\,(n_j + 1), \quad n_i = \sum_{j>i} n_{ij} + \sum_{j<i} n_{ji} \\
Q_{ij} &= (n_{ij} + 1)\,(n + \frac{5}{2}), \quad n = \sum_{i<j} n_{ij}
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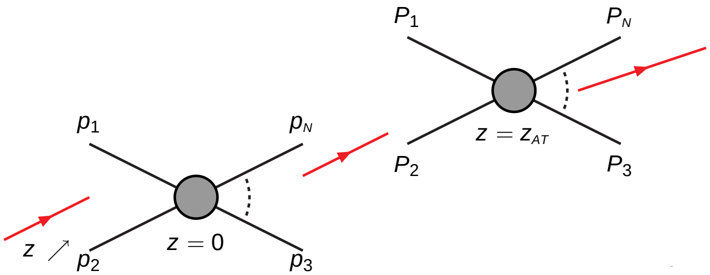
\end{aligned}$$

Diffeomorphisms of Feynman diagrams

$$P_i(z) = T_{ij}(z)\, p_j, \quad \text{with} \quad \sum P_i = \sum p_i = 0, \quad T_{ij}(0) = \delta_{ij}$$



Classification

M $\longrightarrow$ physical

- maps $D(0)$ into $D(z)$ which is singular at $z_{AT} \in R$

$$s_{ij} \quad \longrightarrow \quad S_{ij}(z) \in \mathrm{Phys}_z$$

- no restriction on s_{ij}

M $\longrightarrow$ unphysical

- maps $D(0)$ into $D(z)$ which is singular at $z_{AT} \in R$

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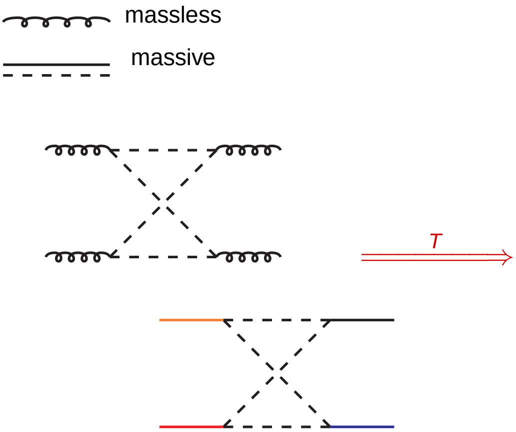
Mappings: I



Mappings: S-I

Solution

$$P_i \quad = \quad (1-z)\,p_i + z\,p_{i+2} \quad \mod 4$$

transf. invariants

$$\begin{aligned} M_i^2 &= z\,(1-z)\,u,\ S = (1-2\,z)^2\,s,\ T = (1-2\,z)^2\,t,\ U = u \\ r &= z^2 - z \\ r_{AT} &= \frac{1}{2\,u^2}\left[4\,m^2 s + ut + \sqrt{s\,(4m^2-u)\,(4m^2 s + ut)}\,\right] \end{aligned}$$

Return

Mappings: S-I

Solution

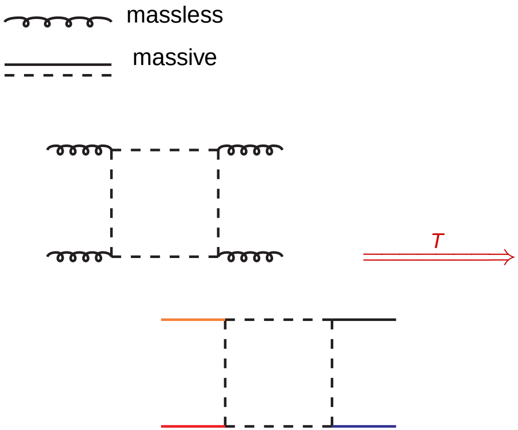
$$P_i = (1-z)\,p_i + z\,p_{i+2} \quad \mathrm{mod}\ 4$$

transf. invariants

$$\begin{aligned} M_i^2 &= z\,(1-z)\,u,\; S = (1-2\,z)^2\,s,\; T = (1-2\,z)^2\,t,\; U = u \\ r &= z^2 - z \\ r_{AT} &= \frac{1}{2\,u^2}\left[4\,m^2 s + ut + \sqrt{s\,(4m^2-u)\,(4m^2 s + ut)}\,\right] \end{aligned}$$

Return

Mappings: II



Mappings: S-IIa

Solution

$$P_i = p_i + (-1)^i \, (p_1 + p_3) \, z$$

$$\begin{aligned} M_i^2 &= u\, r, \; S = s, \; T = t, \; U = (1 + 4\, r)\, u \\ r &= z^2 - z \\ r_{AT} &= \frac{1}{2\, u^2} \left[4\, m^2 u + \sqrt{u^2\, (4m^2 - s)\, (4m^2 - t)} \right] \end{aligned}$$

unphysical, $P_{ij}^2 \notin R_{\mathrm{phys}}$

requires $s < 4\, m^2$

Mappings: S-IIb

Solution

$$
\begin{aligned}
P_{1,4} &= p_{1,4} + (p_1 + p_2)\, z\,, \quad P_{2,3} = p_{2,3} - (p_1 + p_2)\, z\,, \\
M^2_{1,3} &= z\,(z+1)\,s \quad M^2_{2,4} = z\,(z-1)\,s \\
S &= s, \quad U = u, \quad T = (1 + 4\,z^2)\,t \\
z^2_{AT} &= \frac{1}{2}\left[1 - \frac{1}{s}\sqrt{u\,(4m^2 - s)}\,\right]
\end{aligned}
$$

unphysical, $P^2_{ij} \notin R_{\rm phys}$

requires $s > 4\,m^2$ and $u < 4\,m^2 - s$

Return

General solution for D

If $\exists$ a diagram $\overline{D}$, a transformation $\overline{T}$

$$\overline{D}(z) = \overline{T}(z)\,\overline{D}, \quad \overline{T}(0) = I, \quad \overline{D}(z_{AT}) \text{ singular } z_{AT} \in R$$

Map D

$$\begin{aligned} D &\to D\left(z, z_{AT}\right) \\ D\left(z, z_{AT}\right) &= T_1\left(z, z_{AT}\right) D + T_2\left(z, z_{AT}\right) \overline{D}(0) \\ T_1\left(0, z_{AT}\right) &= I, \quad T_2\left(0, z_{AT}\right) = 0 \\ T_1\left(z_{AT}, z_{AT}\right) &= 0, \quad T_2\left(z_{AT}, z_{AT}\right) = I \end{aligned}$$

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T_1\left(z_{AT}, z_{AT}\right) &= 0, \quad T_2\left(z_{AT}, z_{AT}\right) = I
\end{aligned}$$

Solution for direct box $gggg \to 0$

Derive $(T_1 \oplus T_2) \otimes \overline{T}$

$$
\begin{aligned}
P_i &= \Big[f_1 + f_2\,(1 - z_{AT})\Big]\,p_i + f_2\,z_{AT}\,p_{i+2}, \quad \bmod 4 \\
f_1 &= 1 - \frac{z}{z_{AT}} \qquad f_2 = 1 - f_1
\end{aligned}
$$

Or

1. direct box $\longrightarrow$ crossed box
2. crossed box $\longrightarrow$ singular crossed box

Solution for direct box $gggg \to 0$

Derive $(T_1 \oplus T_2) \otimes \overline{T}$

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\begin{aligned}
P_i &= \Big[f_1 + f_2\,(1 - z_{AT})\Big]\, p_i + f_2\, z_{AT}\, p_{i+2}, \qquad \mathrm{mod}\, 4 \\
f_1 &= 1 - \frac{z}{z_{AT}} \qquad\qquad f_2 = 1 - f_1
\end{aligned}
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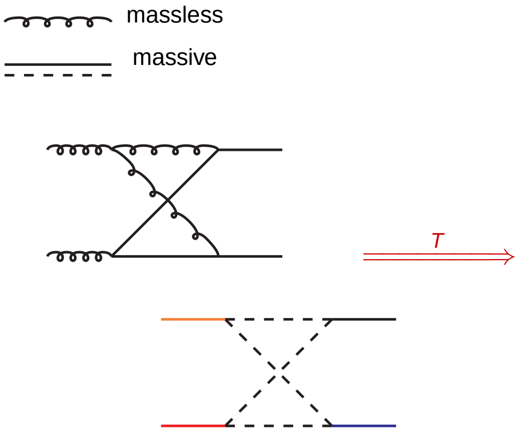
Or

1. direct box $\longrightarrow$ crossed box
2. crossed box $\longrightarrow$ singular crossed box

$gg\bar{t}t \longrightarrow 0$



Solution for $gg\bar{t}t \to 0$

Requires shift on internal masses

$$p \quad \to \quad P = T_p(z)\,p \quad \text{and} \quad m \to M = T_m(z)\,m$$

T_p

$$\begin{aligned} P_1 &= (1-z)\,p_1 + z\left(p_3 + \frac{z}{z_{AT}}K\right) \quad P_2 = (1-z)\,p_2 + z\left(p_4 - \frac{z}{z_{AT}}K\right) \\ P_3 &= z\,p_1 + (1-z)\left(p_3 + \frac{z}{z_{AT}}K\right) \quad P_4 = z\,p_2 + (1-z)\left(p_4 - \frac{z}{z_{AT}}K\right) \end{aligned}$$

T_m

$$\begin{aligned} T_m &= \mathrm{diag}\left(\frac{z}{z_{AT}}, \frac{z}{z_{AT}}, 1, 1\right) \\ K_\mu &= k\,\epsilon\,(\mu, p_1, p_2, p_3) \quad k^2 = -4\,\frac{m}{s}\left[s\,t + \left(t - m^2\right)^2\right]^{-1} \end{aligned}$$

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ODE in z with IBP

ODE for boxes

$$
\begin{aligned}
D_0(\{n\}) &= \frac{\mu^\epsilon}{i\,\pi^2}\int d^n q\,\frac{1}{\prod_{i=0,3}(i)^{n_i}}, \\
D_0(i) &= D_0(1,\ldots,2,\ldots,1) \quad D_0 = D_0(1,\ldots,1) \\
\frac{d}{dz}D_0 &= 2\,zs\left[D_0(2)+D_0(4)\right] + \text{triangles}
\end{aligned}
$$

IBP $\longrightarrow$

$$
D_0(i) = M^{-1}_{ij}\,d_j \quad \det M(z_{AT}) = 0
$$

where d_i contains D_0 or triangles.

ODE in z with IBP

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$$
\begin{aligned}
D_0(\{n\}) &= \frac{\mu^\epsilon}{i\,\pi^2}\int d^n q\,\frac{1}{\prod_{i=0,3}(i)^{n_i}}\,,\\
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D_0(i) = M_{ij}^{-1}\,d_j \qquad \det M(z_{AT}) = 0
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exa

ODE in z with IBP

ODE for boxes

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\begin{aligned}
D_0(\{n\}) &= \frac{\mu^\epsilon}{i\,\pi^2}\int d^n q\,\frac{1}{\prod_{i=0,3}(i)^{n_i}}, \\
D_0(i) &= D_0(1,\,\dots,2,\,\dots,1) \quad D_0 = D_0(1,\,\dots,1) \\
\frac{d}{dz}D_0 &= 2\,zs\left[D_0(2)+D_0(4)\right] + \text{triangles}
\end{aligned}
$$

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D_0(i) = M^{-1}_{ij}\,d_j \quad \det M(z_{AT}) = 0
$$

where d_i contains D_0 or triangles.

ODE in $r = z^2 - z$

exa

ODE

$$\frac{d}{dr} D_0(r) = C_4^{-1}(r) \left[X(r) D_0(r) + D_{\text{rest}}(r) \right]$$

where C_4 is the Caley determinant.

We have

$$\frac{d}{dr} C_4 = -2 X(r) \quad \rightsquigarrow$$

$$D_0(r) = \frac{D_{\text{sing}}}{(r - r_{AT})^{1/2}} + D_{\text{reg}}(r)$$

ODE in $r = z^2 - z$

exa

ODE

$$\frac{d}{dr} D_0(r) = C_4^{-1}(r) \left[X(r) D_0(r) + D_{\mathrm{rest}}(r) \right]$$

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ODE for $H \to gg$; I

Amplitude

There is one form factor F_D that can be written, without reduction, as $F_D = \sum_i F_i$

$$F_1 = \frac{1}{2}\int d^n q \frac{M_H^2 - 2\, m_t^2}{(0)(1)(2)} \quad F_2 = -\, 2\int d^n q \frac{q \cdot p_1}{(0)(1)(2)}$$

$$(n-2)\, F_3 = \int \frac{d^n q}{(0)(1)(2)} \Big[(6-n)\, q^2 + \frac{16}{M_H^2}\, q \cdot p_1 q \cdot p_2 \Big]$$

Mapping

A mapping is needed; suppose that $M_H^2 < 4\, m_t^2$

ODE for $H \to gg$; I

Amplitude

There is one form factor $F_{\scriptscriptstyle D}$ that can be written, without reduction, as $F_{\scriptscriptstyle D} = \sum_i F_i$

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Mapping

A mapping is needed; suppose that $M_H^2 < 4\, m_t^2$

ODE for $H \to gg$; II

Mapping $p_{1,2} \to P_{1,2}$

$$T = \begin{pmatrix} z & 1-z \\ 1-z & z \end{pmatrix} \qquad B \;\to\; M_H^2\,\frac{C}{G}$$

$$C \;=\; r^2 + \mu_t^2\,(1+4\,r) \quad G = -\frac{1}{4}\,M_H^2\,(1+4\,r)$$

$$r = z\,(z-1) \ \text{ and } \ \mu_t^2\,M_H^2 = m_t^2$$

ODE for $H \to gg$; III

Solution

$$
\begin{aligned}
r_{AT} &= -2\mu_t^2 \left[1 + \sqrt{1 - \frac{1}{4\,\mu_t^2}}\,\right] \\
-\infty < \ r_{AT} &< -\frac{1}{2}
\end{aligned}
$$

Solution for

the amplitude is needed at $r = 0$

ODE for $H \to gg$; III

Solution

$$r_{AT} = -2\mu_t^2 \left[1 + \sqrt{1 - \frac{1}{4\,\mu_t^2}}\,\right]$$

$$-\infty < r_{AT} < -\frac{1}{2}$$

Solution for

the amplitude is needed at $r = 0$

ODE for $H \to gg$; IV

Less simple but non-singular (in R)

$$T_p = \begin{pmatrix} 1-z & z & 0 \\ 0 & 1-z & z \\ z & 0 & 1-z \end{pmatrix}$$

$$M_i^2 = \left(1 - \frac{z}{z_{AT}}\right) m^2 + \frac{z}{z_{AT}} \overline{M}_i^2$$

$\overline{M}_i$ free parameters to satisfy

$$P_1^2 < (M_1 + M_2)^2 \qquad P_2^2 > (M_2 - M_3)^2$$

$$(P_1 + P_2)^2 < (M_1 + M_3)^2$$

ODE for $H \to gg$; IV

Less simple but non-singular (in R)

$$T_p = \begin{pmatrix} 1-z & z & 0 \\ 0 & 1-z & z \\ z & 0 & 1-z \end{pmatrix}$$

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$\overline{M}_i$ free parameters to satisfy

$$\begin{aligned} P_1^2 &< (M_1 + M_2)^2 \quad P_2^2 > (M_2 - M_3)^2 \\ (P_1 + P_2)^2 &< (M_1 + M_3)^2 \end{aligned}$$

ODE for $H \to gg$; V

System of ODE

$$\frac{d}{dr} F_i = X_{ij} F_j + Y_j, \quad X, Y \text{ from IBP}$$

Trading F_3 for F_D $\rightsquigarrow$

$$\frac{d}{dr} F_D - X_{33} F_D + \left(X_{33} - \sum_i X_{i1} \right) F_1 + (X_{33} - X_{22}) F_2 = \sum_i Y_i$$

etc.

ODE for $H \to gg$; V

System of ODE

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Trading F_3 for F_D $\rightsquigarrow$

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etc.

ODE for $H \to gg$; VI

Boundary conditions at AT (factorization)

$$
\begin{aligned}
F_1 &\sim \frac{1}{2}\left(M_H^2 - 2\,m_t^2\right) C_0^{\mathrm{sing}}\,(z_{AT}) \\
F_2 &\sim M_H^2\, z_{AT}\, C_0^{\mathrm{sing}}\,(z_{AT}) \\
F_D &\sim \left[\frac{M_H^2}{8}\,(1+6\,r_{AT}) - m_t^2\,(1+4\,r_{AT})\right] C_0^{\mathrm{sing}}\,(z_{AT})
\end{aligned}
$$

ODE for $H \to gg$; VII

Solution

$$C_0(r) = g(r)\ln\frac{B_3(r)}{M_H^2} + h(r)$$

$$\frac{d}{dr}g = -\frac{2}{1+4r}\,g$$

Boundary

$$g(z_{AT}) = \frac{2\pi i}{M_H^2}\beta(z_{AT}) \qquad \beta^2(r) = 1 - 4\frac{\mu_t^2}{r}$$

the regular part $h(r)$ is computed numerically

ODE for $H \to gg$; VII

Solution

$$
\begin{aligned}
C_0(r) &= g(r)\ln\frac{B_3(r)}{M_H^2} + h(r) \\
\frac{d}{dr}g &= -\frac{2}{1+4\,r}\,g
\end{aligned}
$$

Boundary

$$
g\left(z_{AT}\right) = \frac{2\,\pi i}{M_H^2}\,\beta\left(z_{AT}\right) \qquad \beta^2(r) = 1 - 4\,\frac{\mu_t^2}{r}
$$

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ODE for $H \to gg$; VII

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$$
\begin{aligned}
C_0(r) &= g(r)\ln\frac{B_3(r)}{M_H^2} + h(r) \\
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$$

Boundary

$$
g\,(z_{AT}) = \frac{2\,\pi i}{M_H^2}\,\beta\,(\,z_{AT}\,) \quad \beta^2(r) = 1 - 4\,\frac{\mu_t^2}{r}
$$

the regular part $h(r)$ is computed numerically

General strategy, e.g. for $N = 4$

Define

$$D_{n_0 \dots n_3}(i) \quad = \quad \int d^n q \frac{q \cdot q^{n_0} \ \dots \ q \cdot P_3{}^{n_3}}{(0) \ \dots \ (i)^2 \ \dots \ (3)}$$

which satisfy

$$D_{n_0 \dots n_3}(i) \quad = \quad M_{ij}^{-1} d_{n_0 \dots n_3}(j) + d'_{n_0 \dots n_3}(i)$$

Then

find the minimal set of linear combinations $F = c\, D$ such that $\mathrm{Amp} = \sum F$ with $\{F\}$ closed under d/dz.

General strategy, e.g. for $N = 4$

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$$D_{n_0 \ldots n_3}(i) \quad = \quad \int d^n q \frac{q \cdot q^{n_0} \ \ldots \ q \cdot P_3{}^{n_3}}{(0) \ \ldots \ (i)^2 \ \ldots \ (3)}$$

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which satisfy

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Then

find the minimal set of linear combinations $F = c\, D$ such that $\mathrm{Amp} = \sum F$ with $\{F\}$ closed under d/dz.

Extension to multi-loop

Equal mass two-loop sunset à la Remiddi

- with $m = 1, p^2 = x$ shift $x \to z\, x$

$$x\, z\, (x\, z + 1)\, (x\, z + 9)\, \frac{d^2}{dz^2} S(x,z) =$$

$$P(x,z)\, \frac{d}{dz} S(x,z) + Q(x,z)\, S(x,z) + R(x,z)$$

AT solution

$z_{AT} = -x^{-1}$ (Warning: AT = pseudo-threshold); for different masses, map

$$m_i \quad \to \quad M_i = \frac{z - z_{AT}}{1 - z_{AT}}\, m_i + \frac{1 - z}{1 - z_{AT}}\, m$$

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Equal mass two-loop sunset à la Remiddi

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$$x\,z\,(x\,z+1)\,(x\,z+9)\,\frac{d^2}{dz^2}S(x,z) =$$

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Conclusions

Recapitulation

A proposal for solving a simpler problem by concentrating on a single variable deformation of the amplitude.

Refrain

In LL04 I mentioned the word anomalous threshold,
Peter Zerwas told me 'that shows your age'
perhaps he was wrong . . .
perhaps not . . .
but then others will fall away . . .

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