Singular Values of Elliptic Integrals in Quantum Field Theory

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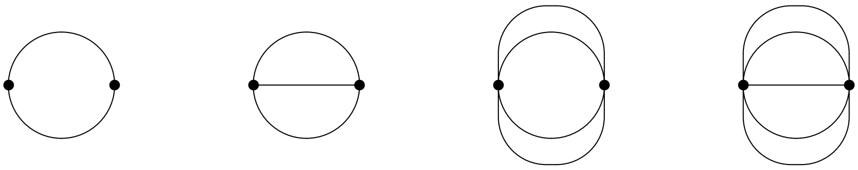
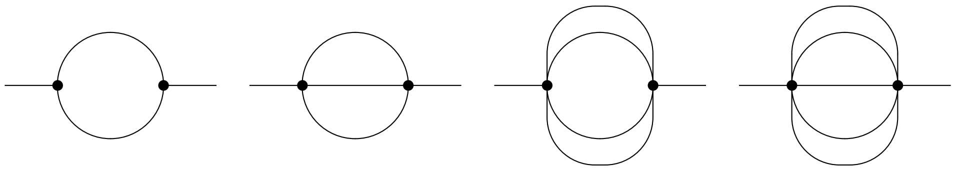
1 Introduction

I shall consider the analytical structure of the simplest massive multi-loop diagrams in QFT, namely vacuum bubbles with 2 vertices:



V_2 V_3 V_4 V_5

and the sunrise diagrams:



S_3 S_4 S_5 S_6

that result from cutting a single line, in 2 spacetime dimensions.

Was sind and was sollen die Zahlen?

2 Two Bessel functions

The no-loop sunrise diagram is, of course, just the propagator in momentum space:

$$S_2(a,w) = \int_0^\infty t\,K_0(at)J_0(wt)\,\mathrm{d}t = \frac{1}{a^2+w^2}$$

and the one-loop vacuum bubble

$$V_2(a,b) = \int_0^\infty t\,K_0(at)K_0(bt)\,\mathrm{d}t = \int_0^\infty w S_2(a,w)S_2(b,w)\,\mathrm{d}w = \frac{\log(a/b)}{a^2-b^2}$$

comes from integrating a pair of propagators, with l'Hôpital's rule giving $V_2(1,1) = \frac{1}{2}$ in the equal-mass case. The distribution

$$\int_0^\infty w J_0(wt_1)J_0(wt_2)\,\mathrm{d}w = 2\delta(t_1^2 - t_2^2)$$

takes us from momentum to position space, in 2 euclidean dimensions.

3 Three Bessel functions

The one-loop sunrise diagram

$$S_3(a,b,w) = \int_0^\infty t\, K_0(at)K_0(bt)J_0(wt)\, \mathrm{d}t = \int_{a+b}^\infty \frac{2v\, D_3(a,b,v)}{v^2+w^2}\, \mathrm{d}v$$

has a discontinuity

$$D_3(a,b,c) = \frac{1}{\sqrt{(a+b+c)(a-b+c)(a+b-c)(a-b-c)}}$$

given by the reciprocal of a Källén function. The dispersion integral is easily evaluated, to give

$$S_3(a,b,w) = 2\,\mathrm{arctanh}\left(\frac{\sqrt{w^2+(a-b)^2}}{\sqrt{w^2+(a+b)^2}}\right) D_3(a,b,\mathrm{i}w)$$

and the equal-mass on-shell result, with $a=b=1$ and $w=\mathrm{i}$, is

$$S_3(1,1,\mathrm{i}) = \frac{2\arctan(1/\sqrt{3})}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}} = L_{-3}(1)$$

where

$$L_{-3}(s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} = \sum_{k=0}^{\infty} \left(\frac{1}{(3k+1)^s} - \frac{1}{(3k+2)^s} \right)$$

is the Dirichlet L-function with the real character $\chi_{-3}(n)$ given by the Legendre–Jacobi–Kronecker symbol $(D|n)$ for discriminant $D = -3$.

The two-loop vacuum diagram

$$V_3(a,b,c) = \int_0^\infty t\, K_0(at) K_0(bt) K_0(ct)\, \mathrm{d}t$$

gives a symmetric combination

$$V_3(a,b,c) = \frac{L_3(a,b,c) + L_3(b,c,a) + L_3(c,a,b)}{4} D_3(a,b,c)$$

of dilogarithms

$$\begin{aligned} L_3(a,b,c) &= \mathrm{Li}_2\left(\frac{(a^2+b^2-c^2)D_3(a,b,c)+1}{(a^2+b^2-c^2)D_3(a,b,c)-1} \right) \\ &- \mathrm{Li}_2\left(\frac{(a^2+b^2-c^2)D_3(a,b,c)-1}{(a^2+b^2-c^2)D_3(a,b,c)+1} \right) \end{aligned}$$

found by Davydychev and Tausk. In the equal-mass case we obtain

$$V_3(1,1,1) = \frac{3}{4}\,L_{-3}(2) = \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{-1}{27}\right)^n \sum_{k=1}^{5}\frac{v_k}{(6n+k)^2}$$

with a vector of coefficients $v = [9, -9, -12, -3, 1]$ giving a highly convergent sum, discovered (and proven) in the course of my investigation of 3-loop vacuum diagrams in 4 dimensions, where quadrilogarithms of the sixth root of unity, $(1+\sqrt{-3})/2$, were encountered.

3.1 Enumerating walks on a honeycomb lattice

The integers

$$a_k = \sum_{p+q+r=k}\left(\frac{k!}{p!\,q!\,r!}\right)^2 = \sum_{j=0}^{k}\binom{k}{j}^2\binom{2j}{j}$$

enumerate closed walks, of length $2k$, on a two-dimensional hexagonal lattice. They are generated by the cube of a Bessel function:

$$I_0^3(2t) = \sum_{k=0}^{\infty} a_k \left(\frac{t^k}{k!}\right)^2$$

from which the recurrence relation

$$(k+1)^2 a_{k+1} - (10k^2 + 10k + 3) a_k + 9k^2 a_{k-1} = 0$$

results. Remarkably, they also determine the sequence of odd moments

$$s_{3,2k+1} = \int_0^\infty t^{2k+1} I_0(t)\, K_0^2(t)\, \mathrm{d}t$$

whose first term is $s_{3,1} = S_3(1,1,\mathrm{i}) = L_{-3}(1)$. To prove this, one shows that $s_{3,3} = \frac{4}{3}\, s_{3,1}$, by differentiations of $S_3(1,1,w)$ with respect to the magnitude w of the external momentum, before going to the on-shell point $w = \mathrm{i}$. Then the result

$$s_{3,2k+1} = L_{-3}(1) \left(\frac{2^k k!}{3^k} \right)^2 a_k$$

follows from the recursion for $s_{3,2k+1}$. Later we will see remarkable relations between Feynman diagrams and lattice Green functions.

4 Four Bessel functions

The 2-loop sunrise diagram S_4 (which is supposed to look like sunrise) may be constructed by folding one instance of S_3 with the discontinuity of another, to obtain

$$S_4(a,b,c,w) = \int_{a+b}^{\infty} 2uD_3(a,b,u)S_3(u,c,w)\,\mathrm{d}u$$

from which we obtain an evaluation of the equal-mass on-shell value

$$S_4(1,1,1,\mathrm{i}) = \int_2^{\infty} \frac{4\,\mathrm{arctanh}\left(\sqrt{\frac{u-2}{u+2}}\right)}{u(u^2-4)}\,\mathrm{d}u = \int_0^1 \frac{2y\log(y)}{y^4-1}\,\mathrm{d}y = \frac{\pi^2}{16}$$

by the substitution $u = y + 1/y$.

The 3-loop vacuum diagram

$$V_4(a,b,c,d) = \int_0^{\infty} wS_3(a,b,w)S_3(c,d,w)\,\mathrm{d}w$$

obtained by integrating over a pair of one-loop graphs and is easily evaluated in the equal-mass case by the substitution $w = 1/y - y$:

$$V_4(1,1,1,1) = \int_0^{\infty} \frac{4\,\mathrm{arctanh}^2\left(\frac{w}{\sqrt{w^2+4}}\right)}{w(w^2+4)}\,\mathrm{d}w = \int_0^1 \frac{4y\log^2(y)}{1-y^4}\,\mathrm{d}y = \frac{7}{8}\zeta(3)$$

4.1 Enumerating walks on a diamond lattice

The integers

$$b_k = \sum_{p+q+r+s=k} \left(\frac{k!}{p!\,q!\,r!\,s!}\right)^2 = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2k-2j}{k-j} \binom{2j}{j}$$

enumerate closed walks, of length $2k$, on a three-dimensional diamond lattice. They are generated by the fourth power of a Bessel function:

$$I_0^4(2t) = \sum_{k=0}^{\infty} b_k \left(\frac{t^k}{k!}\right)^2$$

and determine the sequence of odd moments

$$s_{4,2k+1} = \int_0^\infty t^{2k+1} I_0(t)\, K_0^3(t)\, \mathrm{d}t$$

whose first term is $s_{4,1} = S_4(1,1,1,\mathrm{i}) = \pi^2/16$. The result

$$s_{4,2k+1} = \frac{\pi^2}{16} \left(\frac{k!}{4^k}\right)^2 b_k$$

then follows from evaluation of $s_{4,3} = \frac{1}{4} s_{4,1}$ and the use of recurrence relations. This relation to the diamond lattice will be exploited as we move up to 5 Bessel functions.

5 Five Bessel functions

The 3-loop sunrise diagram may be written as an integral

$$S_5(a,b,c,d,w) = \int_{a+b+c}^{\infty} 2v\, D_4(a,b,c,v) S_3(v,d,w)\,\mathrm{d}v$$

over the discontinuity D_4 of

$$S_4(a,b,c,w) = \int_{a+b+c}^{\infty} \frac{2v\, D_4(a,b,c,v)}{v^2+w^2}\,\mathrm{d}v$$

which is given by a complete elliptic integral $\mathbf{K}$ of the first kind, and hence by an arithmetic-geometric mean:

$$\begin{aligned} D_4(a,b,c,d) &= \int_{a+b}^{d-c} 2u\, D_3(a,b,u) D_3(u,c,d)\,\mathrm{d}u \\ &= \frac{2\mathbf{K}\left(\frac{Q(a,b,c,-d)}{Q(a,b,c,d)}\right)}{Q(a,b,c,d)} = \frac{\pi}{\mathrm{AGM}\left(Q(a,b,c,d),\ 4\sqrt{abcd}\right)} \end{aligned}$$

where

$$Q(a,b,c,d) = \sqrt{(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)}$$

Then in the equal-mass on-shell case we obtain

$$S_5(1,1,1,1,\mathrm{i}) = \int_0^\infty t\, I_0(t)\, K_0^4(t)\,\mathrm{d}t = \int_0^{\frac{1}{3}} \frac{2D(y)}{\sqrt{1-4y^2}} \operatorname{arctanh}\left(\frac{\sqrt{1-2y}}{\sqrt{1+2y}}\right) \mathrm{d}y$$

as an integral over

$$D(y) = \frac{2\pi y}{\mathrm{AGM}\left(\sqrt{(1+3y)(1-y)^3}, \sqrt{16y^3}\right)}$$

Similar the equal-mass 4-loop vacuum diagram gives

$$V_5(1,1,1,1,1) = \int_0^\infty t\, K_0^5(t)\,\mathrm{d}t = \int_0^{\frac{1}{3}} \frac{D(y)L(y)}{\sqrt{1-4y^2}}\,\mathrm{d}y$$

where $D(y)$ is now folded with the dilogarithm

$$L(y) = -\mathrm{Li}_2\left(\frac{1-\sqrt{1-4y^2}}{2}\right) + \frac{1}{2}\log^2\left(\frac{1-\sqrt{1-4y^2}}{2}\right) - \log^2(y) + \frac{\pi^2}{12}$$

I have computed 200,000 decimal digits of V_5.

5.1 Conjectural evaluations with one external particle

$$\frac{\sqrt{5}}{2}\int_0^\infty t\,I_0(t)K_0^4(t)\,\mathrm{d}t \stackrel{?}{=} \frac{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{480}$$

$$\frac{\sqrt{5}}{2}\int_0^\infty t^3 I_0(t)K_0^4(t)\,\mathrm{d}t \stackrel{?}{=} \frac{13\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{30^3} - \frac{\Gamma\left(\frac{7}{15}\right)\Gamma\left(\frac{11}{15}\right)\Gamma\left(\frac{13}{15}\right)\Gamma\left(\frac{14}{15}\right)}{15}$$

5.2 Proven evaluations with two external particles

$$\frac{\pi}{\sqrt{3}}\int_0^\infty t\,I_0^2(t)K_0^3(t)\,\mathrm{d}t = \frac{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{480}$$

$$\frac{\pi}{\sqrt{3}}\int_0^\infty t^3 I_0^2(t)K_0^3(t)\,\mathrm{d}t = \frac{13\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{30^3} + \frac{\Gamma\left(\frac{7}{15}\right)\Gamma\left(\frac{11}{15}\right)\Gamma\left(\frac{13}{15}\right)\Gamma\left(\frac{14}{15}\right)}{15}$$

5.3 Conjectural sum rule

$$\int_0^\infty t\, I_0(t) \left(K_0(t) - \frac{2\pi}{\sqrt{15}} I_0(t) \right) K_0^3(t)\, \mathrm{d}t \stackrel{?}{=} 0$$

with a dispersive version

$$\int_0^{\frac{1}{3}} \frac{D(y)}{\sqrt{1-4y^2}} \left(2 \operatorname{arctanh}\left(\frac{\sqrt{1-2y}}{\sqrt{1+2y}} \right) - \frac{2\pi}{\sqrt{15}} \right) \mathrm{d}y \stackrel{?}{=} 0$$

and a non-dispersive version

$$\int_0^\infty \frac{E(x)}{\sqrt{4+x^2}} \left(2 \operatorname{arcsinh}\left(\frac{x}{2} \right) - \frac{2\pi}{\sqrt{15}} \right) \mathrm{d}x \stackrel{?}{=} 0$$

where

$$E(x) = \frac{\pi}{\mathrm{AGM}(z, z^*)} = \frac{\pi}{\mathrm{AGM}(\Re z, |z|)}\,, \quad z = \sqrt{(1-\mathrm{i}x)(3+\mathrm{i}x)^3}$$

5.4 A modular identity from QFT

$$\frac{\theta_2^3(q^{3/2})}{\theta_2(q^{1/2})} = \theta_3^2(q) \left\{ \frac{\theta_3(q^3)}{\theta_3(q)} - \frac{\theta_3^3(q^3)}{\theta_3^3(q)} \right\}$$

where

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}, \quad \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

This was derived by combining two evaluations in terms of $E(x)$ with a result from Davydychev and Delbourgo for the area of the Dalitz plot.

5.5 Doubling of internal and external masses

$$\int_0^\infty t\, I_0^2(t) K_0^2(t) K_0(2t)\, \mathrm{d}t = \frac{1}{3} \int_0^\infty t\, I_0(2t) I_0(t) K_0^3(t)\, \mathrm{d}t = \frac{\Gamma^6\left(\frac{1}{3}\right)}{64\pi^2 2^{2/3}}$$

proven by combining contour integration with an evaluation by Watson in 1939 of a lattice Green function.

5.6 Lattice Green functions

$$W_j(z) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3}{1 - z\,w_j(\theta_1, \theta_2, \theta_3)}$$

face-centred cubic:

$$w_1(\theta_1, \theta_2, \theta_3) = \frac{\cos\theta_1\cos\theta_2 + \cos\theta_2\cos\theta_3 + \cos\theta_3\cos\theta_1}{3}$$

simple cubic:

$$w_2(\theta_1, \theta_2, \theta_3) = \frac{\cos\theta_1 + \cos\theta_2 + \cos\theta_3}{3}$$

body-centred cubic:

$$w_3(\theta_1, \theta_2, \theta_3) = \cos\theta_1\cos\theta_2\cos\theta_3$$

diamond:

$$w_4(\theta_1, \theta_2, \theta_3) = \frac{1 + \cos\theta_1\cos\theta_2 + \cos\theta_2\cos\theta_3 + \cos\theta_3\cos\theta_1}{4}$$

In 1939, Watson evaluated $W_j(1)$ for $j = 1, 2, 3$ in terms of squares of elliptic integrals at singular values. QFT chooses the diamond value $W_4\left(\frac{1}{4}\right)$.

5.7 Expansions of lattice Green functions

$$
\begin{aligned}
W_1(z) &= \sum_{k=0}^{\infty} f_k \left(\frac{z}{12}\right)^k \\
W_2(z) &= \sum_{k=0}^{\infty} \binom{2k}{k} a_k \left(\frac{z}{6}\right)^{2k} \\
W_3(z) &= \sum_{k=0}^{\infty} \binom{2k}{k}^3 \left(\frac{z}{8}\right)^{2k} \\
W_4(z) &= \sum_{k=0}^{\infty} b_k \left(\frac{z}{16}\right)^k
\end{aligned}
$$

with integers

$$
\begin{aligned}
a_k &= \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \\
b_k &= \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2k-2j}{k-j} \binom{2j}{j} \\
f_k &= \sum_{j=0}^{k} \binom{k}{j} (-4)^{k-j} b_j
\end{aligned}
$$

5.8 Relations between lattice Green functions

$$\begin{aligned}\left(\sum_{k=0}^{\infty} a_k\,(-x)^k\right)^2 &= \sum_{k=0}^{\infty} f_k\,\frac{x^k}{(1+3x)^{2k+2}}\\ &= \sum_{k=0}^{\infty} a_k\,\frac{\binom{2k}{k}\,(-x(1+x)(1+9x))^k}{((1-3x)(1+3x))^{2k+1}}\\ &= \sum_{k=0}^{\infty} \frac{\left(\binom{2k}{k}x^k\right)^3}{((1+x)^3(1+9x))^{k+\frac{1}{2}}}\\ &= \sum_{k=0}^{\infty} b_k\,\frac{x^k}{((1+x)(1+9x))^{k+1}}\end{aligned}$$

for sufficiently small x. These relations between lattice Green functions originate, at bottom, in identities obtained by Wilfrid Norman Bailey in 1936, where he generalized Clausen relations of the form $({}_2\mathrm{F}_1)^2 \sim {}_3\mathrm{F}_2$ used in my work on D-dimensional 3-loop QED, and more recently by Grozin in 3-loop HQET. More generally, Bailey's identities enable us to relate elliptic integrals to Appell F_4 double series. In short,

$$\text{Elliptic}^2 \sim \text{Honeycomb}^2 \sim \text{Cubic} \sim \text{Diamond} \sim \text{QFT}$$

5.9 Singular value of an elliptic integral in QFT

In particular, I have proven that

$$K_{15}^2 = \frac{8}{\sqrt{15}-\sqrt{3}} \int_0^\infty t\, I_0^2(t) K_0^3(t)\, \mathrm{d}t$$

where $K_{15} = \mathbf{K}(k_{15})$ is evaluated at the 15th singular value and

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\,\frac{1}{\mathrm{AGM}(1,k')}$$

with $k' = \sqrt{1-k^2}$.
The Nth singular value, k_N, yields the algebraic relation

$$K_N = \mathbf{K}(k_N) = \frac{1}{\sqrt{N}}\,\mathbf{K}(k_N')$$

in which case K_N is reducible to Γ values and an algebraic number. For QFT, at $N = 15$, the algebraic number theory involves only surds:

$$\frac{1}{\sqrt{3}} \int_0^\infty t\, I_0^2(t) K_0^3(t)\, \mathrm{d}t = \frac{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{480\pi} = \frac{\sqrt{5}-1}{8}\,K_{15}^2$$

5.10 The Chowla–Selberg formula

More generally, for every fundamental discriminant $D < -4$,

$$\prod_{-D>k>0} \left[\Gamma\left(\frac{k}{-D}\right)\right]^{(D|k)} = (-2\pi D)^{h(D)} \prod_{[a,b,c]\in H(D)} \frac{1}{a}\left|\eta\left(\frac{b+\sqrt{D}}{2a}\right)\right|^4$$

where the product runs over the strict equivalence classes $[a,b,c]$ of primitive integral binary quadratic forms $ax^2+bxy+cy^2$ with discriminant $D=b^2-4ac$ and

$$\eta(z) = \exp(\pi \mathrm{i} z/12)\prod_{k=1}^{\infty}(1-\exp(2\pi \mathrm{i} k z))$$

is Dirichlet's eta function. The equivalence classes form an Abelian group $H(D)$, by Gauss's composition of quadratic forms, and the order of this group is the class number $h(D)$ of the number field $Q(\sqrt{D})$.

QFT has chosen, via its connection to the diamond lattice, an algebraic multiple of a product of Γ values with $-D=N=15$. I am now computing corresponding algebraic multiples for singular values with $N \gg 15$. The state of the art (as of last week) is $N = 1,242,763$. The previous number theory record was $N = 210$.

6 Six Bessel functions

S_6 yields a number with many relations to previous and other structures:

$$
\begin{aligned}
\frac{12}{\pi^4}\int_0^\infty t\, I_0(t)K_0^5(t)\,\mathrm{d}t &= \int_0^\infty \exp(-4t)I_0^4(t)\,\mathrm{d}t = \sum_{n=0}^\infty \frac{\binom{2n}{n}b_n}{4^{3n+1}} \\
&= \frac{4}{\pi^2}\int_0^\infty t\, I_0^3(t)K_0^3(t)\,\mathrm{d}t = \frac{4}{\pi^2}\int_0^{\frac{1}{3}} D(y)\widetilde{D}(y)\,\mathrm{d}y \\
&= \frac{2}{\pi^2}\int_0^\infty \exp(-4t)I_0^2(t)K_0^2(t)\,\mathrm{d}t = \frac{1}{\sqrt{3}\pi^3}\int_{\frac{1}{3}}^1 \frac{D(y)D\left(\frac{1}{3y}\right)}{y}\,\mathrm{d}y \\
&= \frac{1}{3\pi}\int_{\frac{3}{5}}^1 \frac{W_2(z)}{\sqrt{(1-z)(z-3/5)}}\,\mathrm{d}z
\end{aligned}
$$

with K_2^2 in $W_2\left(\frac{3}{5}\right)$, K_6^2 in $W_2(1)$ and AGM's in

$$
\begin{aligned}
D(y) &= \frac{2\pi y}{\mathrm{AGM}\left(\sqrt{(1+3y)(1-y)^3}, \sqrt{16y^3}\right)} \\
\widetilde{D}(y) &= \frac{1}{\mathrm{AGM}\left(\sqrt{(1+3y)(1-y)^3}, \sqrt{(1-3y)(1+y)^3}\right)}
\end{aligned}
$$

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Stefano Laporta provided the stimulus for this work, by discovering the 15th singular value in 4-dimensional QFT.

I am grateful to the organising committee for this successful meeting and in particular to Tord Riemann, for his Shakespearian comment

Ist es auch Wahnsinn,
So hat es doch Methode

which seems to me to apply rather well to the work reported in

Bailey, Borwein, Broadhurst, Glasser, arXiv.org/0801.0891
Broadhurst, arXiv/0801.4813