Loops and Legs in Quantum Field Theory

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Unitarity Cuts for One-Loop Amplitudes

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On-shell approach for loop amplitudes

- Classic unitarity cuts: loops from trees
- Unitarity method: assume results of reduction
- Generalized unitarity
- D-dimensional unitarity
- Recursion relations for amplitudes

Various other talks will address on-shell methods:

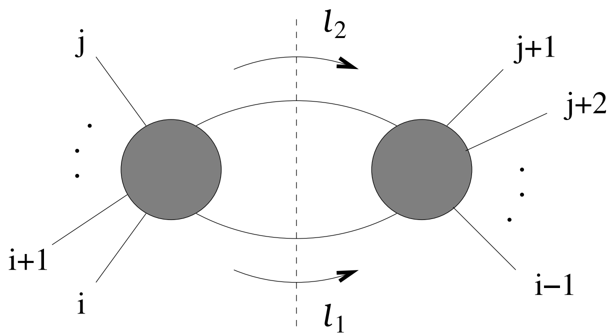
Papadopoulos, Mastrolia, Dunbar, Boels, Badger, Maitre, Kosower

Unitarity Cuts

$$\Delta A^{1-\text{loop}} = \int d\mu \; A^{\text{tree}}_{\text{Left}} \times A^{\text{tree}}_{\text{Right}}$$

where

$$d\mu = d^4\ell_1 \, d^4\ell_2 \, \delta^{(4)}(\ell_1 + \ell_2 - K) \, \delta(\ell_1^2) \, \delta(\ell_2^2)$$

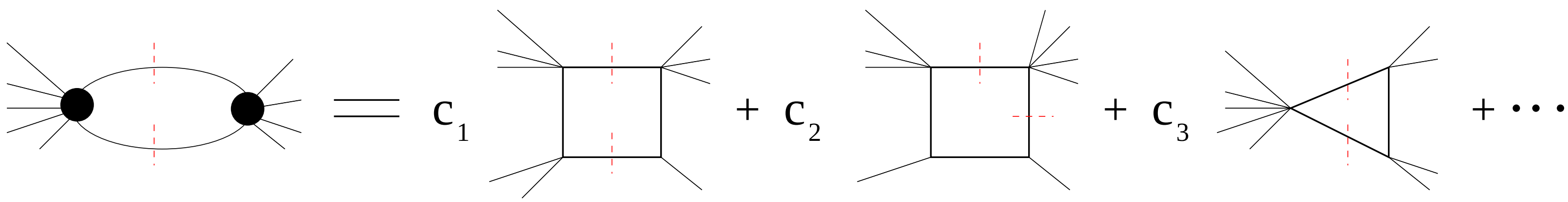


By unitarity, this is the discontinuity of the amplitude across a branch cut, in a kinematic region selecting the cut momentum K. (Cutkosky 1960)

Amplitudes from unitarity cuts

$$C = \Delta A_n^{1-\mathrm{loop}} = \sum c\, \Delta I$$

Tree level data.



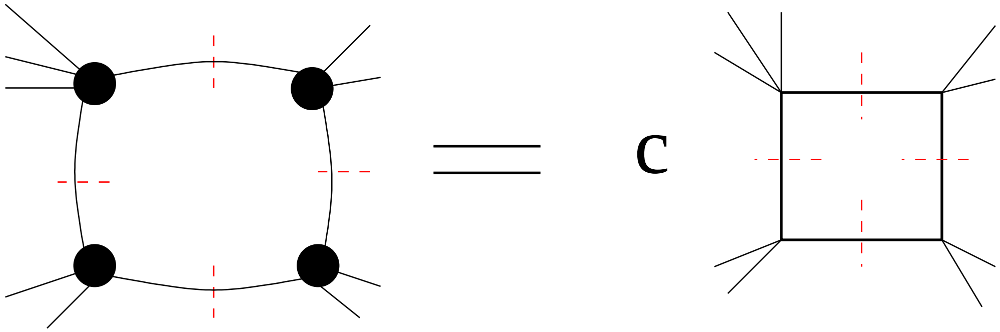
Matching 4-dimensional cuts can suffice to determine reduction coefficients!

(Bern, Dixon, Dunbar, Kosower 1994)

But: we get several coefficients together in the same equation.

Box Coefficients from Quadruple Cuts

(RB, Cachazo, Feng)



Generalized Unitarity: Try replacing all four propagators by delta functions.

This operation isolates any given box.

In four dimensions, these four delta functions localize the integral completely. This computation is very easy!

The loop momentum solution

The box coefficients computed from quadruple cuts are given by

$$c = \frac{1}{2} \sum_{\mathcal{S}} A_1^{\mathrm{tree}} A_2^{\mathrm{tree}} A_3^{\mathrm{tree}} A_4^{\mathrm{tree}}$$

$\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines.

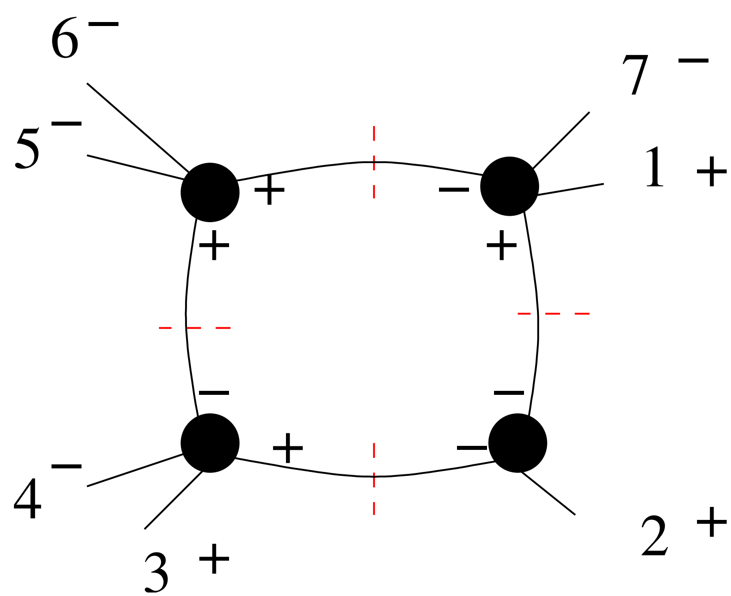
$$\mathcal{S} = \{\, \ell \,|\, \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0 \}$$

Can these equations always be solved?

In complexified momentum space, there are exactly 2 solutions.

(Note: nonvanishing 3-point amplitudes.)

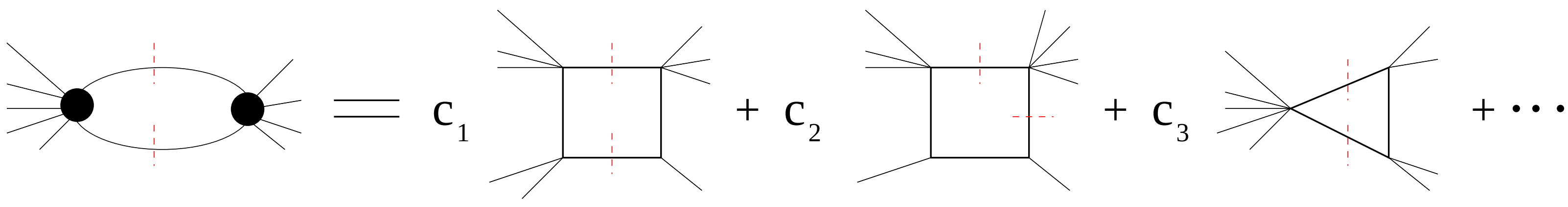
Example: Box Coefficient from Quadruple Cut



$$\text{coeff} = \frac{1}{2} \frac{\langle \ell_1\, \ell_4 \rangle^3}{\langle \ell_1\, 2 \rangle \langle 2\, \ell_4 \rangle} \frac{\langle 4\, \ell_2 \rangle^3}{\langle \ell_2\, \ell_1 \rangle \langle \ell_1\, 3 \rangle \langle 3\, 4 \rangle} \frac{\langle 5\, 6 \rangle^3}{\langle 6\, \ell_3 \rangle \langle \ell_3\, \ell_2 \rangle \langle \ell_2\, 5 \rangle} \frac{\langle \ell_3\, 7 \rangle^3}{\langle 7\, 1 \rangle \langle 1\, \ell_4 \rangle \langle \ell_4\, \ell_3 \rangle}$$

$$= -\frac{[1\,2]^3\, [2\,3]^3\, \langle 5\,6 \rangle^3}{[7\,1]\, [3\,4]\, \langle 5|P_{3,4}|2]\, \langle 6|P_{7,1}|2]\, [2|P_{3,4}P_{5,6}|7]\, [2|P_{7,1}P_{5,6}|4]}$$

Integral Coefficients from Unitarity Cuts



RHS: cuts of master integrals.

Extract coefficients by matching cuts.

Cuts of 4-d Master Integrals

$$\Delta I_2 = \int \langle \ell\, d\ell\rangle\, [\ell\, d\ell]\, \frac{K^2}{\langle \ell | K | \ell]^2}$$

$$\Delta I_3 = \int \langle \ell\, d\ell\rangle\, [\ell\, d\ell]\, \frac{1}{\langle \ell | K | \ell]\, \langle \ell | Q | \ell]}$$

$$\Delta I_4 = \int \langle \ell\, d\ell\rangle\, [\ell\, d\ell]\, \frac{1}{K^2}\, \frac{1}{\langle \ell | Q_1 | \ell]\, \langle \ell | Q_2 | \ell]}$$

$$Q_j \equiv -K_j + \frac{K_j^2}{K^2} K$$

Cutting the Amplitude in 4d

$$C \;=\; c\int d^4\ell \, \frac{\prod_{i=1}^{k+n}(-2\ell\cdot P_i)}{\prod_{j=1}^{k}(\ell-K_j)^2}\;\delta(\ell^2)\;\delta((\ell-K)^2)$$

We define the following vectors:

$$\begin{aligned} Q_j &= -K_j + \frac{{K_j}^2}{K^2}K, \\ R_i &= -P_i. \end{aligned}$$

Then the cut integral can be written as follows:

$$C \;=\; c\int \langle \ell\; d\ell\rangle[\ell\; d\ell]\,\frac{(K^2)^{n+1}}{\langle \ell|K|\ell]^{n+2}}\,\frac{\prod_{i=1}^{k+n}\langle \ell|R_i|\ell]}{\prod_{j=1}^{k}\langle \ell|Q_j|\ell]}$$

Finish by identifying poles and residues.

We have given the results in general form. (RB, Feng)

See also: Forde

Box coefficients

$$C[K_r, K_s, K] = \frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{j=1}^{k+n}\langle P_{sr,1}|R_j|P_{sr,2}]}{\langle P_{sr,1}|K|P_{sr,2}]^{n+2}\prod_{t=1,t\neq i,j}^{k}\langle P_{sr,1}|Q_t|P_{sr,2}]} + \{P_{sr,1}\leftrightarrow P_{sr,2}\}\right)$$

$$P_{sr,1} = Q_s + \left(\frac{-2Q_s\cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2}\right)Q_r$$

$$P_{sr,2} = Q_s + \left(\frac{-2Q_s\cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2}\right)Q_r$$

$$\Delta_{sr} = (2Q_s\cdot Q_r)^2 - 4Q_s^2Q_r^2$$

Triangle coefficients

$$C[K_s, K] = \frac{(K^2)^{1+n}}{2} \frac{1}{(\sqrt{\Delta_s})^{n+1}} \frac{1}{(n+1)! \langle P_{s,1}\, P_{s,2} \rangle^{n+1}}$$
$$\times \frac{d^{n+1}}{d\tau^{n+1}} \left(\frac{\prod_{j=1}^{k+n} \langle P_{s,1} - \tau P_{s,2} | R_j Q_s | P_{s,1} - \tau P_{s,2} \rangle}{\prod_{t=1, t\neq s}^{k} \langle P_{s,1} - \tau P_{s,2} | Q_t Q_s | P_{s,1} - \tau P_{s,2} \rangle} + \{ P_{s,1} \leftrightarrow P_{s,2} \} \right) \Bigg|_{\tau=0} .$$

$$P_{s,1} = Q_s + \left(\frac{-2Q_s \cdot K + \sqrt{\Delta_s}}{2K^2} \right) K$$
$$P_{s,2} = Q_s + \left(\frac{-2Q_s \cdot K - \sqrt{\Delta_s}}{2K^2} \right) K$$
$$\Delta_s = (2Q_s \cdot K)^2 - 4Q_s^2 K^2$$

Bubble coefficients

$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(\mathcal{B}^{(r;a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0}$$

$$\mathcal{B}^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n![\eta|\eta' K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j (K + s\eta) | \ell \rangle}{\langle \ell\, \eta \rangle^{n+1} \prod_{p=1}^{k} \langle \ell | Q_p (K + s\eta) | \ell \rangle} \right) \Bigg|_{|\ell\rangle \to |K - \tau\eta'|\eta],\tau=0}$$

$$\mathcal{B}^{(r;b;1)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,1} - \tau P_{r,2} | \eta | P_{r,1} \right]^{t+1}}{\left\langle P_{r,1} - \tau P_{r,2} | K | P_{r,1} \right]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,1} - \tau P_{r,2} | Q_r \eta | P_{r,1} - \tau P_{r,2} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,1} - \tau P_{r,2} | R_j (K + s\eta) | P_{r,1} - \tau P_{r,2} \rangle}{\langle P_{r,1} - \tau P_{r,2} | \eta K | P_{r,1} - \tau P_{r,2} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,1} - \tau P_{r,2} | Q_p (K + s\eta) | P_{r,1} - \tau P_{r,2} \rangle} \right) \Bigg|_{\tau=0}$$

$$\mathcal{B}^{(r;b;2)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,2} - \tau P_{r,1} | \eta | P_{r,2} \right]^{t+1}}{\left\langle P_{r,2} - \tau P_{r,1} | K | P_{r,2} \right]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,2} - \tau P_{r,1} | Q_r \eta | P_{r,2} - \tau P_{r,1} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,2} - \tau P_{r,1} | R_j (K + s\eta) | P_{r,2} - \tau P_{r,1} \rangle}{\langle P_{r,2} - \tau P_{r,1} | \eta K | P_{r,2} - \tau P_{r,1} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,2} - \tau P_{r,1} | Q_p (K + s\eta) | P_{r,2} - \tau P_{r,1} \rangle} \right) \Bigg|_{\tau=0}$$

D-dimensional unitarity

Orthogonal decomposition within Four Dimensional Helicity (FDH) scheme.

(Mahlon; Bern, Chalmers; Bern, Morgan)

$$\int d^{4-2\epsilon}\ell_{4-2\epsilon} = \int d^{-2\epsilon}\ell_{-2\epsilon}\int d^4\ell_4$$
$$= \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)}\int_0^1 du\, u^{-1-\epsilon}\int d^4\widetilde{\ell}.$$

where $\ell^2_{-2\epsilon} = \frac{K^2}{4}u.$

The integral over u will remain. The u-dependence is controlled.

(Anastasiou, RB, Feng, Kunszt, Mastrolia)

$$\Delta A = \int_0^1 du\, u^{-1-\epsilon}\int d^4\ell\ \delta(\ell^2)\,\delta(\sqrt{1-u}\,K^2 - 2K\cdot\ell)$$

Problem is reduced to a standard 4-d cut integral.

(Cf. methods by Ossola, Papadopoulos, Pittau; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov)

D-dimensional unitarity algorithm

$$\Delta A = \int_0^1 du\, u^{-1-\epsilon} \int d^4\ell\ \delta(\ell^2)\, \delta(\sqrt{1-u}\, K^2 - 2K\cdot\ell)$$

1. 4d cut: get u-dependent coefficients of master integrals.

   u-dependence is polynomial. (RB, Feng, Yang; RB, Feng, Mastrolia)

2. Treat polynomial u-dependence of integrand. Two choices:

   (a) For each term in the polynomial, use shift identities to get coefficients of 4d master integrals.

   (b) Use dimensionally shifted master integrals.

   The u-integral is not done explicitly.

Here, u is like the $\widetilde{q}^2$ of Ossola, Papadopoulos, Pittau; or the s_e^2 of Giele, Kunszt, Melnikov.

Coefficients are polynomials in u

The (maximum) degrees are the following:

Pentagon: 0

Box: $[(n+2)/2]$

Triangle: $[(n+1)/2]$

Bubble: $[n/2]$

$$C = c\int d^{4-2\epsilon}p \, \frac{\prod_{i=1}^{k+n}(-2p\cdot P_i)}{\prod_{j=1}^{k}(p-K_j)^2}\delta(p^2)\delta((p-K)^2)$$

The terms in the polynomials

Analytically,

$$C(u) = \sum_{s=0}^{d} \left(\frac{1}{s!} \frac{d^s C(u)}{du^s} \bigg|_{u \to 0} \right) u^s .$$

Or, numerically,

1. Define

$$C_k \equiv C(u_k),$$

for $(k = 0, ..., d-1)$, where

$$u_k = \; e^{-2\pi i k/d} \; .$$

2. Then,

$$C(u) = \sum_{s=0}^{d} \left(\frac{1}{d} \sum_{k=0}^{d-1} \, C_k \; e^{2\pi i s k/d} \right) u^s .$$

Dimensional shift identities

(Anastasiou, RB, Feng, Kunszt, Mastrolia)

$$\text{Bub}^{(n)} = F^{(n)}_{2\to 2}\text{Bub}^{(0)}$$

$$\text{Tri}^{(n)} = F^{(n)}_{3\to 3}\text{Tri}^{(0)} + F^{(n)}_{3\to 2}\text{Bub}^{(0)}$$

$$\text{Box}^{(n)} = F^{(n)}_{4\to 4}\text{Box}^{(0)} + F^{(n)}_{4\to 3}\text{Tri}^{(0)} + F^{(n)}_{4\to 2}\text{Bub}^{(0)}$$

$$F^{(n)}_{2\to 2} = \frac{(-\epsilon)_{\frac{3}{2}}}{(n-\epsilon)_{\frac{3}{2}}}, \quad F^{(n)}_{3\to 3} = \frac{-\epsilon}{n-\epsilon}(1-Z^2)^n,$$

$$F^{(n)}_{4\to 4} = \frac{(-\epsilon)_{\frac{1}{2}}}{(n-\epsilon)_{\frac{1}{2}}}\left(\frac{B}{A}\right)^n,$$

$$F^{(n)}_{3\to 2} = \frac{(-\epsilon)_{\frac{3}{2}}}{n-\epsilon}\sum_{k=1}^{n}\frac{2Z(1-Z^2)^{n-k}}{(k-\epsilon)_{\frac{1}{2}}}$$

$$F^{(n)}_{4\to j} = \frac{D+(Z^2-1)C}{(n-\epsilon)_{\frac{1}{2}}ZA}\sum_{k=1}^{n}\left(\frac{B}{A}\right)^{n-k}(k-1-\epsilon)_{\frac{1}{2}}F^{(k-1)}_{3\to j}$$

Cuts of D-dimensional Master Integrals

$$\Delta I_2 \;=\; \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle\, [\ell\, d\ell] \quad K^2 \sqrt{1-u} \frac{1}{\langle \ell | K | \ell]^2}$$

$$\Delta I_3 \;=\; \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle\, [\ell\, d\ell] \quad \sqrt{1-u} \frac{1}{\langle \ell | K | \ell]\, \langle \ell | Q | \ell]}$$

$$\Delta I_4 \;=\; \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle\, [\ell\, d\ell] \quad \frac{\sqrt{1-u}}{K^2} \frac{1}{\langle \ell | Q_1 | \ell]\, \langle \ell | Q_2 | \ell]}$$

$$\Delta I_4 = \frac{1}{2K^2} \frac{\sqrt{1-u}}{\sqrt{(Q_1 \cdot Q_2)^2 - Q_1^2 Q_2^2}} \ln \frac{Q_1 \cdot Q_2 + \sqrt{(Q_1 \cdot Q_2)^2 - Q_1^2 Q_2^2}}{Q_1 \cdot Q_2 - \sqrt{(Q_1 \cdot Q_2)^2 - Q_1^2 Q_2^2}}$$

$$\begin{aligned} \Delta I_5 = \frac{\sqrt{1-u}}{(K^2)^2} \Bigg(& \frac{S[Q_3, Q_2, Q_1, K]}{4\sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}} \ln \frac{Q_3 \cdot Q_2 + \sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}}{Q_3 \cdot Q_2 - \sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}} \\ & + \frac{S[Q_3, Q_1, Q_2, K]}{4\sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}} \ln \frac{Q_3 \cdot Q_1 + \sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}}{Q_3 \cdot Q_1 - \sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}} \\ & + \frac{S[Q_2, Q_1, Q_3, K]}{4\sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}} \ln \frac{Q_2 \cdot Q_1 + \sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}}{Q_2 \cdot Q_1 - \sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}} \Bigg) . \end{aligned}$$

i

$$S[Q_i, Q_j, Q_k, K] = \frac{T_1}{T_2}$$

$$T_1 = -8 \det \begin{pmatrix} K \cdot Q_k & Q_i \cdot K & Q_j \cdot K \\ Q_i \cdot Q_k & Q_i^2 & Q_i \cdot Q_j \\ Q_j \cdot Q_k & Q_i \cdot Q_j & Q_j^2 \end{pmatrix} ; \quad T_2 = -4 \det \begin{pmatrix} Q_k^2 & Q_i \cdot Q_k & Q_j \cdot Q_k \\ Q_i \cdot Q_k & Q_i^2 & Q_i \cdot Q_j \\ Q_j \cdot Q_k & Q_i \cdot Q_j & Q_j^2 \end{pmatrix} .$$

Cutting the Amplitude in D dimensions

$$C = c \int d^{4-2\epsilon}p \, \frac{\prod_{i=1}^{k+n}(-2p \cdot P_i)}{\prod_{j=1}^{k}(p - K_j)^2} \; \delta(p^2) \; \delta((p-K)^2)$$

Let us define the following four-vectors:

$$Q_j = -(\sqrt{1-u})K_j + \frac{{K_j}^2 - (1-\sqrt{1-u})(K_j \cdot K)}{K^2} K,$$

$$R_i = -(\sqrt{1-u})P_i - \frac{(1-\sqrt{1-u})(P_i \cdot K)}{K^2} K.$$

Then the cut integral can be written as follows:

$$C = c \int_0^1 du \, u^{-1-\epsilon} \int \langle \ell \, d\ell \rangle [\ell \, d\ell] (\sqrt{1-u}) \frac{(K^2)^{n+1}}{\langle \ell | K | \ell]^{n+2}} \frac{\prod_{i=1}^{k+n} \langle \ell | R_i | \ell]}{\prod_{j=1}^{k} \langle \ell | Q_j | \ell]}$$

Coefficient formulas look the same, now in terms of u-dependent Q_j, R_i.

We have checked 4- and 5-gluon examples to all orders in ϵ. (Anastasiou, RB, Feng, Kunszt, Mastrolia; RB, Feng, Yang. Original results from Bern, Dixon, Dunbar, Kosower.)

Box coefficients

$$C[K_r, K_s, K] = \frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{j=1}^{k+n}\langle P_{sr,1}|R_j|P_{sr,2}]}{\langle P_{sr,1}|K|P_{sr,2}]^{n+2}\prod_{t=1,t\neq i,j}^{k}\langle P_{sr,1}|Q_t|P_{sr,2}]} + \{P_{sr,1} \leftrightarrow P_{sr,2}\}\right)$$

$$P_{sr,1} = Q_s + \left(\frac{-2Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2}\right) Q_r$$

$$P_{sr,2} = Q_s + \left(\frac{-2Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2}\right) Q_r$$

$$\Delta_{sr} = (2Q_s \cdot Q_r)^2 - 4Q_s^2 Q_r^2$$

Pentagon coefficients

(RB, Feng, Yang)

If $k = 3$:

$$C[K_i, K_j, K_t, K] = (K^2)^{3+n} \prod_{s=1}^{n+3} \beta_s^{(q_i, q_j, q_t; p_s)}$$

If $k \geq 4$:

$$C[K_i, K_j, K_t, K] = (K^2)^{3+n} \frac{\prod_{s=1}^{n+3} \beta_s^{(q_i, q_j, q_t; p_s)}}{\prod_{w=1, w \neq i,j,t}^{k} \gamma_w^{(K_i, K_j; K_w, K_t)}}.$$

$$\alpha_j \equiv \frac{K_j^2 - K_j \cdot K}{K^2}$$

$$\beta_s^{(q_i, q_j, q_t; p_s)} \equiv (\beta_s - \sum_{h=i,j,k} a_h^{(q_i, q_j, q_t; p_s)} \alpha_h)$$

$$\gamma_s^{(K_i, K_j; K_s, K_t)} \equiv \frac{K_i^2 \epsilon(K, K_j, K_s, K_t) + K_j^2 \epsilon(K_i, K, K_s, K_t) + K_s^2 \epsilon(K_i, K_j, K, K_t) + K_t^2 \epsilon(K_i, K_j, K_s, K)}{K^2 \epsilon(K_i, K_j, K, K_t).}$$

Triangle coefficients

$$C[K_s, K] = \frac{(K^2)^{1+n}}{2} \frac{1}{(\sqrt{\Delta_s})^{n+1}} \frac{1}{(n+1)!\langle P_{s,1}\, P_{s,2}\rangle^{n+1}}$$

$$\times \frac{d^{n+1}}{d\tau^{n+1}} \left(\frac{\prod_{j=1}^{k+n} \langle P_{s,1} - \tau P_{s,2}|R_j Q_s|P_{s,1} - \tau P_{s,2}\rangle}{\prod_{t=1,t\neq s}^{k} \langle P_{s,1} - \tau P_{s,2}|Q_t Q_s|P_{s,1} - \tau P_{s,2}\rangle} + \{P_{s,1} \leftrightarrow P_{s,2}\} \right)\Bigg|_{\tau=0} .$$

$$P_{s,1} = Q_s + \left(\frac{-2Q_s \cdot K + \sqrt{\Delta_s}}{2K^2} \right) K$$

$$P_{s,2} = Q_s + \left(\frac{-2Q_s \cdot K - \sqrt{\Delta_s}}{2K^2} \right) K$$

$$\Delta_s = (2Q_s \cdot K)^2 - 4Q_s^2 K^2$$

Bubble coefficients

$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(\mathcal{B}^{(r;a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0}$$

$$\mathcal{B}^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n![\eta|\eta' K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j (K+s\eta) | \ell \rangle}{\langle \ell\, \eta \rangle^{n+1} \prod_{p=1}^{k} \langle \ell | Q_p (K+s\eta) | \ell \rangle} \right) \Bigg|_{|\ell\rangle \to |K - \tau\eta'|\eta], \tau=0}$$

$$\mathcal{B}^{(r;b;1)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,1} - \tau P_{r,2} | \eta | P_{r,1} \right]^{t+1}}{\left\langle P_{r,1} - \tau P_{r,2} | K | P_{r,1} \right]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,1} - \tau P_{r,2} | Q_r \eta | P_{r,1} - \tau P_{r,2} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,1} - \tau P_{r,2} | R_j (K+s\eta) | P_{r,1} - \tau P_{r,2} \rangle}{\langle P_{r,1} - \tau P_{r,2} | \eta K | P_{r,1} - \tau P_{r,2} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,1} - \tau P_{r,2} | Q_p (K+s\eta) | P_{r,1} - \tau P_{r,2} \rangle} \right) \Bigg|_{\tau=0}$$

$$\mathcal{B}^{(r;b;2)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,2} - \tau P_{r,1} | \eta | P_{r,2} \right]^{t+1}}{\left\langle P_{r,2} - \tau P_{r,1} | K | P_{r,2} \right]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,2} - \tau P_{r,1} | Q_r \eta | P_{r,2} - \tau P_{r,1} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,2} - \tau P_{r,1} | R_j (K+s\eta) | P_{r,2} - \tau P_{r,1} \rangle}{\langle P_{r,2} - \tau P_{r,1} | \eta K | P_{r,2} - \tau P_{r,1} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,2} - \tau P_{r,1} | Q_p (K+s\eta) | P_{r,2} - \tau P_{r,1} \rangle} \right) \Bigg|_{\tau=0}$$

These coefficients are polynomials in u.

Proof is constructive. For alternate formulas with more transparent u-dependence, see arXiv:0803.3147. (RB, Feng, Yang)

Incorporating Masses

(RB, Feng; RB, Feng, Mastrolia)

$$C \quad = \quad c\int d^{4-2\epsilon}p\,\frac{\prod_{i=1}^{k+n}(-2p\cdot P_i)}{\prod_{j=1}^{k}((p-K_j)^2-m_j^2)}\delta(p^2-M_1^2)\delta((p-K)^2-M_2^2)$$

We define the following four-vectors:

$$Q_j \quad = \quad -\left((1-2z)+\frac{M_1^2-M_2^2}{K^2}\right)K_j+\frac{K_j^2+M_1^2-m_j^2-2zK\cdot K_j}{K^2}K$$

$$R_i \quad = \quad -\left((1-2z)+\frac{M_1^2-M_2^2}{K^2}\right)P_i-\frac{z(2P_i\cdot K)}{K^2}K,$$

where

$$z=\frac{\alpha-\beta\sqrt{1-u}}{2}$$

$$\alpha=\frac{K^2+M_1^2-M_2^2}{K^2},$$

$$\beta=\frac{\sqrt{(K^2)^2+(M_1^2)^2+(M_2^2)^2-2K^2M_1^2-2K^2M_2^2-2M_1^2M_2^2}}{K^2}$$

Cut amplitude:

$$\int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)^{n+1}}{\langle \ell | K | \ell]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j | \ell]}{\prod_{i=1}^{k} \langle \ell | Q_i | \ell]}$$

Cut masters:

$$\begin{aligned}
\Delta I_2 &= \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)}{\langle \ell | K | \ell]^2} \\
\Delta I_3 &= \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{1}{\langle \ell | K | \ell]\, \langle \ell | Q | \ell]} \\
\Delta I_4 &= \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)^{-1}}{\langle \ell | Q_1 | \ell]\, \langle \ell | Q_2 | \ell]}
\end{aligned}$$

Again, the formulas for integral coefficients will look the same!

See also: Kilgore

We have reproduced cuts for $gg \to gg$ and $gg \to gH$ with massive fermion loops. (RB, Feng, Mastrolia. Original results from Bern, Morgan; Rozowsky.)

Coefficients of cut-free integrals are to be fixed by universal divergent behavior, or other constraints.

How were these formulas derived?

$$C \;=\; c\int d^4\ell \, \frac{\prod_{i=1}^{k+n}(-2\ell\cdot P_i)}{\prod_{j=1}^{k}(\ell-K_j)^2}\delta(\ell^2)\delta((\ell-K)^2)$$

Change to spinor variables: $\ell_{a\dot{a}} = t\,\lambda_a\tilde{\lambda}_{\dot{a}}$. (Cachazo, Svrček, Witten)

$$\int d^4\ell\;\delta(\ell^2)\;\delta((\ell-K)^2)\;(\bullet) = \int_0^\infty dt\,t\int\langle\lambda\,d\lambda\rangle\,[\tilde{\lambda}\,d\tilde{\lambda}]\,\delta((t\lambda\tilde{\lambda}-K)^2)(\bullet)$$

Use 2nd delta function to perform t-integral.

$$C \;=\; \int\langle\ell\,d\ell\rangle[\ell\,d\ell]\,\frac{(K^2)^{n+1}}{\langle\ell|K|\ell]^{n+2}}\,\frac{\prod_{i=1}^{k+n}\langle\ell|R_i|\ell]}{\prod_{j=1}^{k}\langle\ell|Q_j|\ell]}$$

$$C = \int \langle \ell\, d\ell\rangle [\ell\, d\ell] \frac{(K^2)^{n+1}}{\langle \ell|K|\ell]^{n+2}} \frac{\prod_{i=1}^{k+n} \langle \ell|R_i|\ell]}{\prod_{j=1}^{k} \langle \ell|Q_j|\ell]}$$

Split the factors in the denominator with partial fractions.

$$\frac{\prod_{j=1}^{k-1} [a_j\ \ell]}{\prod_{i=1}^{k} \langle \ell|Q_i|\ell]} = \sum_{i=1}^{k} \frac{1}{\langle \ell|Q_i|\ell]} \frac{\prod_{j=1}^{k-1} [a_j|Q_i|\ell\rangle}{\prod_{m=1,m\neq i}^{k} \langle \ell|Q_m Q_i|\ell\rangle} \qquad \text{box \& pentagon}$$

$$\frac{\prod_{j=1}^{n-1} [a_j\ \ell]}{\langle \ell|K|\ell]^n \langle \ell|Q|\ell]} = \frac{\prod_{j=1}^{n-1} [a_j|Q|\ell\rangle}{\langle \ell|KQ|\ell\rangle^{n-1}} \frac{1}{\langle \ell|K|\ell]\, \langle \ell|Q|\ell]} \qquad \text{triangle}$$

$$+ \sum_{p=0}^{n-2} (-1)^{n-p} \frac{\prod_{j=1}^{n-p-2} [a_j|Q|\ell\rangle\, [a_{n-p-1}|K|\ell\rangle \prod_{t=n-p}^{n-1} [a_t\ \ell]}{\langle \ell|K|\ell]^{p+2}\, \langle \ell|QK|\ell\rangle^{n-p-1}} \qquad \text{bubble}$$

Multiple poles lead to derivatives.

Alternate formula: triangle coefficients without derivatives

$$
\begin{aligned}
C[K_s, K]_{n=-2} &= 0 \\
C[K_s, K]_{n=-1} &= \frac{1}{2}\left(\frac{\prod_{j=1}^{k-1} \langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1, t\neq s}^{k} \langle P_{s,1}|Q_t|P_{s,2}]} \right) \\
C[K_s, K]_{n=0} &= \frac{K^2}{2\Delta_s}\left[\frac{\prod_{j=1}^{k} \langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1, t\neq s}^{k} \langle P_{s,1}|Q_t|P_{s,2}]} \left(\sum_{j=1}^{k} \frac{(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)}{\langle P_{s,1}|R_j|P_{s,2}]} \right.\right. \\
&\quad \left.\left. - \sum_{t=1, t\neq s}^{k} \frac{(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)}{\langle P_{s,1}|Q_t|P_{s,2}]} \right) + \{P_{s,1} \leftrightarrow P_{s,2}\} \right]
\end{aligned}
$$

$$
\begin{aligned}
&C[K_s, K]_{n=1} = \\
&\frac{(K^2)^2}{4\Delta_s^2}\left[\frac{\prod_{j=1}^{k+1}\langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1,t\neq s}^{k}\langle P_{s,1}|Q_t|P_{s,2}]}\left[\left(\sum_{j=1}^{k+1}\frac{(2Q_s\cdot K)(2R_j\cdot Q_s)-2Q_s^2(2R_j\cdot K)}{\langle P_{s,1}|R_j|P_{s,2}]}\right.\right.\right. \\
&\left.-\sum_{t=1,t\neq s}^{k}\frac{(2Q_s\cdot K)(2Q_t\cdot Q_s)-2Q_s^2(2Q_t\cdot K)}{\langle P_{s,1}|Q_t|P_{s,2}]}\right)^2 \\
&+\sum_{j=1}^{k+1}\frac{-[(2Q_s\cdot K)(2R_j\cdot Q_s)-2Q_s^2(2R_j\cdot K)]^2+2Q_s^2K^2\langle P_{s,1}|R_j|P_{s,2}]\langle P_{s,2}|R_j|P_{s,1}]}{\langle P_{s,1}|R_j|P_{s,2}]^2} \\
&-\sum_{t=1,t\neq s}^{k}\frac{-[(2Q_s\cdot K)(2Q_t\cdot Q_s)-2Q_s^2(2Q_t\cdot K)]^2+2Q_s^2K^2\langle P_{s,1}|Q_t|P_{s,2}]\langle P_{s,2}|Q_t|P_{s,1}]}{\langle P_{s,1}|Q_t|P_{s,2}]^2} \\
&+\{P_{s,1}\leftrightarrow P_{s,2}\}]
\end{aligned}
$$

$$C[K_s,K]_{n=2} \quad = \quad \frac{(K^2)^3}{12\Delta_s^3}\left[\frac{\prod_{j=1}^{k+2}\langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1,t\neq s}^{k}\langle P_{s,1}|Q_t|P_{s,2}]}(\mathcal{A}^3+3\mathcal{A}\mathcal{B}+\mathcal{C})+\{P_{s,1}\leftrightarrow P_{s,2}\}\right]$$

$$\mathcal{A} \quad = \quad \sum_{j=1}^{k+2}\frac{(2Q_s\cdot K)(2R_j\cdot Q_s)-2Q_s^2(2R_j\cdot K)}{\left\langle P_{s,1}|R_j|P_{s,2}\right]}-\sum_{t=1,t\neq s}^{k}\frac{(2Q_s\cdot K)(2Q_t\cdot Q_s)-2Q_s^2(2Q_t\cdot K)}{\left\langle P_{s,1}|Q_t|P_{s,2}\right]}$$

$$\mathcal{B} \quad = \quad -\sum_{j=1}^{k+2}\frac{[(2Q_s\cdot K)(2R_j\cdot Q_s)-2Q_s^2(2R_j\cdot K)]^2+2Q_s^2K^2\left\langle P_{s,1}|R_j|P_{s,2}\right]\left\langle P_{s,2}|R_j|P_{s,1}\right]}{\left\langle P_{s,1}|R_j|P_{s,2}\right]^2}$$

$$+\sum_{t=1,t\neq s}^{k}\frac{[(2Q_s\cdot K)(2Q_t\cdot Q_s)-2Q_s^2(2Q_t\cdot K)]^2-2Q_s^2K^2\left\langle P_{s,1}|Q_t|P_{s,2}\right]\left\langle P_{s,2}|Q_t|P_{s,1}\right]}{\left\langle P_{s,1}|Q_t|P_{s,2}\right]^2}$$

$$\mathcal{C} \quad = \quad \sum_{j=1}^{k+2}\frac{[(2Q_s\cdot K)(2R_j\cdot Q_s)-2Q_s^2(2R_j\cdot K)]^3-3Q_s^2K^2\left\langle P_{s,1}|R_j|P_{s,2}\right]\left\langle P_{s,2}|R_j|P_{s,1}\right]}{\left\langle P_{s,1}|R_j|P_{s,2}\right]^2}$$

$$\frac{2[(2Q_s\cdot K)(2R_j\cdot Q_s)-2Q_s^2(2R_j\cdot K)]}{\left\langle P_{s,1}|R_j|P_{s,2}\right]}-\sum_{t=1,t\neq s}^{k}\frac{2[(2Q_s\cdot K)(2Q_t\cdot Q_s)-2Q_s^2(2Q_t\cdot K)]}{\left\langle P_{s,1}|Q_t|P_{s,2}\right]}$$

$$\frac{[(2Q_s\cdot K)(2Q_t\cdot Q_s)-2Q_s^2(2Q_t\cdot K)]^2-3Q_s^2K^2\left\langle P_{s,1}|Q_t|P_{s,2}\right]\left\langle P_{s,2}|Q_t|P_{s,1}\right]}{\left\langle P_{s,1}|Q_t|P_{s,2}\right]^2}$$

Summary

- Coefficients of master integrals can be obtained without explicit reduction in a D-dimensional unitarity method.

- General formulas available for the massless case and in D dimensions.