Hypergeometric functions with rational arguments

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Outline

- Motivation
- Algorithm for expanding hypergeometric functions (HF's) about half-integer parameters
- Performances of the `HypExp` package
- Examples
- Conclusions and Outlook

Motivation

- HF appear in many branches of science (Maths, Physics, Engeneering, Economics, ...)
- In particle physics:

  HF appear in loop and phase space integrals in dimensional regularization

  $\Rightarrow$ Task of expanding HF about their parameters

- First: Focus on expansion about integer parameters.

  Algorithms developed and implemented *[Moch,Uwer,Weinzierl;Weinzierl; Kalmykov,Ward,Yost;Moch,Uwer;Maître,TH]*

- Recent focus also on expansion about half-integer parameters,

  mostly due to massive particles in loop and phase space integrals

  [Broadhurst,Fleischer,Tarasov;Davydychev,Tausk;Davydychev,Grozin;Davydychev,Kalmykov;Fleischer,Jegerlehner,Tarasov]
  [Jegerlehner,Kalmykov,Veretin;Jegerlehner,Kalmykov;Schröder,Vuorinen;Tarasov;Bejdakic,Schröder;Grozin,Maître,TH;Argeri,Mastrolia]

- Algorithms for expansion about integer and half-integer parameters developed and implemented in computer algebra systems *[Weinzierl;Kalmykov;Kalmykov,Ward,Yost]*
- Here: Alternative algorithm for expansion of HF about half-integer parameters and its implementation in the Mathematica package `HypExp`

Strategy of the Algorithm

- Definition
  - Consider a HF ${}_PF_{P-1}(\{A_i\};\{B_j\};x)$. It is of type P^r_s if, at $\epsilon=0$, r out of the A_i and s out of the B_j are half-integers and the rest integers.
- Reduction
  - Express a HF of a given type in terms of integration- and differentiation-operators acting on one specific HF of the same type (= the basis function of this type)
- Expansion of the basis function
  - Freedom in choice of basis function for each type of HF
- Integration and differentiation routines
  - Find integration and differentiation routines which act on the expanded basis function
- Other, related algorithms:
  - Reduction to a set of basis functions by means of recurrence relations *[Kalmykov;Kalmykov,Ward,Yost;Maître,TH]*
  - Nested (harmonic and binomial) sums approach *[Gonzalez-Arroyo,Lopez,Yndurain]* *[Vermaseren;Blümlein;Moch,Uwer,Weinzierl;Weinzierl]*

Reduction

- Define $\prod_{j}^{a:a} = 1, \quad \prod_{j}^{a:b} f(j) = \prod_{j=a}^{b-1} f(j) \quad \text{if} \quad a < b, \quad \prod_{j}^{a:b} f(j) = \prod_{j=b}^{a-1} \frac{1}{f(j)} \quad \text{if} \quad a > b\,,$

  so that $\quad \Gamma(b) = \Gamma(a) \prod_{j}^{a:b} (j) \quad \text{if} \quad a - b \in \mathbb{Z}$

- Furthermore, $\quad J^{+}(j)[f](x) = \frac{1}{x^j} \int_0^x \mathrm{d}x' x'^{j-1} f(x') \equiv J^{+}(j,1)[f](x)$

$$J^{-}(j)[f](x) = \frac{1}{x^{j-1}} \frac{\mathrm{d}}{\mathrm{d}x} x^j f(x) \equiv J^{-}(j,1)[f](x)$$

$$J^{\pm}(j,n)[f](x) \equiv \left(J^{\pm}(j)\right) \left[J^{\pm}(j,n-1)[f]\right](x)$$

so that $\quad \frac{x^i}{(i+j)^n} \equiv J^{+}(j,n)[y^i](x) \quad \text{and} \quad i^n x^i = J^{-}(0,n)[y^i](x)$

[see also Moch,Uwer,Weinzierl;Maître,TH]

- Consider HF of type 2_1^1 and start from

$${}_2F_1(A_1, A_2; B_1; x) = 1 + \frac{\Gamma(B_1)}{\Gamma(A_1)\Gamma(A_2)} \sum_{i=1}^{\infty} \frac{\Gamma(A_1+i)\Gamma(A_2+i)}{\Gamma(B_1+i)\Gamma(i+1)} x^i\,.$$

with $\quad A_1 = a_1 + \frac{1}{2} + \alpha_1 \epsilon, \quad A_2 = a_2 + \alpha_2 \epsilon, \quad B_1 = b_1 + \frac{1}{2} + \beta_1 \epsilon$

Reduction cont'd

- Transform Γ-functions according to

$$
{}_2F_1(a_1+\tfrac{1}{2}+\alpha_1\epsilon, a_2+\alpha_2\epsilon; b_1+\tfrac{1}{2}+\beta_1\epsilon; x) = 1+\frac{\Gamma(\frac{1}{2}+\beta_1\epsilon)}{\Gamma(\frac{1}{2}+\alpha_1\epsilon)\Gamma(1+\alpha_2\epsilon)}\frac{\prod_j^{0:b_1}(j+\frac{1}{2}+\beta_1\epsilon)}{\prod_j^{0:a_1}(j+\frac{1}{2}+\alpha_1\epsilon)\prod_j^{1:a_2}(j+\alpha_2\epsilon)}
$$

$$
\times\sum_{i=1}^{\infty}\underbrace{\frac{\prod_j^{0:a_1}(i+j+\frac{1}{2}+\alpha_1\epsilon)\prod_j^{1:a_2}(i+j+\alpha_2\epsilon)}{\prod_j^{0:b_1}(i+j+\frac{1}{2}+\beta_1\epsilon)}}_{D}\frac{\Gamma(i+\frac{1}{2}+\alpha_1\epsilon)\Gamma(i+1+\alpha_2\epsilon)}{\Gamma(i+\frac{1}{2}+\beta_1\epsilon)\Gamma(i+1)}x^i
$$

- Decomposition of D into partial fractions w.r.t. i yields

$$
D = \sum_{j\geq 0,n}\frac{C^+_{j,n}}{(i+j+\gamma\epsilon)^n}+\sum_{j<0,n}\frac{C^+_{j,n}}{(i+j+\gamma\epsilon)^n}+\sum_{j,n}\frac{C^{1/2}_{j,n}}{(i+\frac{1}{2}+j+\gamma\epsilon)^n}+\sum\nolimits_n C^-_n i^n
$$

with $C^+_{j,n}$, C^-_n and $C^{1/2}_{j,n}$: polynomials in ϵ.

- First and third sum: Expand denominator in ϵ and write expression in terms of $J^+(j,n)$ and $J^+(j+\frac{1}{2},n)$
- Last sum: Express $i^n\,x^i$ in terms of $J^-(0,n)$
- Second sum: Requires more work, but conceptually straightforward

Reduction cont'd

- Final formula reads

$$
{}_2F_1(a_1+\tfrac{1}{2}+\alpha_1\epsilon, a_2+\alpha_2\epsilon; b_1+\tfrac{1}{2}+\beta_1\epsilon; x) = 1 + \frac{\prod_j^{0:b_1}(j+\frac{1}{2}+\beta_1\epsilon)}{\prod_j^{0:a_1}(j+\frac{1}{2}+\alpha_1\epsilon)\prod_j^{1:a_2}(j+\alpha_2\epsilon)}
$$

$$
\times\Big[\sum_{j\geq 0,n}\tilde{C}^+_{j,n}J^+(j,n) + \sum_{j<0,n,\gamma}\tilde{C}^+_{j,n,\gamma}J^+(j,n,\gamma)
$$

$$
+\sum_{j,n}\tilde{C}^{1/2}_{j,n}J^+(j+\tfrac{1}{2},n) + \textstyle\sum_n C^-_n J^-(0,n)\Big]B
$$

- Relation most useful on the level of the expansion in ϵ (due to $\tilde{C}$ and due to B)
- Basis function

$$
B \equiv \frac{\Gamma(\frac{1}{2}+\beta_1\epsilon)}{\Gamma(\frac{1}{2}+\alpha_1\epsilon)\Gamma(1+\alpha_2\epsilon)}\sum_{i=1}^{\infty}\frac{\Gamma(i+\frac{1}{2}+\alpha_1\epsilon)\Gamma(i+1+\alpha_2\epsilon)}{\Gamma(i+\frac{1}{2}+\beta_1\epsilon)\Gamma(i+1)}x^i
$$

$$
= {}_2F_1(\tfrac{1}{2}+\alpha_1\epsilon, 1+\alpha_2\epsilon, \tfrac{1}{2}+\beta_1\epsilon, x) - 1
$$

- Generalization to other types 2^r_s and to higher ${}_PF_{P-1}$ straightforward
- This part of the algorithm is universal (type independent)

Expansion of the basis functions

- We make the ansatz

$$B = g(x)\left[1 + \sum_{j=1}^{\infty} \epsilon^j \sum_{s_1,\ldots s_j=+,0,-} c(s_1,\ldots,s_j;x) H_{s_1,\ldots,s_j}(f(x))\right]$$

with $f(x) = \sqrt{x}$ for HFs of type P_i^i

and $f(x) = i\sqrt{\frac{x}{1-x}}$ or $f(x) = \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}$ for HFs of type $P_{i\pm1}^i$.

[Weinzierl;Jegerlehner,Kalmykov,Veretin;Davydychev,Kalmykov;Kalmykov;Kalmykov,Ward,Yost]

- $g(x)$ is HF with $\epsilon \to 0$
- Properties of the coefficients $c(s_1,\ldots,s_j;x)$
  - Homogeneous of order j in the α_i, β_i
  - Symmetric in α's and β's corresponding to equal a's and b's
  - Must reduce to the coefficient of a reduced HF in the limit as one of the A's becomes equal to one of the B's

Expansion of the basis functions cont'd

- Also make ansatz for x-dependence of $c(s_1, ..., s_j; x)$.
  Ansatz depends on $f(x)$, but is not complicated:
  E. g. for type 2_1^1: constant or $\sqrt{x} \cdot$ constant,
  depending on number (even, odd) of "+" weights in $s_1, \ldots, s_j$.
- Insert complete ansatz for B in differential equation for the HF ${}_PF_{P-1}$

$$\mathcal{D}B = 0$$

- This yields (after possible variable changes)

$$\sum_{j=0}^{\infty} \epsilon^j \sum_{l} \sum_{s_1,\ldots s_l=+,0,-} \mathcal{C}(s_1, ..., s_l) H_{s_1,\ldots,s_l}(y) = 0$$

- Differential equation is satisfied if all the coefficients $\mathcal{C}(s_i)$ vanish
- Coefficients $c(s_1, ..., s_j; x)$ can be extracted from these conditions
- This part of the algorithm is a case-by-case approach

Integration and Differentiation Routines

- Task: Carry out explicitly int. and diff. operators on expanded basis function
- HPL's are iterated integration over rational functions
  $\Rightarrow$ well-suited for carrying out int. and diff. procedures
- Difficulties
  - Securing the cancellation of $1/x$ divergences at lower integration limit
  - Integration of $1/\sqrt{x} \cdot \mathrm{HPL}[..., f(x)]$
  - introduction of two new weights $\dfrac{1}{\sqrt{1-t^2}}$ and $\dfrac{1}{t\sqrt{1-t^2}}$

    whose contributions cancel in the end
- Conceptually rather clear, implementation a bit tedious

Performances of the Package

- Expansion of all ${}_PF_{P-1}$ about integer parameters to arbitrary order in ϵ, both for general argument x and $x = 1$
- Expansion about half-integer parameters of types

$$2^2_1, \quad 2^1_1, \quad 2^1_0, \quad 2^0_1, \qquad 3^3_2, \quad 3^2_2, \quad 3^1_1, \quad 3^1_0, \quad 3^0_1, \qquad 4^1_1, \quad 4^3_3$$

also to arbitrary order in ϵ, both for general argument x and $x = 1$

```
Timing[Collect[HypExp[
  HypergeometricPFQ[{1/2 + ε, 1 - 3 ε, 1 - 2 ε}, {1/2 - 4 ε, 1 + ε}, x], ε, 2], ε, Simplify]]
```

$$\Big\{0.147978\ \text{Second}, \frac{1}{1-x}\Bigg(\Bigg(-15\log^2\Big(\frac{\sqrt{x}+1}{1-\sqrt{x}}\Big) + 30\sqrt{x}\log(1-x)\log\Big(\frac{\sqrt{x}+1}{1-\sqrt{x}}\Big) + 18\log^2(1-x) + 24\,\text{HPL}(\{0,\text{minus}\},\sqrt{x}) + 40\sqrt{x}\,\text{HPL}(\{0,\text{plus}\},\sqrt{x}) + 55\sqrt{x}\,\text{HPL}(\{\text{minus},\text{plus}\},\sqrt{x})\Bigg)\epsilon^2\Bigg) + \frac{\Big(5\sqrt{x}\log\big(\frac{\sqrt{x}+1}{1-\sqrt{x}}\big) + 6\log(1-x)\Big)\epsilon}{1-x} + \frac{1}{1-x}\Big\}$$

```
Timing[Collect[
  HypExp[HypergeometricPFQ[{1/2 + ε, 1 - 3 ε, 1 - 2 ε}, {1/2 - 4 ε, ε}, 1], ε, 1], ε, Expand]]
```

$$\Big\{4.31634\ \text{Second}, \Big(\frac{19}{32} + \frac{15\pi^2}{8} - \frac{75\log(2)}{4} + 45\log^2(2)\Big)\epsilon + \frac{15\log(2)}{2} - \frac{9}{16} + \frac{5}{8\epsilon}\Big\}$$

Performances of the Package cont'd

- $\texttt{HypExpInt}[\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,x] = \int_0^1 du\, \frac{u^{\chi_1}\ \ln^{\chi_2}(u)\ \ln^{\chi_3}(1-u)\ \ln^{\chi_4}(1-xu)}{(ux-1)^{\chi_5}}$

$$\chi_i \in \mathbb{N}^{\geq 0} \quad \text{and}$$

$$\chi_2+\chi_3+\chi_4+1-\delta_{\chi_5,0} \leq 5$$

```
HypExpInt[1, 0, 2, 1, 2, x]
```

$$\frac{\log^4(1-x)}{6x^2} - \frac{\log(x)\log^3(1-x)}{3x^2} - \frac{\log^3(1-x)}{(x-1)x^2} + \frac{(\pi^2 x - \pi^2 - 6)\log^2(1-x)}{6(x-1)x^2} + \frac{2\log(x)\log^2(1-x)}{(x-1)x^2} + \frac{2(3\zeta(3)x - 3\zeta(3) - \pi^2)\log(1-x)}{3(x-1)x^2} - \frac{2\,\mathrm{Li}_2(x)}{(x-1)x^2} + \frac{4\,\mathrm{Li}_3(1-x)}{(x-1)x^2} + \frac{2\,\mathrm{Li}_3(x)}{(x-1)x^2} - \frac{2\,\mathrm{Li}_4(1-x)}{x^2} - \frac{2\,\mathrm{Li}_4(x)}{x^2} - \frac{2\,\mathrm{Li}_4(\frac{x}{x-1})}{x^2} + \frac{\pi^4 x - 180\zeta(3) - \pi^4}{45(x-1)x^2}$$

- $\texttt{HypExpU}[n,m,p] = \int_0^1 du\, \ln^n(u)\cdot\ln^m(1-u)\cdot u^p$

$$n, m \in \mathbb{N}^{\geq 0}, \quad p \in \mathbb{Z}, \quad m+p \geq 0$$

```
HypExpU[3, 4, -3]
```

$$-\frac{33\pi^4}{20} - \frac{\pi^6}{12} + 135\zeta(3) - 33\pi^2\zeta(3) + 2\pi^4\zeta(3) + 54\zeta(3)^2 + 378\zeta(5) + 48\pi^2\zeta(5) - 720\zeta(7)$$

- Update of `Series`

```
Series[Log[1 + ϵ] HypergeometricPFQ[{1/2 + ϵ, 1 - 3 ϵ, 1 - 2 ϵ}, {1/2 - 4 ϵ, 1 + ϵ}, x],
  {ϵ, 0, 2}] // Simplify
```

$$\frac{\epsilon}{1-x} + \frac{\left(-10\sqrt{x}\log\left(\frac{\sqrt{x}+1}{1-\sqrt{x}}\right) - 12\log(1-x) + 1\right)\epsilon^2}{2x-2} + O(\epsilon^3)$$

- Further features: Argument transformation rules for polylogarithms; Expansion of incomplete `Beta` function; Libraries

New weights

- Harmonic Polylogs of weights -1, 0, 1 (or +, -, 0) are not sufficient to describe the expansion of any HF with half-integer parameters
- Expample: Consider HF of type 3^{1}_{2}

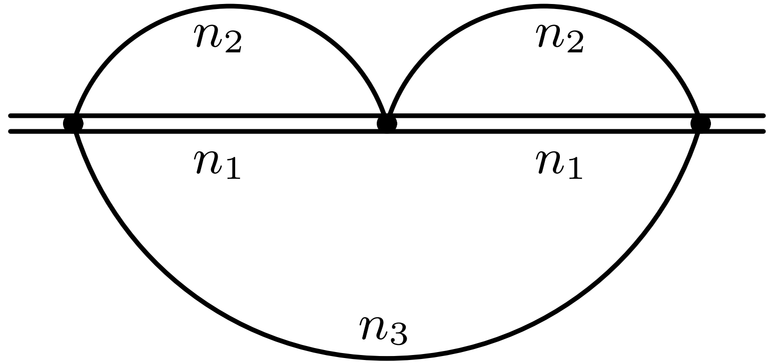
$$
\begin{aligned}
{}_3F_2\left(\tfrac{1}{2}+\alpha_1\epsilon, \alpha_2\epsilon, 1+\alpha_3\epsilon; \tfrac{1}{2}+\beta_1\epsilon, \tfrac{1}{2}+\beta_2\epsilon; x\right) = \\
& 1 - \alpha_2\epsilon\omega H(+;\omega) - \alpha_2\epsilon^2\big(-(\alpha_2+\alpha_3)\,\omega H(-,+;\omega) + \ldots\big) \\
+ \quad & \alpha_2\epsilon^3\big(\alpha_2\alpha_3\omega H(+,+,+;\omega) + \ldots\big) \\
+ \quad & 4\epsilon^3\alpha_2\,(\alpha_1-\beta_1)\,(\alpha_1-\beta_2)\,\omega H\left(\frac{1}{2},\frac{1}{2},+;\omega\right) + \mathcal{O}(\epsilon^4)\ ,
\end{aligned}
$$

where $\omega = i\sqrt{x/(1-x)}$

- We have to define new weights for the HPLs: $f_{\frac{1}{2}}(t) = \dfrac{1}{t\sqrt{1-t^2}}$.
- Not a limitation to the algorithm, but a limitation of the current implementation, as these weights are not supported by the package HPL.

Example 1

- Three-loop on-shell HQET propagator diagram with massive quark loops



$$I_{n_1 n_2 n_3} = \frac{1}{i\pi^{d/2}} \int \frac{I^2_{n_1 n_2}(p_0)\, d^d p}{(1 - p^2 - i0)^{n_3}}$$

$$\begin{aligned} I_{n_1 n_2}(p_0) &= \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{(-2(k_0 + p_0) - i0)^{n_1} (1 - k^2 - i0)^{n_2}} \\ &\stackrel{p_0 \le 0}{=} f(n_1, n_2, \epsilon)\; {}_2F_1 \left(\begin{array}{c} \frac{1}{2}n_1, \frac{1}{2}n_1 + n_2 - 2 + \epsilon \\ n_1 + n_2 - \frac{3}{2} + \epsilon \end{array} \middle| \, 1 - p_0^2 \right) \end{aligned}$$

- Crucial step: Express square of ${}_2F_1$ function as a linear combination of ${}_3F_2$ (Clausen identity)
- Other steps: Wick rotation $p_0 = i\, p_{0,E}$; introduction of $z = 1/(1 + p_{0,E}^2)$; argument transformation $z \to 1/z$ in ${}_3F_2$ functions.

Example 1 cont'd

- Result: $$\frac{I_{122}}{\Gamma^3(1+\epsilon)} = -\frac{1}{2\epsilon^2(1+2\epsilon)}\left[\frac{(1+2\epsilon)\Gamma^2(1-\epsilon)\Gamma^4(1+2\epsilon)\Gamma(1-2\epsilon)\Gamma^2(1+3\epsilon)}{\Gamma^4(1+\epsilon)\Gamma(1+4\epsilon)\Gamma(1-4\epsilon)\Gamma(2+6\epsilon)}\right.$$ $$\left.-\frac{2\,\Gamma^2(1-\epsilon)\Gamma^3(1+2\epsilon)(1+2\epsilon)}{\Gamma^2(1+\epsilon)\Gamma(1-2\epsilon)\Gamma(2+4\epsilon)}\,{}_3F_2\left(\begin{array}{c}\frac{1}{2},1+2\epsilon,-\epsilon\\ \frac{3}{2}+2\epsilon,1-\epsilon\end{array}\middle|\,1\right)+{}_4F_3\left(\begin{array}{c}1,\frac{1}{2}-\epsilon,1+\epsilon,-2\epsilon\\ \frac{3}{2}+\epsilon,1-\epsilon,1-2\epsilon\end{array}\middle|\,1\right)\right]$$

[Grozin,Maître,TH]

  - $I_{n_1 n_2 n_3}$ known explicitly, consistency checks with reduction formula of

[Grozin,A.Smirnov,V.Smirnov]

- Expansion: $$\frac{I_{122}}{\Gamma^3(1+\epsilon)} = \frac{\pi^2}{3(1+6\epsilon)}\left[1-\pi^2\epsilon^2+48\zeta_3\epsilon^3-\frac{38\pi^4}{15}\epsilon^4-48\left(\pi^2\zeta_3-30\zeta_5\right)\epsilon^5\right.$$ $$\left.+\left(1152\zeta_3^2-\frac{4793\pi^6}{945}\right)\epsilon^6+\left(39312\zeta_7-\frac{608\pi^4\zeta_3}{5}-1440\pi^2\zeta_5\right)\epsilon^7+\mathcal{O}(\epsilon^8)\right]$$

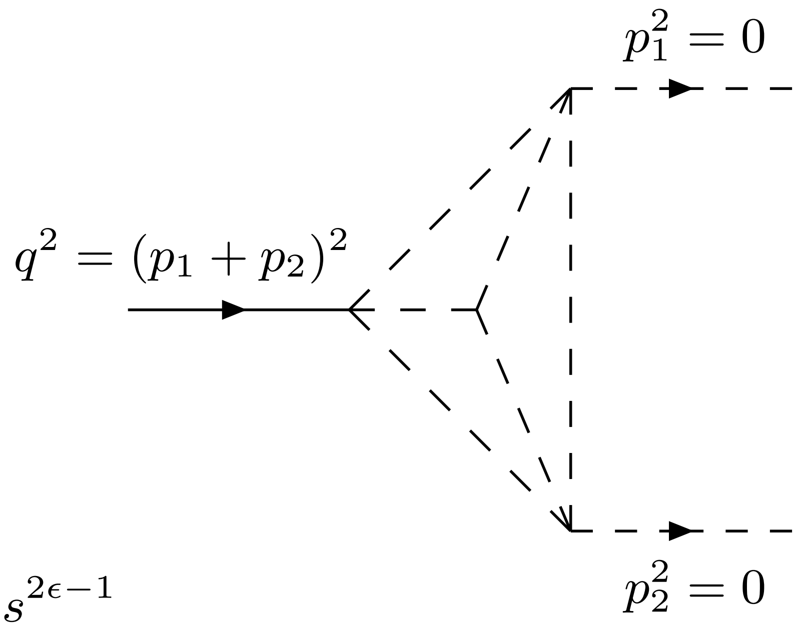
[Grozin,Maître,TH]

  - Three leading terms of I_{111} already known [Grozin,A.Smirnov,V.Smirnov;Pak]

- Conjecture: $$\frac{I_{122}}{\Gamma^3(1+\epsilon)} = \frac{\pi^2}{3}\frac{\Gamma^3(1+2\epsilon)\Gamma^2(1+3\epsilon)}{\Gamma^6(1+\epsilon)\Gamma(2+6\epsilon)}$$???

Example 2

- Master integral for massless three-loop form factors



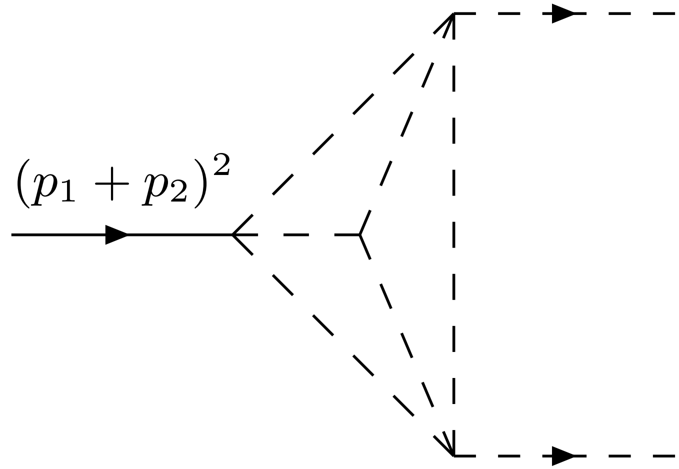
$$A_{6,2} = \left[-q^2 - i\eta\right]^{-3\epsilon} \mathcal{F}(\epsilon) \int_0^1 ds\,(1-s)^{-3\epsilon} s^{2\epsilon-1}$$

$$\times \left(\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} - s^{-\epsilon}\, {}_2F_1(\epsilon, -\epsilon; 1-\epsilon; s) \right) {}_3F_2 \left(1-3\epsilon, 1-2\epsilon, 1-\epsilon; 2-4\epsilon, \frac{3}{2} - 2\epsilon; -\frac{(s-1)^2}{4s} \right)$$

- Expansion in ϵ
- Simplification of HPL arguments: $x \equiv -\frac{(1-s)^2}{4s} \quad \Leftrightarrow \quad i\sqrt{\frac{x}{1-x}} = -\frac{1-s}{1+s}$
- Integration of HPL's and rational functions over s

Example 2

- Master integral for massless three-loop form factors



$$
\begin{aligned}
A_{6,2} \quad &= \quad \frac{\mathcal{N}}{(1-5\epsilon)(1-4\epsilon)\epsilon} \\
&\times \quad \Bigg[-2\zeta(3) - \epsilon\,\frac{7\pi^4}{180} + \epsilon^2\left(\frac{2}{3}\pi^2\zeta(3) - 10\zeta(5)\right) \\
&\quad +\epsilon^3\left(\frac{163\pi^6}{7560} + 76\zeta(3)^2\right) + \epsilon^4\left(\frac{55}{18}\pi^4\zeta(3) + \frac{445\zeta(7)}{2}\right) \\
&\quad +\epsilon^5\left(-\frac{744}{5}\zeta(5,3) + 1000\zeta(3)\zeta(5) - 22\pi^2\zeta(3)^2 + \frac{802183\pi^8}{4536000}\right) + \mathcal{O}(\epsilon^6)\Bigg]
\end{aligned}
$$

- Numerical check with twofold Mellin-Barnes representation and its evaluation with `MB`

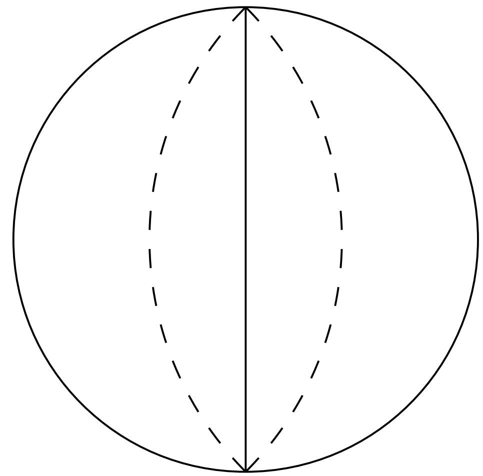
[Czakon]

Example 3

- Four-loop tadpole with three massive lines
- Start from 1-dimensional MB representation

[Gluza,Kajda,Riemann]

- Derive the following closed form



$$T(1,1,1,1,1) = \frac{2^{3-4\epsilon}\, e^{4\epsilon\gamma_E}\, \pi\, \Gamma^2(1-\epsilon)}{\sin(\pi\epsilon)\,\Gamma(2-\epsilon)} \left[\frac{\sqrt{\pi}\,\Gamma(\epsilon)\Gamma(-1+2\epsilon)\Gamma(-2+3\epsilon)}{\Gamma(2-\epsilon)\Gamma(-\frac{1}{2}+2\epsilon)} \right.$$
$$\times {}_3F_2(\epsilon, -1+2\epsilon, -2+3\epsilon\,;\, 2-\epsilon, -\tfrac{1}{2}+2\epsilon\,;\, \tfrac{1}{4})$$
$$\left. - \frac{\Gamma(-\frac{1}{2}+\epsilon)\Gamma(-2+3\epsilon)\Gamma(-3+4\epsilon)}{\Gamma(-\frac{3}{2}+3\epsilon)}\, {}_3F_2(-1+2\epsilon, -2+3\epsilon, -3+4\epsilon\,;\, \epsilon, -\tfrac{3}{2}+3\epsilon\,;\, \tfrac{1}{4}) \right]$$
$$= \frac{1}{4\epsilon^4} + \frac{1}{\epsilon^3} + \left(\frac{97}{48} + \frac{\pi^2}{12}\right)\frac{1}{\epsilon^2} + \left(\frac{833}{288} + \frac{\pi^2}{3} - \frac{\zeta_3}{3}\right)\frac{1}{\epsilon} + \frac{4177}{432}$$
$$+ \frac{97\pi^2}{144} - \frac{4\zeta_3}{3} + \frac{\pi^4}{12} + \frac{1}{1728}\left[99 + 16\pi^2 - 24\,\psi^{(1)}(\tfrac{1}{3})\right]^2 + \mathcal{O}(\epsilon)$$

- Checked to agree with earlier findings

[Boughezal,Czakon;Faißt,Maierhöfer,Sturm;Gluza,Kajda,Riemann]

Conclusions

- We presented a new algorithm for expanding certain classes of HF's about half-integer parameters

- Implementation in the already existing `Mathematica` package `HypExp`

- Application to physical problems
  - Confirmation of previous calculations and
  - derivation of new results

- The `HypExp` package is user-friendly and publicly available from

  `http://www-theorie.physik.unizh.ch/~maitreda/HypExp/`

Outlook

- Extension to expansion about other fractions p/q ?
  - So far, this case has only little physical significance
  - Some poineering work was done by S. Weinzierl:
    - Expansion of ‘balanced’ fractions

$$\frac{\Gamma(n+a_1-\frac{p_1}{q_1}+b_1\epsilon)}{\Gamma(n+c_1-\frac{p_1}{q_1}+d_1\epsilon)}\,\frac{\Gamma(n+a_2-\frac{p_2}{q_2}+b_2\epsilon)}{\Gamma(n+c_2-\frac{p_2}{q_2}+d_2\epsilon)}\cdots$$

    - as well as single ‘unbalanced’ rational numbers in numerator or denominator

$$\frac{\Gamma(n+1-\frac{p}{q}+b\,\epsilon)}{\Gamma(n+1+d\,\epsilon)} \quad \text{or} \quad \frac{\Gamma(n+a+b\,\epsilon)}{\Gamma(n+c-\frac{p}{q}+d\,\epsilon)}\cdots$$

- Extension not a problem for the reduction part of the algorithm.

  Problems: Expansion of the basis functions?

  Application of integration and differentiation operators?