New Results in four and five Loop QED and QCD calculations

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SFB TR9

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LOOPS & LEGS 2008

Outline

0. Five-loop massless correlators: calculational status

1. * Vector (new!) and scalar correlators to order α_s^4 for generic number of quark flavours N_f

2. $(g-2)_\mu$: new analytical results at order α^5

3. Progress in computing QED β-function in 5 loops

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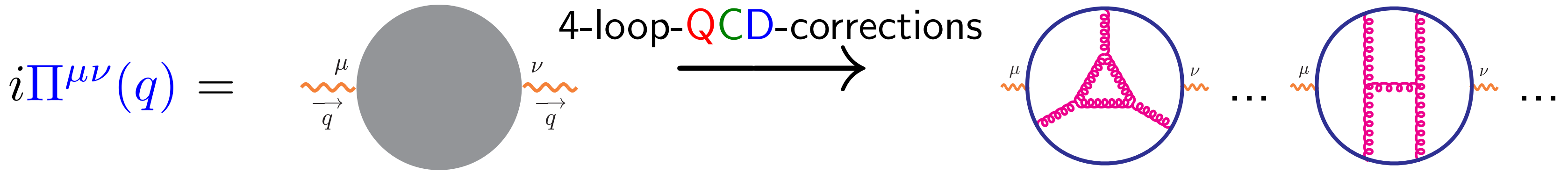
* applications of the result for $\sigma_{tot}(e^+e^- \to hadrons)$ and τ-decays will be discussed in the talk by J. H. Kühn tomorrow

- Correlator of two currents:

$$\Pi^{\mu\nu}(q,j) = i\int dx\, e^{iqx}\langle 0|T j^{\mu}(x) j^{\nu}(0)|0\rangle = (-g^{\mu\nu}q^2 + q^{\mu}q^{\nu})\Pi(q^2)$$

here: $j^{\mu}(x)$ electromagnetic heavy vector current ((as an example)

- Diagrammatically:



- Two limits are of interest:
  - Expansion of the diagrams in the external momentum around $q^2 = 0$

    $\Longrightarrow$ vacuum diagrams (tadpoles) $\longrightarrow$ front edge: 4 loops

  - High energy limit

    $\Longrightarrow$ massless propagators $\longrightarrow$ front edge:

    5 loops /for absorptive part only/
    4 loops /in general/

Tool Box *

- **IRR / Vladimirov, (78)/ + IR R^* -operation /K. Ch., Smirnov (1984)/ + resolved combinatorics /K. Ch., (1997)/**
- **reduction to Masters: "direct and automatic" construction of CF's through $1/D$ expansion—made with BAICER—within the Baikov's representation for Feynman integrals**[1]
- **all 4-loop master p-integrals are known analytically /P. Baikov and K.Ch. (2004)/**
- **computing time and required resources: could be huge (the price for full automatization)**; to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .) and MANY computers... (see below)

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* NO IBP identities are use at any step!

[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

$$R(s) \approx \Im\,\Pi(s - i\delta)$$

$$3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx}\langle 0|T[\;\; j_\mu^v(x) j_\mu^v(0)\;]|0\rangle dx$$

To conveniently sum the RG-logs one uses the Adler function:

$$D(Q^2) = Q^2 \frac{d}{d\,Q^2}\Pi(q^2) = Q^2 \int \frac{R(s)}{(s+Q^2)^2}\, ds$$

or $(a_s \equiv \alpha_s/\pi)$

$$R(s) = \frac{1}{2\pi i}\int_{-s-i\delta}^{-s+i\delta} dQ^2 \frac{D(Q^2)}{Q^2} = D(s) - \pi^2 \frac{\beta_0^2 d_0}{3} a_s^3 + \ldots$$

$$d_4 = n_f^3 \left[-\tfrac{6131}{5832} + \tfrac{203}{324}\zeta_3 + \tfrac{5}{18}\zeta_5\right]$$ (“renormalon” chain /M. Beneke 1993/)

$$+ n_f^2 \left[\tfrac{1045381}{15552} - \tfrac{40655}{864}\zeta_3 + \tfrac{5}{6}\zeta_3^2 - \tfrac{260}{27}\zeta_5\right]$$ /Baikov, Kühn, K.Ch. (2002)/

$$+ n_f \left[-\frac{13044007}{10368} + \frac{12205}{12}\zeta_3 - 55\,\zeta_3^2 + \frac{29675}{432}\zeta_5 + \frac{665}{72}\zeta_7\right]$$

$$+ \left[\frac{144939499}{20736} - \frac{5693495}{864}\zeta_3 + \frac{5445}{8}\zeta_3^2 + \frac{65945}{288}\zeta_5 - \frac{7315}{48}\zeta_7\right]$$

Interesting features:

1. irrationals up to ζ_7 (understandable from the structure of the masters)

2. no ζ_4 and/or ζ_6 (expected but mysterious!)

2005 - 2008: Three $\mathcal{O}(\alpha_s^4)$ Calculations

used CPU time (measured in terms of a one 3 GH PC)

- **QCD: scalar $R^{SS}(s)$: ($\approx$ 5 years)**
- **Quenched QED: β-function ($\approx$ 25 years)**
- **QCD: $R^{VV}(s)$ ($\approx$ 60 years)**

In Reality

calculations of $R^{VV}(s)$ were made mainly on HP XC4000 supercomputer of the Karlsruhe University (claster of Dual Core AMD Opteron 2.6 GH). It took about 60 CPU years, but only around 5 calendar months (up to $160 = (20$ "octets" of processors running parallel FORM) were simultaneously used).

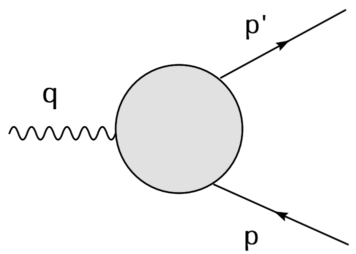
To compare: scalar $R^{SS}(s)$ took about 5 CPU years and QQED about 25 CPU years, with calendar time about 1 year for each (but with older processors and less developed software).

- Dirac theory predicts for a lepton $l = e, \mu, \tau$:

$$\vec{\mu}_l = g_l \left(\frac{e}{2m_l c} \right) \vec{s}, \quad g_l = 2$$

- quantum loops $\Longrightarrow$ deviations from $g_l = 2$:

$$g_l = 2\ (1 + a_l)$$



$$= \bar{u}(p') \left[\gamma^\mu F_{\rm E}(q^2) + i\,\frac{\sigma^{\mu\nu} q_\nu}{2m_\mu} F_{\rm M}(q^2) \right] u(p)$$

- in the static limit

$$F_E(0) = 1; \quad F_M(0) = a_\mu$$

The present experimental value is terrifically accurate:

$$a_\mu = 116592080(63) \cdot 10^{-11}$$

0.5 parts per billion!! (E821: Final Report: PRD73 (2006) 072003)

The current theory prediction shows an "interesting but not yet conclusive discrepancy[1]" of about 3 σ (or 1.8 σ if ones switches to the τ-data in order to describe the hadronic effects)

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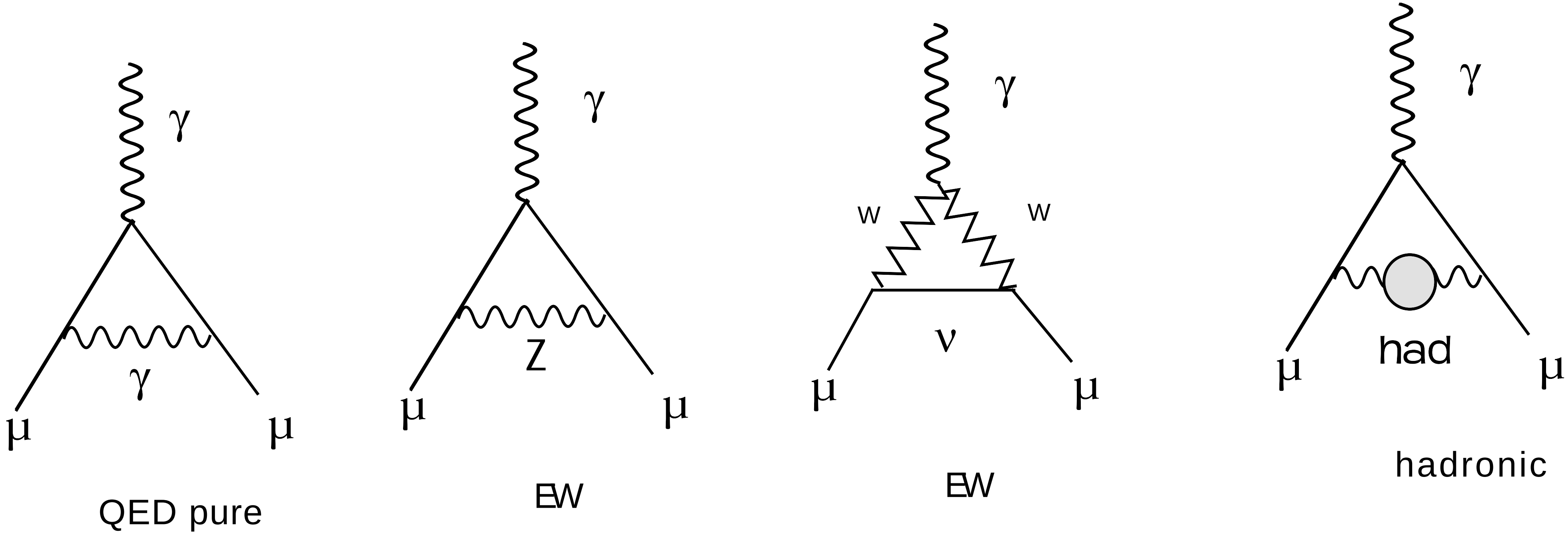
[1] A. Höcker, W. Marciano, PDG, July of 2007

Higher order corections to a_μ are basically classified into 4 classes:



The QED part is known to 4 loops[1] (and numericaly leading terms in 5 !)
The EW part is known to 2 loops

The Hadronic part is known but with limited accuracy ...

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[1] and four loops contribute as much as $378.5 \cdot 10^{-11}$!!! (cmp. to experimental uncertainty $\sim 70 \cdot 10^{-11}$!)

The QED contribution to a_μ

a_μ^{QED} = $(1/2)(\alpha/\pi)$ Schwinger 1948

+ $0.765857410\ (27)\ (\alpha/\pi)^2$

Sommerfield; Petermann; Suura & Wichmann '57; Elend '66; MP '04

+ $24.05050964\ (43)\ (\alpha/\pi)^3$

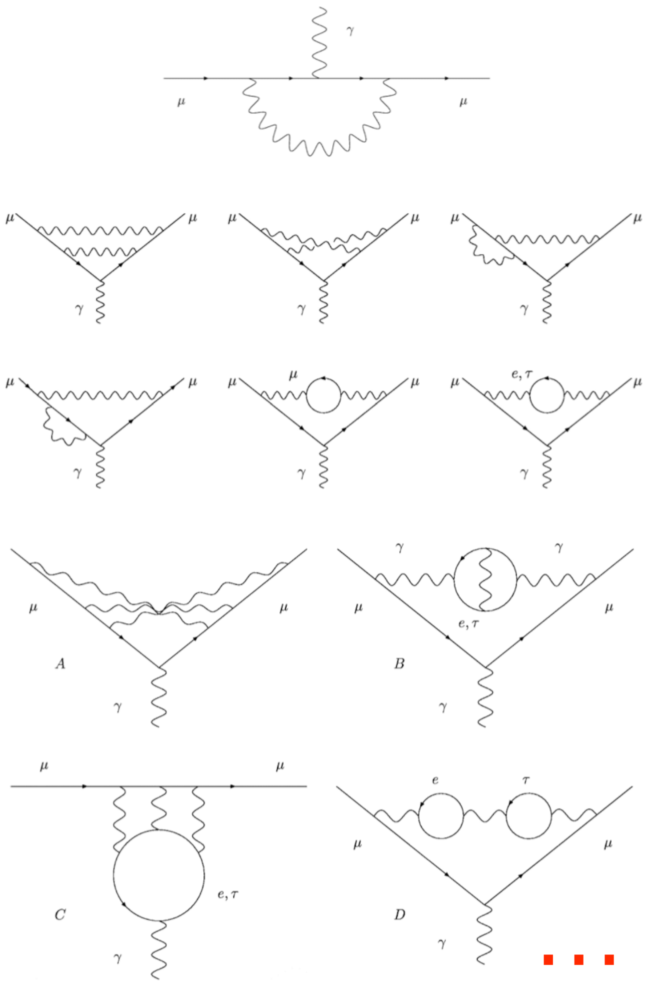
Barbieri, Laporta, Remiddi … ; Czarnecki, Skrzypek; MP '04;
Friot, Greynat, de Rafael '05

+ $130.805\ (8)\ (\alpha/\pi)^4$ Revised!

Kinoshita & Lindquist '81, … , Kinoshita & Nio '04, '05; Aoyama, Hayakawa, Kinoshita & Nio, June 2007

+ $663\ (20)\ (\alpha/\pi)^5$ In progress

Kinoshita et al. '90, Yelkhovsky, Milstein, Starshenko, Laporta, Karshenboim,…, Kataev, Kinoshita & Nio March '06.



Adding up, I get:

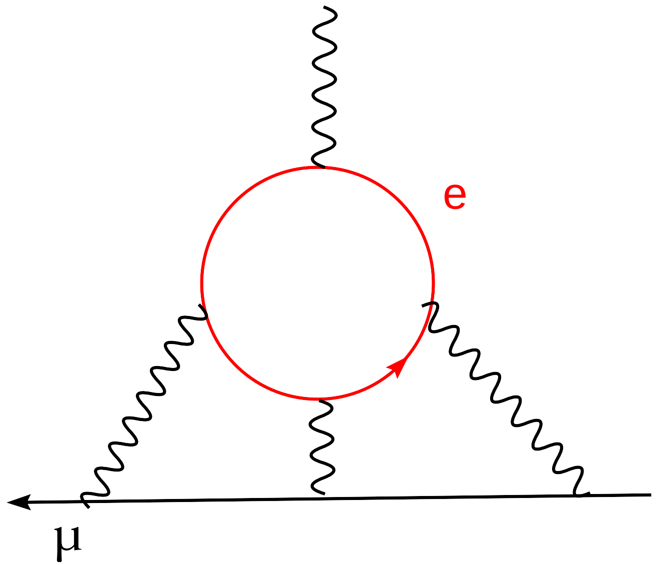
a_μ^{QED} = 116584718.10 (14) (08) x 10^{-11}

mainly from 5-loop unc ↲ ↳ from new $\delta\alpha$

with α = 1/137.035999068 (96) [0.7 ppb]

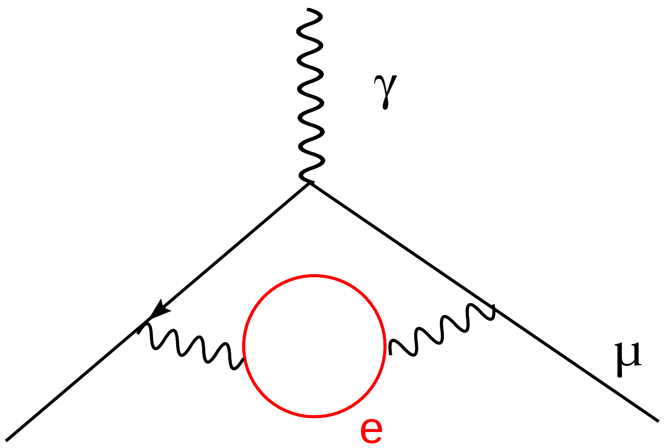
There are 2 types of numerically leading enhanced with logs of large ratio $\frac{M_\mu}{m_e} = 206.7682838$ pure QED contributions:

LL: light by light



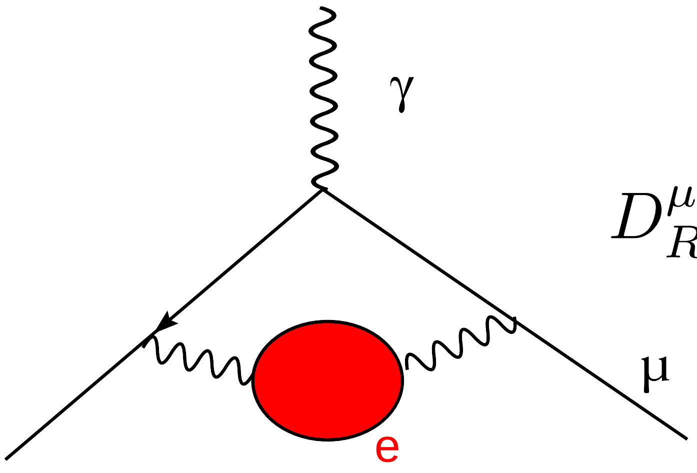
$$\left(\frac{2\pi^2}{3}\ln(\frac{M_\mu}{m_e}) + \ldots\right)\left(\frac{\alpha}{\pi}\right)^3$$

VP: vacuum polarization:



$$\left(\frac{1}{3}\ln(\frac{M_\mu}{m_e}) + \ldots\right)\left(\frac{\alpha}{\pi}\right)^2$$

We will consider mixed VP contributions:



$D^{\mu\nu}_R(q^2, m_e^2, \alpha)$

$$D^{\mu\nu}_R(-q^2, m_e^2, \alpha) = \frac{-ig_{\mu\nu}}{q^2}\left(d_R = \frac{1}{1+\Pi_R(-q^2, m_e^2, \alpha)}\right)$$

= the photon propagator composed from

electron* loops and photon exchanges *only

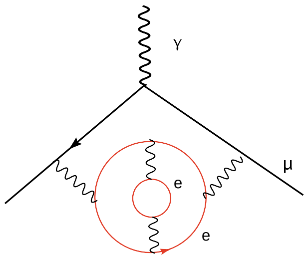
As is well-known (B.E Lautrup and E.de Rafael NPN 70 (1974) 317)

$$a_\mu(M/m, \alpha) = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x(1-x)\left[d_R\left(\frac{x^2}{1-x}\frac{M^2}{m^2}, \alpha\right) - 1\right] \qquad \star$$

Important: $\Pi_R(Q^2/m^2, \alpha) = \Pi_R^\infty(Q^2/m^2, \alpha) + \mathcal{O}(m^2/Q^2)$ **and if one uses only** Π_R^∞ **in (** $\star$ **) then the resulting error in** a_μ **will only be of order** m/M **or higher!**

Some instructive hystorical examples

In 1991 R, Faustov *et al.*, found (using some results from[1]) that the contribution of



+ two more

$$a_\mu = [0.923 + \mathcal{O}(\frac{m_e}{M_\mu})] \left(\frac{\alpha}{\pi}\right)^4$$

was in disagreement to the numeric result by

(T. Kinoshita, B. Nizik, Y.Okamoto (1990)) $a_\mu = 1.4416(18) \left(\frac{\alpha}{\pi}\right)^4$ $(\star)$

The problem came from a theoretical error in [1]. After its correction (T. Kinoshita, H Kawai, Y.Okamoto (1991)) the new result is in good agreeement to $(\star)$

$$a_\mu = [1.452570 + \mathcal{O}(\frac{m_e}{M_\mu})] \left(\frac{\alpha}{\pi}\right)^4$$

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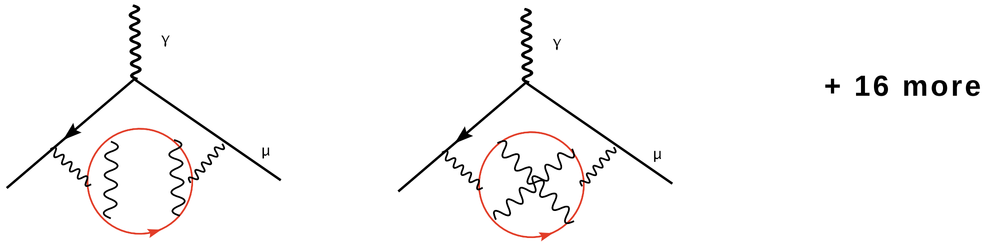
[1] J. Calmet and E. De Rafael, (1975)

Hystory: continuation I

quenched approximation for the photon propagator:



Initial asymptotic result (D. Broadhurst, A. K. Lataev, O. Tarasov (1994)) was in disagreement to that by (T. Kinoshita, B. Nizik, Y.Okamoto (1990))

$$-0.29087^{asym} \quad \Longleftrightarrow \quad -0.7945(202)^{num}$$

after recalculation (with a new, improved integration routine VEGAS instead of RIWARD and much better statistics) the result changed to (T. Kinoshita, (1993):

$$-0.2415(19)$$

Hystory: continuation II

Two years later (1995) D. Broadhurst and P. Baikov improved the asymptoic result (by combining the asymptoic and threshold result via a new Padè-based method) and arrived to

$$-0.230696(5)^{impr.\ asym.,1995} \quad \Longleftrightarrow \quad -0.2415(19)^{num,1993}$$

"The problem was traced to round-off errors caused by insufficient number of effective digits in real*8 arithmetic in carrying out the renormalization ... This was resolved by going over to the real *16 arithmetics." /Kinoshita, Nio, (2004)/

The new result (T. Kinoshita, M Nio (1999)) is in good agreeement to the improved asymptotic one:

$$-0.230696(5)^{impr.\ asym.,1995} \quad \Longleftrightarrow \quad -0.230\ 596\ (416)^{num,1999}$$

On-Shell versus $\overline{\text{MS}}$ Schemes for α

Traditionally in calculations of a_ℓ everybody uses the classical OS-scheme: all lepton masses are **on-shell** ones and the charge renormalization is fixed by the condition:

$$\Pi^{OS}(Q=0,m,\alpha^{OS}) \equiv 0$$

In QED different schemes are easily related through the *scheme invariant concept* of the *invariant charge*:

$$\frac{\alpha^{OS}}{1+\Pi^{OS}(Q,m,\alpha^{OS})} = \frac{\overline{\alpha}}{1+\overline{\Pi}(Q,m,\overline{\alpha})}$$

Thw important facts:

- The $\overline{\text{MS}}$-renormalzed photon self-energy $\overline{\Pi}(Q,m,\overline{\alpha})$ does not have any $\ln(m_e)$ terms at large Q (they apear only $\mathcal{O}(m_e^4/Q^4)$ level)
- It does cointain $\ln(m_e/\mu)$ at $Q=0$

Thus, in order to consruct Π_R^{OS} in four-loop one needs 3 pieces:

- the value of $\overline{\Pi}(Q^2, m = 0, \overline{\alpha})$ at 4 loops $\Longleftarrow$ easlily computable with BAICER (not so many dots) (P. Baikov, 2000-2008)

- $\overline{\Pi}(Q = 0, m, \overline{\alpha})$ at 4 loops $\Longleftarrow$ easlily computable these days (not so many dots)

  As a by-product we also get the 4-loop relation between OS- and $\overline{\text{MS}}$ renormalized α:

$$\alpha^{OS} = \frac{\overline{\alpha}}{1 + \overline{\Pi}(Q^2 = 0, m, \overline{\alpha})}$$

- the $\overline{\text{MS}}$ $\Longleftrightarrow$ On-Shell relation for quark masses at 3 loops (M. Steinhauser, K.Ch, (1999); K. Melnikov, T. van Ritbergen (2000))

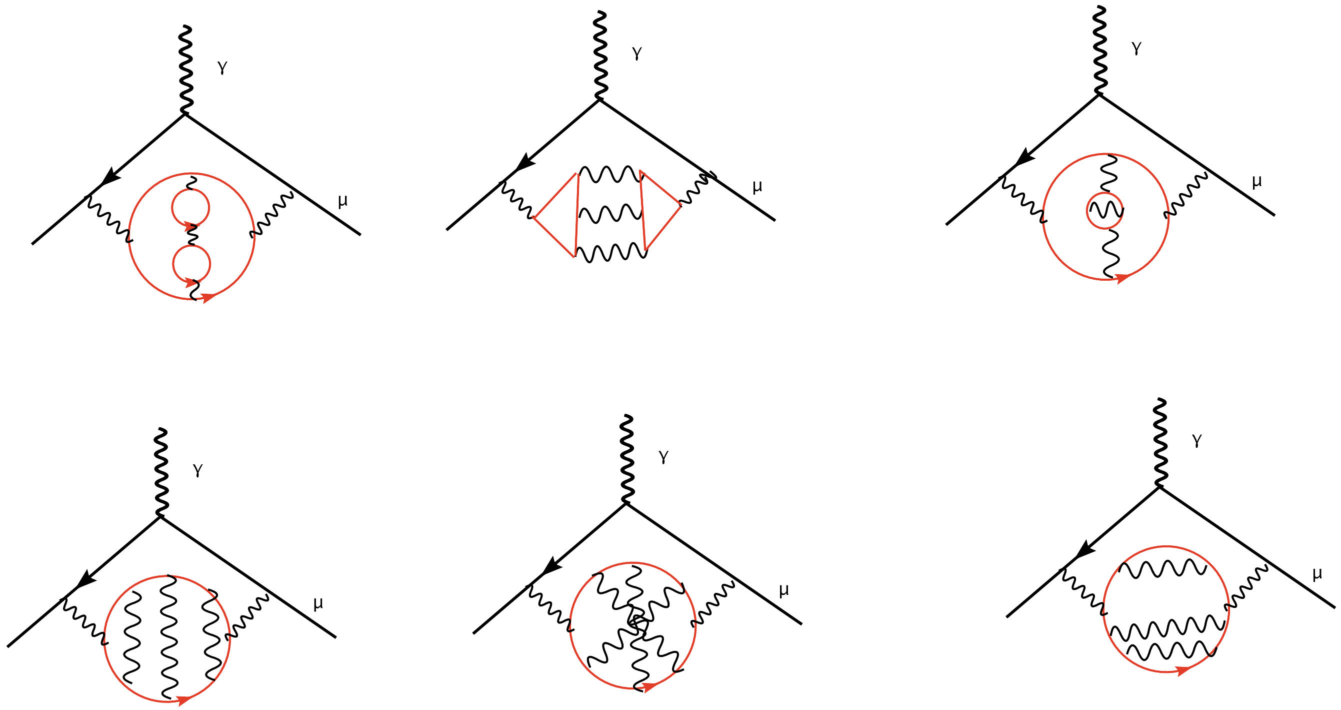
The four-loop piece of Π_R^{OS} reads ($L_{Qm} = \ln\frac{Q^2}{m_e^2}, \quad a_n = \mathrm{PolyLog}[n, 1/2]$)

$$
\begin{aligned}
\Pi_4^{OS} = & \\
+ \quad & n_l \Big[-\frac{71189}{34560} - \frac{157}{72}\pi^2 - \frac{59801}{129600}\pi^4 + \frac{6559}{320}\zeta_3 - \frac{1}{24}\pi^2\zeta_3 - \frac{1603}{120}\zeta_5 \\
& - \frac{35}{4}\zeta_7 + \frac{23}{128}\ell_{Qm} + \frac{59}{12}\pi^2\ln 2 + \frac{106}{675}\pi^4\ln 2 - \frac{1559}{1080}\pi^2\ln^2 2 + \frac{32}{135}\pi^2\ln^3 2 \\
& + \frac{1559}{1080}\ln^4 2 - \frac{32}{225}\ln^5 2 + \frac{1559}{45}a_4 + \frac{256}{15}a_5 \Big] \\
+ \quad & n_l^2 \Big[\frac{20099}{4725} - \frac{179}{324}\pi^2 - \frac{9491}{28800}\pi^4 + \frac{1054517}{50400}\zeta_3 - \frac{5}{3}\zeta_3^2 - \frac{95}{18}\zeta_5 \\
& - \frac{7}{18}\ell_{Qm} - \frac{11}{12}\zeta_3\ell_{Qm} + \frac{5}{3}\zeta_5\ell_{Qm} - \frac{1}{48}\ell_{Qm}^2 + \frac{16}{27}\pi^2\ln 2 - \frac{1001}{720}\pi^2\ln^2 2 \\
& + \frac{1001}{720}\ln^4 2 + \frac{1001}{30}a_4 \Big] \\
+ \quad & n_l^3 \Big[\frac{75259}{68040} + \frac{8}{405}\pi^2 - \frac{15109}{22680}\zeta_3 - \frac{5}{9}\zeta_5 - \frac{151}{162}\ell_{Qm} + \frac{19}{27}\zeta_3\ell_{Qm} \\
& + \frac{11}{72}\ell_{Qm}^2 - \frac{1}{9}\zeta_3\ell_{Qm}^2 - \frac{1}{108}\ell_{Qm}^3 \Big],
\end{aligned}
\qquad (1)
$$

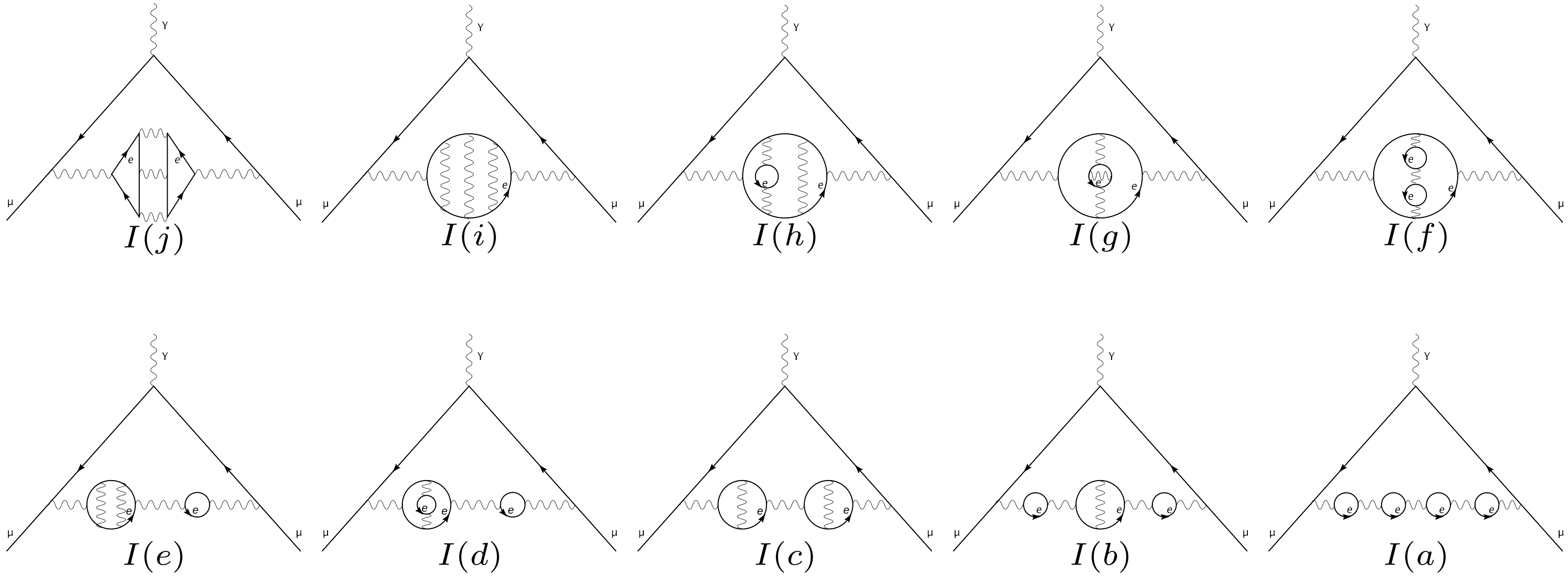
The resulting contributions to the a_μ *coming from genuily 4-loop terms in the photon propagator* read ($L_{Mm} = \ln \frac{M_\mu}{m_e}$)

$$\left(\frac{\alpha}{\pi}\right)^5 \Big[n_l^3 \left(-0.591 + 0.546 L_{Mm} - 0.177 L_{Mm}^2 + 0.037 L_{Mm}^3\right) \\ + n_l^2 \left((0.924 - 0.342 L_{Mm} + 0.0417 L_{Mm}^2) + n_l (1.210 - 0.1796 L_{Mm})\right) \Big] \tag{1}$$

they correspond to diagrams:



+ about a fifty more



Subset	analytical	numerical
$I(j)$	-1.21429	–
$I(i)$	+0.25237	–
$I(g) + I(h)$	+1.50112	–
$I(f)$	+2.89019	+2.88598(9) ✓
$I(c)$	+4.81759	+4.74212(14) ✓
$I(d)$	+7.44918	+7.45270(88) ✓
$I(e)$	-1.33141	-1.20841(70) ✓
$I(b)$	+27.7188	+27.69038(30) ✓
$I(a)$	+20.1832	+20.14293(23) ✓

(all numerics from M. Nio, T. Aoyama, M. Hayakawa and T. Kinoshita (2007) and T. Kinoshita and M. Nio, (2006) Numerically all (VP,mixed) 5-loop terms summed together lead to a tiny contribution:

$$a_\mu(\text{VP,mixed,4-loops}) = \left(\frac{\alpha}{\pi}\right)^5 \left[3.429 + \mathcal{O}(\frac{m_e}{M_\mu})\right]$$

$$= 2.31896 \cdot 10^{-13} + \mathcal{O}(\frac{m_e}{M_\mu})$$

Status of the QED β-function at five loops

Recently we have computed the $\beta^{\overline{\mathsf{MS}}}(\overline{\alpha})$ in the quenched approximation $(A = A(\mu) = \frac{\alpha(\mu)}{4\,\pi})$

$$\beta^{\mathrm{qQED}} = n_l \left[\frac{4}{3} A + 4\,A^2 - 2\,A^3 - 46\,A^4 + \left(\frac{4157}{6} \quad + \quad 128\,\zeta_3\right) A^5\right]$$

At the moment we are finishing the remaining five-loops diagrams proportional to n_l^2 and n_l^3, the "renormalon" type contribution of order n_l^4 is simple and available since long . . .

Conclusions

- numerical and analytical methods help each other in computing the QED contributions to $(g-2)_\mu$
- the asymptotic contribution to the VP part of $(g-2)_\mu$ in order α^5 is is computed and supports purely numerical result of the Kinoshita group
- conversion formula for $\alpha^{OS}/\alpha^{\overline{\mathrm{MS}}}$ is evaluated to four loops: it means that one could reexpress any (QED!) $\mathcal{O}(\alpha^5)$ result in terms of the runnung $\alpha^{\overline{\mathrm{MS}}}$
- together with (soon to be completed) QED β-function in 5 loops this opens a way to investigate the five loop QED result with RG methods (see, e.g. A. Kataev, (2006)).