Two-Loop Heavy-Flavor Contribution to Bhabha Scattering

Roberto BONCIANI

Institute for Theoretical Physics
University of Zürich
CH-8057 Zürich, Switzerland

In collaboration with:

A. Ferroglia and A. A. Penin

Plan of the Talk

- Introduction
- NNLO QED corrections:
  - Photonic contributions
  - Electron-Loop contributions
  - Heavy-Fermion-Loop contributions
- Numerics
- Summary

Why Bhabha Scattering?

- Bhabha Scattering is a fundamental process for e^+e^- collider physics because it is chosen for the precise evaluation of the Luminosity:

$$L = \frac{N}{\sigma_{th}}$$

  where N is the measured number of Bhabha events and σ_{th} is the Bhabha cross section calculated from theory.

  Since L enters as a normalization factor in the cross section measurements a process in which δL is as small as possible is needed.

- Bhabha scattering is a process with a large cross section and it is QED dominated $\Rightarrow$
  - it allows precise experimental measurements (large statistics);
  - it allows precise theoretical calculation of the cross section $\Longrightarrow$ radiative corrections under control at the level of NNLO.

Small and Large Angle Bhabha Scattering

Small and Large Angle Bhabha Scattering

- Small-Angle

  SABS is important for high-energy accelerators, as for instance LEP or the future ILC.

  For LEP, luminometers were located between 1.4° and 2.9°

  For ILC, they will be located between 0.7° and 2.3°

  The small angle region makes in such a way that the weak contribution can be neglected (the Born with a Z^0 exchanged is already at the level of 0.1%)

Small and Large Angle Bhabha Scattering

- Small-Angle

  SABS is important for high-energy accelerators, as for instance LEP or the future ILC.

  For LEP, luminometers were located between 1.4° and 2.9°

  For ILC, they will be located between 0.7° and 2.3°

  The small angle region makes in such a way that the weak contribution can be neglected (the Born with a Z^0 exchanged is already at the level of 0.1%)

- Large-Angle

  LABS is important for low-energy accelerators (meson factories), as for instance DAΦNE.

  The KLOE experiment has luminometers located between 55° and 125°

  The small energy makes in such a way that the weak contributions also in this case are negligible. At 10 GeV they are at the level of 0.1%

Experimental Accuracy Requirements

Experimental Accuracy Requirements

DAΦNE – VEPP-2M	$\Rightarrow$	0.1 %
LEP	$\Rightarrow$	0.3-0.5 %
ILC	$\Rightarrow$	$\sim$ 0.01 %

Experimental Accuracy Requirements

DAΦNE – VEPP-2M	$\Rightarrow$	0.1 %
LEP	$\Rightarrow$	0.3-0.5 %
ILC	$\Rightarrow$	$\sim$ 0.01 %

This needs devoted Monte Carlos with two ingredients under control:

Experimental Accuracy Requirements

DAΦNE – VEPP-2M	$\Rightarrow$	0.1 %
LEP	$\Rightarrow$	0.3-0.5 %
ILC	$\Rightarrow$	$\sim$ 0.01 %

This needs devoted Monte Carlos with two ingredients under control:

- Parton shower
- matching with fixed-order Radiative Corrections (at the level of NNLO?)

Experimental Accuracy Requirements

DAΦNE – VEPP-2M	$\Rightarrow$	0.1 %
LEP	$\Rightarrow$	0.3-0.5 %
ILC	$\Rightarrow$	$\sim$ 0.01 %

This needs devoted Monte Carlos with two ingredients under control:

- Parton shower
- matching with fixed-order Radiative Corrections (at the level of NNLO?)

BABAYAGA	C. M. Carloni Calame, *et al.*, Nucl. Phys. B 584 (2000) 459
BHLUMI	S. Jadach, *et al.*, Comput. Phys. Commun. 102 (1997) 229
BHWIDE	S. Jadach, *et al.*, Phys. Lett. B 390 (1997) 298
MCGPJ	A. B. Arbuzov, *et al.*, JHEP 9710 (1997) 001
SABSPV	M. Cacciari, *et al.*, Comput. Phys. Commun. 90 (1995) 301
BHAGENF	F. A. Berends, *et al.*, Nucl. Phys. B 228 (1983) 537

Experimental Accuracy Requirements

DAΦNE – VEPP-2M	$\Rightarrow$	0.1 %
LEP	$\Rightarrow$	0.3-0.5 %
ILC	$\Rightarrow$	$\sim$ 0.01 %

This needs devoted Monte Carlos with two ingredients under control:

- Parton shower
- matching with FIXED-ORDER RADIATIVE CORRECTIONS (at the level of NNLO?)

BABAYAGA	**C. M. Carloni Calame,** *et al.*, **Nucl. Phys. B 584 (2000) 459**
BHLUMI	**S. Jadach,** *et al.*, **Comput. Phys. Commun. 102 (1997) 229**
BHWIDE	**S. Jadach,** *et al.*, **Phys. Lett. B 390 (1997) 298**
MCGPJ	**A. B. Arbuzov,** *et al.*, **JHEP 9710 (1997) 001**
SABSPV	**M. Cacciari,** *et al.*, **Comput. Phys. Commun. 90 (1995) 301**
BHAGENF	**F. A. Berends,** *et al.*, **Nucl. Phys. B 228 (1983) 537**

Cross Section in the Literature

Cross Section in the Literature

- One-loop corrections
  - QED and EW corr. (Consoli ’79, Böhm-Denner-Hollik ’88, Greco ’88, Caffo-Gatto-Remiddi ’88)

Cross Section in the Literature

- One-loop corrections
  - QED and EW corr. (Consoli '79, Böhm-Denner-Hollik '88, Greco '88, Caffo-Gatto-Remiddi '88)
- Two-loop QED corrections
  - Leading Log-enhanced corr. (virtual and real) for SABS and LABS (Faldt-Osland '94, Arbuzov-Fadin-Kuraev-Lipatov-Merenkov-Trentadue '95-'97)
  - Virtual corr. to the cross section with $m = 0$ (Bern-Dixon-Ghinkulov '00)
  - Log-enhanced photonic contributions (Glover-Tausk-van der Bij '01)
  - $N_F = 1$ with $m_e \neq 0$ (B.-Ferroglia-Mastrolia-Remiddi-van der Bij '04-'05)
  - Constant term of photonic corrections not suppressed by the ratio m^2/s (Penin '05)
  - HF contr. in the small-m_f limit (Actis-Czakon-Gluza-Riemann '07, Becher-Melnikov '07)
  - HF contribution: complete analytic dep. on m_f (B.-Ferroglia-Penin '07)
  - HF and H contribution: num. with disp. rel. (Actis-Czakon-Gluza-Riemann '07)

Cross Section in the Literature

- One-loop corrections
  - QED and EW corr. (Consoli '79, Böhm-Denner-Hollik '88, Greco '88, Caffo-Gatto-Remiddi '88)
- Two-loop QED corrections
  - Leading Log-enhanced corr. (virtual and real) for SABS and LABS (Faldt-Osland '94, Arbuzov-Fadin-Kuraev-Lipatov-Merenkov-Trentadue '95-'97)
  - Virtual corr. to the cross section with $m = 0$ (Bern-Dixon-Ghinkulov '00)
  - Log-enhanced photonic contributions (Glover-Tausk-van der Bij '01)
  - $N_F = 1$ with $m_e \neq 0$ (B.-Ferroglia-Mastrolia-Remiddi-van der Bij '04-'05)
  - Constant term of photonic corrections not suppressed by the ratio m^2/s (Penin '05)
  - HF contr. in the small-m_f limit (Actis-Czakon-Gluza-Riemann '07, Becher-Melnikov '07)
  - HF contribution: complete analytic dep. on m_f (B.-Ferroglia-Penin '07)
  - HF and H contribution: num. with disp. rel. (Actis-Czakon-Gluza-Riemann '07)
- Two-loop EW corrections
  - Log-enhanced corr. (Bardin-Hollik-Riemann '90, Fadin-Lipatov-Martin-Melles '00, Jantzen-Kühn-Moch-Penin-Smirnov '01-'05)

Differential Cross Section

Differential Cross Section

The goal is the calculation of the differential Cross Section:

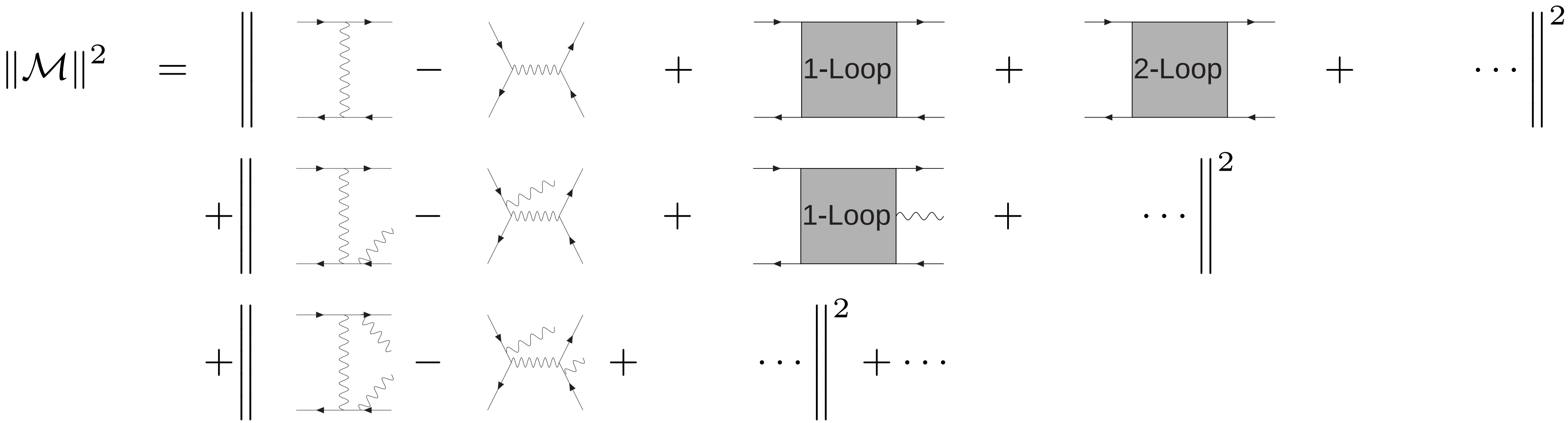
$$\frac{d\sigma}{d\Omega} \sim \|\mathcal{M}\|^2$$

Differential Cross Section

The goal is the calculation of the differential Cross Section:

$$\frac{d\sigma}{d\Omega} \sim \|\mathcal{M}\|^2$$

where:

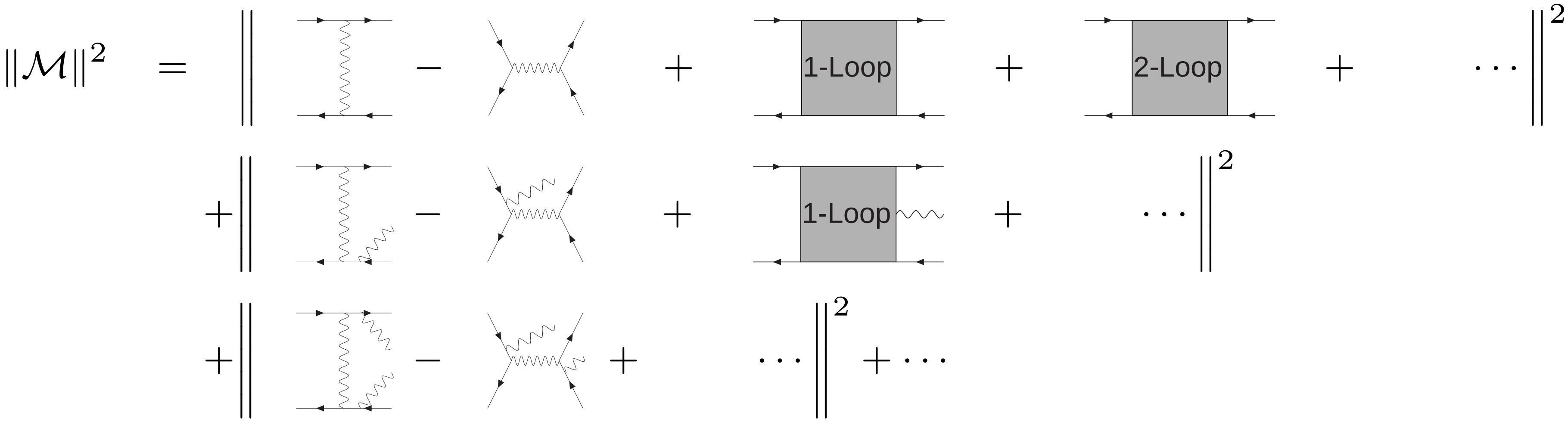


Differential Cross Section

The goal is the calculation of the differential Cross Section:

$$\frac{d\sigma}{d\Omega} \sim \|\mathcal{M}\|^2$$

where:



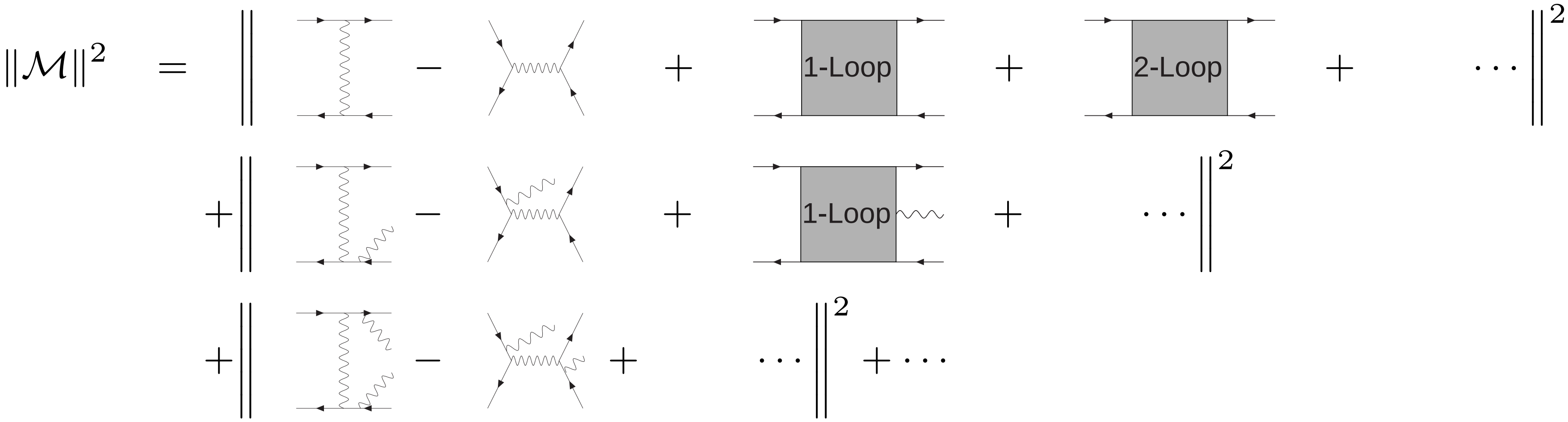
$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

Differential Cross Section

The goal is the calculation of the differential Cross Section:

$$\frac{d\sigma}{d\Omega} \sim \|\mathcal{M}\|^2$$

where:



$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

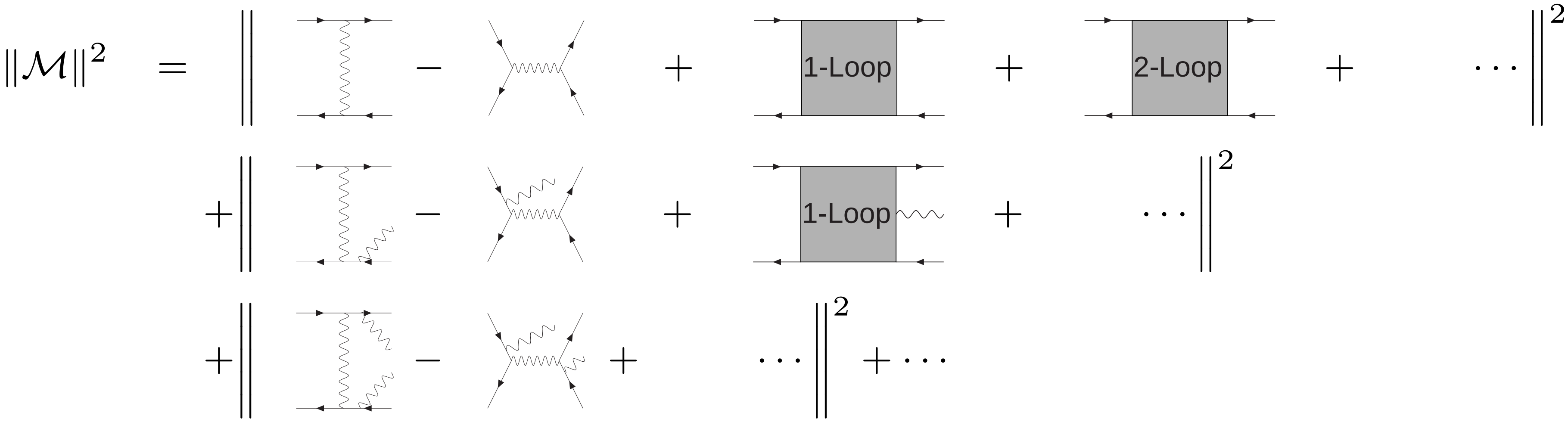
$$\frac{d\sigma_0(s,t,m^2)}{d\Omega} = \frac{\alpha^2}{s}\left\{\frac{1}{s^2}\left[st + \frac{s^2}{2} + (t-2m^2)^2\right] + \frac{1}{t^2}\left[st + \frac{t^2}{2} + (s-2m^2)^2\right] + \frac{1}{st}\left[(s+t)^2 - 4m^4\right]\right\}$$

Differential Cross Section

The goal is the calculation of the differential Cross Section:

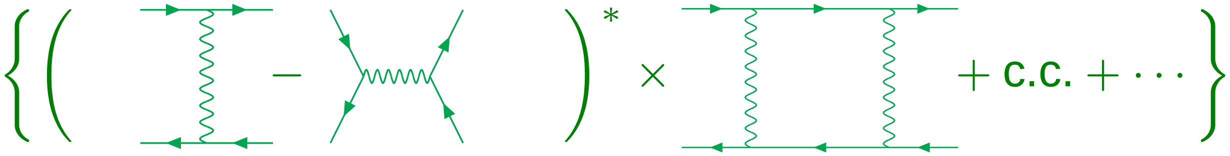
$$\frac{d\sigma}{d\Omega} \sim \|\mathcal{M}\|^2$$

where:



$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Big\{ \ldots + \text{c.c.} + \cdots \Big\}$$

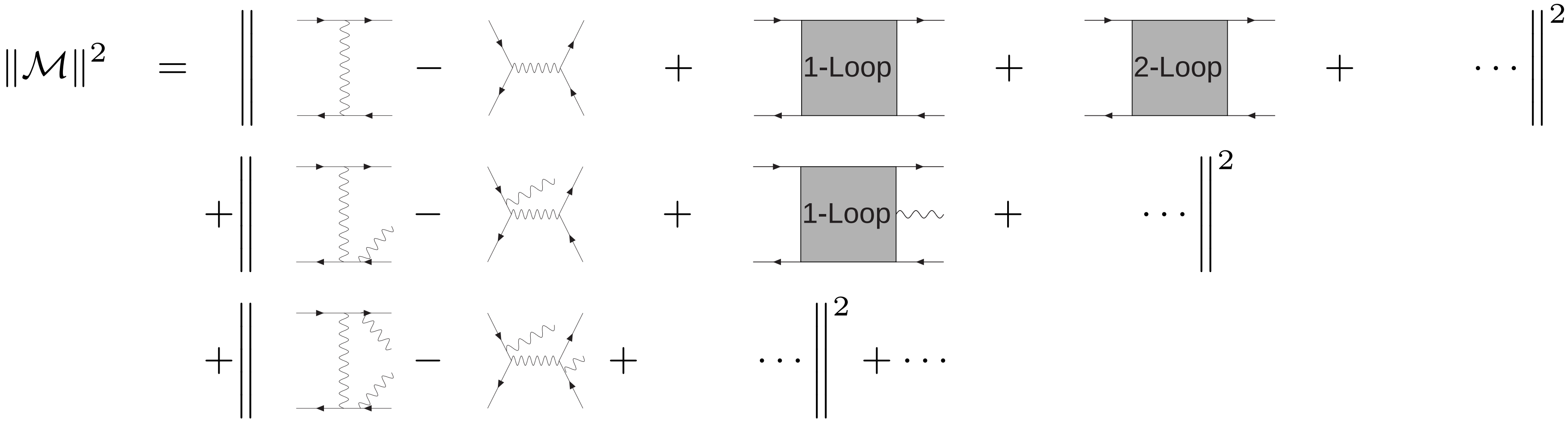


Differential Cross Section

The goal is the calculation of the differential Cross Section:

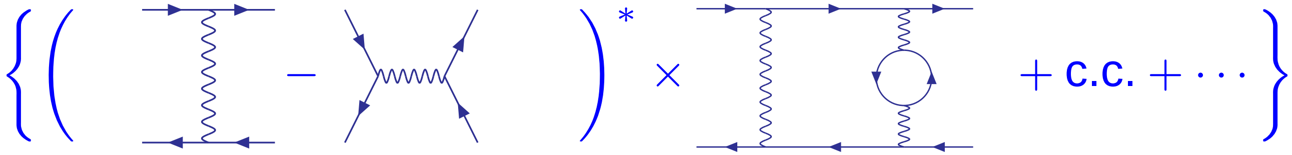
$$\frac{d\sigma}{d\Omega} \sim \|\mathcal{M}\|^2$$

where:



$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

$$\left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}$$



QED Corrections

We can devide the QED higher-order corrections in three gauge-independent groups:

QED Corrections

We can devide the QED higher-order corrections in three gauge-independent groups:

Pure Photonic Corrections	Electron-Loop Corrections	Heavy-F Loop Corrections

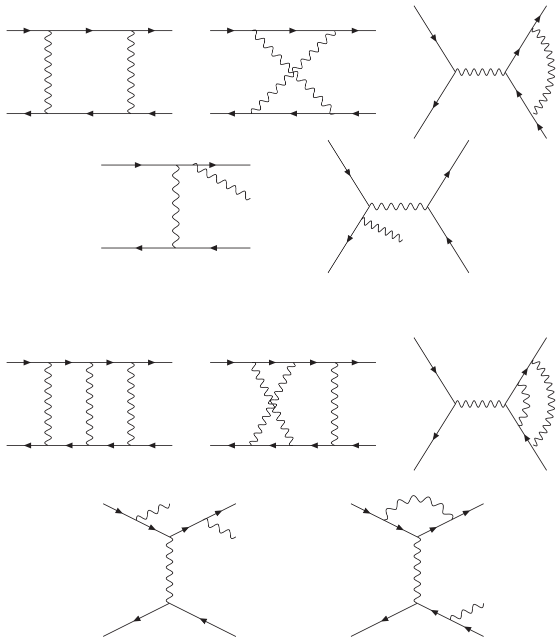
QED Corrections

We can devide the QED higher-order corrections in three gauge-independent groups:

Pure Photonic Corrections

Electron-Loop Corrections

Heavy-F Loop Corrections

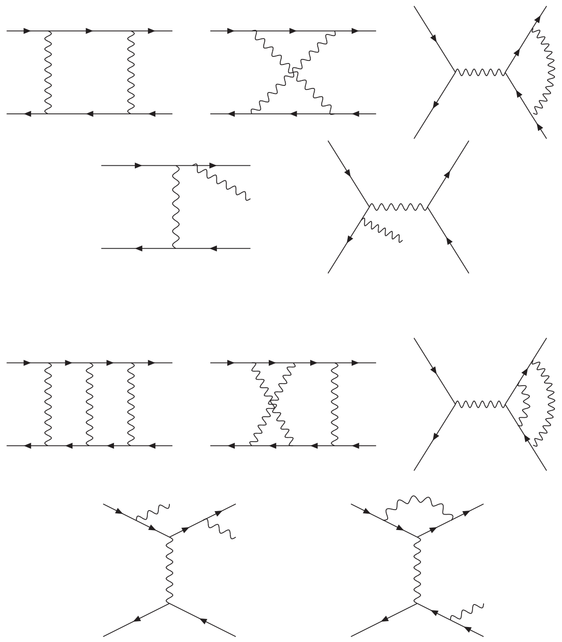


Arbuzov-Fadin-Kuraev-Lipatov-
-Merenkov-Trentadue '95-'97
Glover-Tausk-van der Bij '01
Penin '05
(B.-Ferroglia '05)

QED Corrections

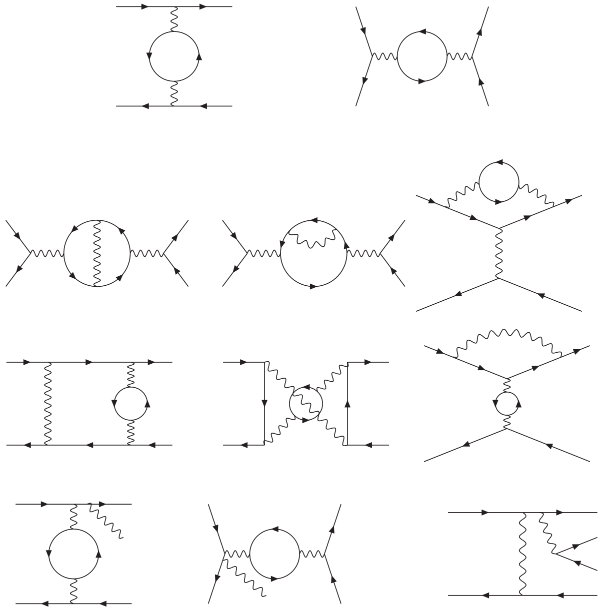
We can devide the QED higher-order corrections in three gauge-independent groups:

Pure Photonic Corrections



Arbuzov-Fadin-Kuraev-Lipatov-
-Merenkov-Trentadue ’95-’97
Glover-Tausk-van der Bij ’01
Penin ’05
(B.-Ferroglia ’05)

Electron-Loop Corrections



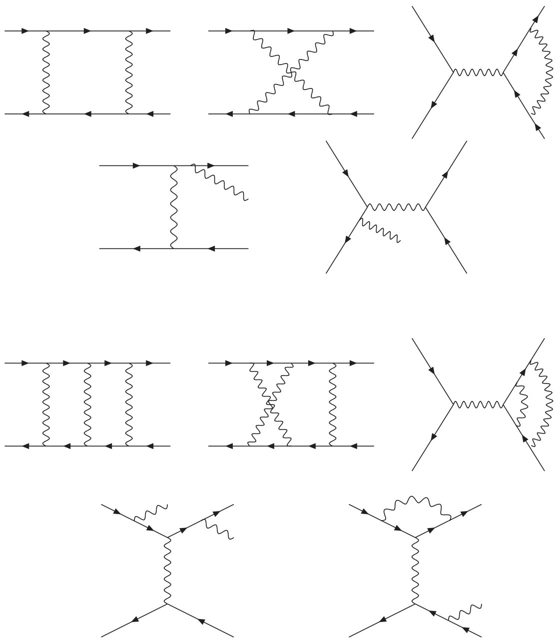
Arbuzov-Kuraev-Merenkov-
-Trentadue ’97
B.-Ferroglia-Mastrolia-Remiddi-
-van der Bij ’04-’05
(B.-Ferroglia ’05)

Heavy-F Loop Corrections

QED Corrections

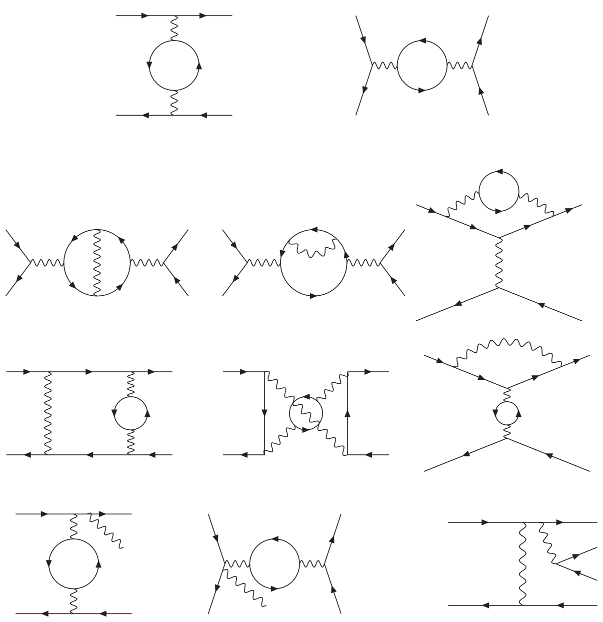
We can devide the QED higher-order corrections in three gauge-independent groups:

Pure Photonic Corrections



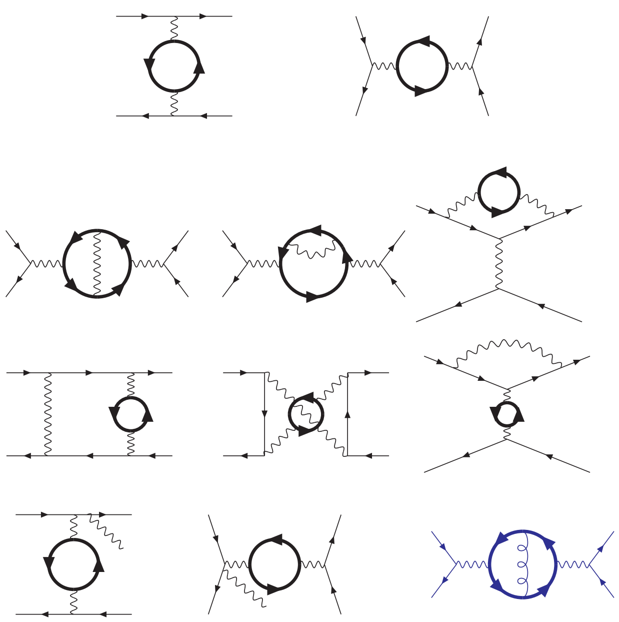
Arbuzov-Fadin-Kuraev-Lipatov-
-Merenkov-Trentadue '95-'97
Glover-Tausk-van der Bij '01
Penin '05
(B.-Ferroglia '05)

Electron-Loop Corrections



Arbuzov-Kuraev-Merenkov-
-Trentadue '97
B.-Ferroglia-Mastrolia-Remiddi-
-van der Bij '04-'05
(B.-Ferroglia '05)

Heavy-F Loop Corrections



Becher-Melnikov '07
Actis-Czakon-Gluza-Riemann '07
B.-Ferroglia-Penin '07

Mass Hierarchy

Mass Hierarchy

The physical problem is characterized by a well defined mass hierarchy

- Low-Energy Acc.

$$m_e^2 \ll m_\mu^2 < m_c^2 \sim m_\tau^2 \sim m_b^2 \sim s,t,u \ll m_t^2$$

- High-Energy Acc.

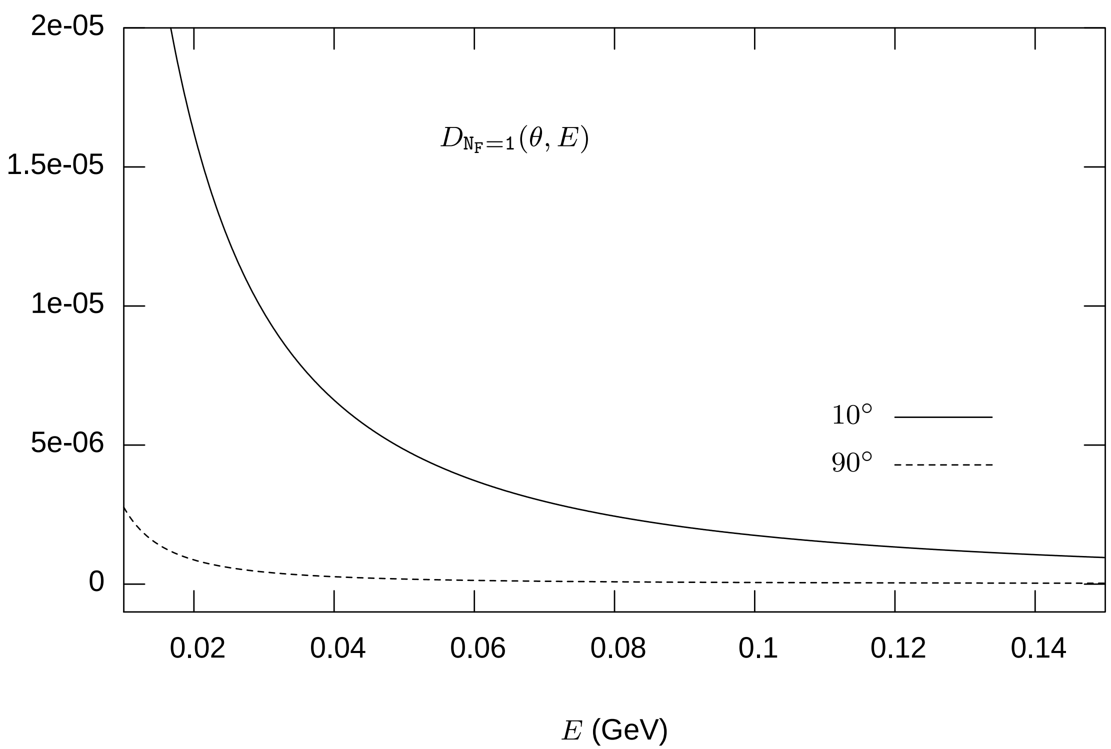
$$m_e^2 \ll m_{light-f}^2 \ll m_t^2 \sim s,t,u$$

The electron mass is always small compared to all the scales in the game

In both cases, therefore, the electron contribution provides the biggest fermionic contribution, followed by the muon

This hierarchy allows to calculate radiative corrections neglecting the mass of the electron, or, better, keeping the mass of the electron only in the log-enhanced terms, as a regulator for the collinear divergences

Mass Hierarchy

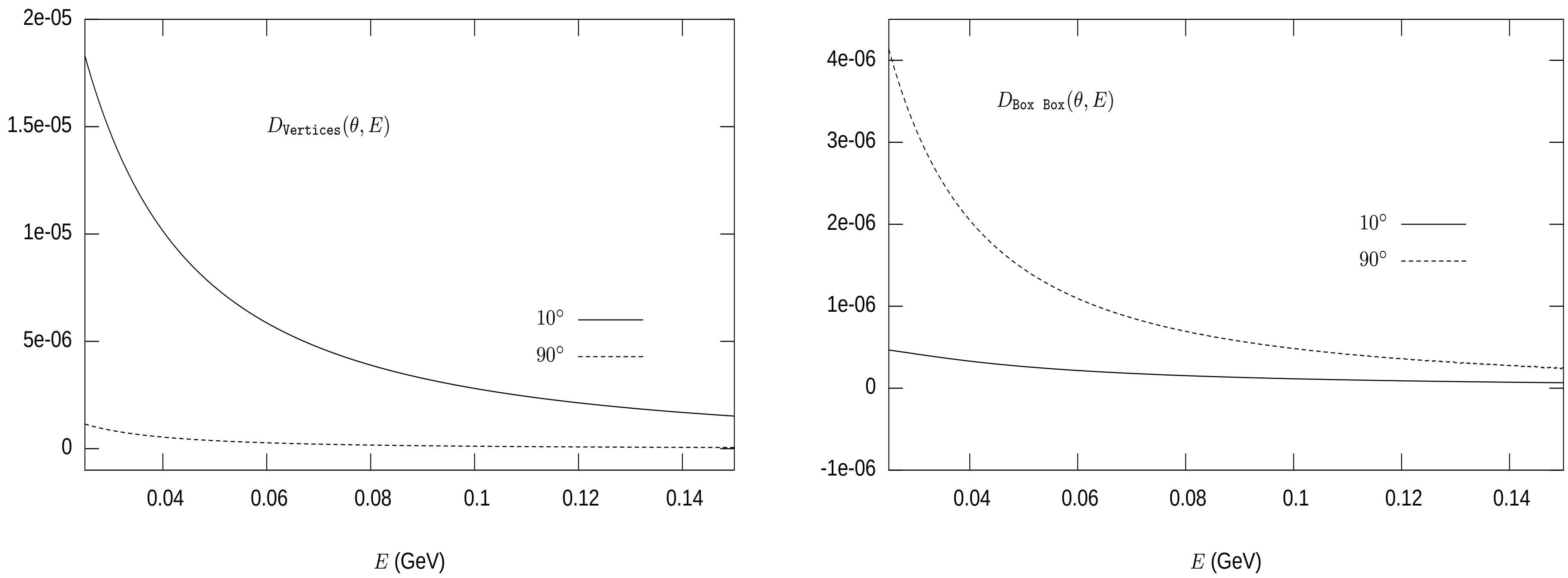


$$D_{\mathrm{N_F}=1}(\theta, E) = \left(\frac{\alpha}{\pi}\right)^2 \left| \left(\frac{d\sigma_2^{(\mathrm{N_F}=1)}}{d\Omega} - \left. \frac{d\sigma_2^{(\mathrm{N_F}=1)}}{d\Omega} \right|_L \right) \right| \left(\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1}{d\Omega} \right)^{-1}$$

The soft-photon energy cut-off is set equal to the beam energy: $\omega = E$
The soft-pair energy cut-off is set equal to the beam energy: $\Omega = E$

R. B. and A. Ferroglia, Phys. Rev. D 72, 056004 (2005)

Mass Hierarchy



$$D_i(\theta, E) = \left(\frac{\alpha}{\pi}\right)^2 \left| \left(\frac{d\sigma_2^{(\text{ph } i)}}{d\Omega} - \frac{d\sigma_2^{(\text{ph } i)}}{d\Omega}\bigg|_L \right) \right| \left(\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1}{d\Omega} \right)^{-1}$$

The soft-photon energy cut-off is set equal to the beam energy: $\omega = E$

R. B. and A. Ferroglia, Phys. Rev. D 72, 056004 (2005)

The Cross Section in the small-m_e limit

... therefore, we can expand the Bhabha scattering Differential Cross Section in series of the electron mass and retain only terms that do not vanish in the limit $m_e^2/s \to 0$.

The Cross Section in the small-m_e limit

... therefore, we can expand the Bhabha scattering Differential Cross Section in series of the electron mass and retain only terms that do not vanish in the limit $m_e^2/s \to 0$.

At the NNLO the Cross Section has the following form:

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)}(\xi)\ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)}(\xi)\ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)}(\xi) + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

where

$$\xi = \frac{1-\cos\theta}{2}$$

The Cross Section in the small-m_e limit

... therefore, we can expand the Bhabha scattering Differential Cross Section in series of the electron mass and retain only terms that do not vanish in the limit $m_e^2/s \to 0$.

At the NNLO the Cross Section has the following form:

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)}(\xi) \ln^2 \left(\frac{s}{m_e^2} \right) + \delta_2^{(1)}(\xi) \ln \left(\frac{s}{m_e^2} \right) + \delta_2^{(0)}(\xi) + \mathcal{O} \left(\frac{m_e^2}{s} \right)$$

where

$$\xi = \frac{1 - \cos\theta}{2}$$

NOTE: this approximation is not valid in the almost-forward ($|t| < m^2$) and in the almost-backward ($|u| < m^2$) directions, where terms of order m^2/t and m^2/u become important

The Cross Section in the small-m_e limit

... therefore, we can expand the Bhabha scattering Differential Cross Section in series of the electron mass and retain only terms that do not vanish in the limit $m_e^2/s \to 0$.

At the NNLO the Cross Section has the following form:

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)}(\xi) \ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)}(\xi) \ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)}(\xi) + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

where

$$\xi = \frac{1 - \cos\theta}{2}$$

NOTE: this approximation is not valid in the almost-forward ($|t| < m^2$) and in the almost-backward ($|u| < m^2$) directions, where terms of order m^2/t and m^2/u become important

However, alredy at 1° the terms of order m^2/t are TOTALLY negligible.

The Photonic Contribution

$$\frac{d\sigma_{phot}}{d\sigma_0} = \delta^{(2)}_{phot,2} \ln^2\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{phot,1} \ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{phot,0}$$

The Photonic Contribution

$$\frac{d\sigma_{phot}}{d\sigma_0} = \delta^{(2)}_{phot,2} \ln^2\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{phot,1} \ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{phot,0}$$

$\delta^{(2)}_{phot,2}$ known since **Arbuzov-Kuraev-Shaikhatdenov '98**

$\delta^{(2)}_{phot,1}$ known since **Glover-Tausk-van der Bij '01**

$\delta^{(2)}_{phot,0}$ known since **Penin '05**

The Photonic Contribution

$$\frac{d\sigma_{phot}}{d\sigma_0} = \delta^{(2)}_{phot,2} \ln^2 \left(\frac{s}{m_e^2} \right) + \delta^{(2)}_{phot,1} \ln \left(\frac{s}{m_e^2} \right) + \delta^{(2)}_{phot,0}$$

- $\delta^{(2)}_{phot,i}$ Reconstruction from massless CS

For a generic QED/QCD process without closed fermion loops

$$\mathcal{M}^{(m \neq 0)} = \prod_{i \in \{\texttt{all legs}\}} Z_i^{\frac{1}{2}}(m, \epsilon) \mathcal{M}^{(m=0)}$$

where Z is the ratio between the massive and massless Dirac form factor

$$F^{(m \neq 0)}(Q^2) = Z(m, \epsilon)\, F^{(m=0)}(Q^2) + \mathcal{O}(m^2/Q^2)$$

Therefore, starting from the totally massless result of Bern-Dixon-Ghinkulov ’00 one can reconstruct the photonic cross section where the collinear divergences are regulated with the mass of the electron

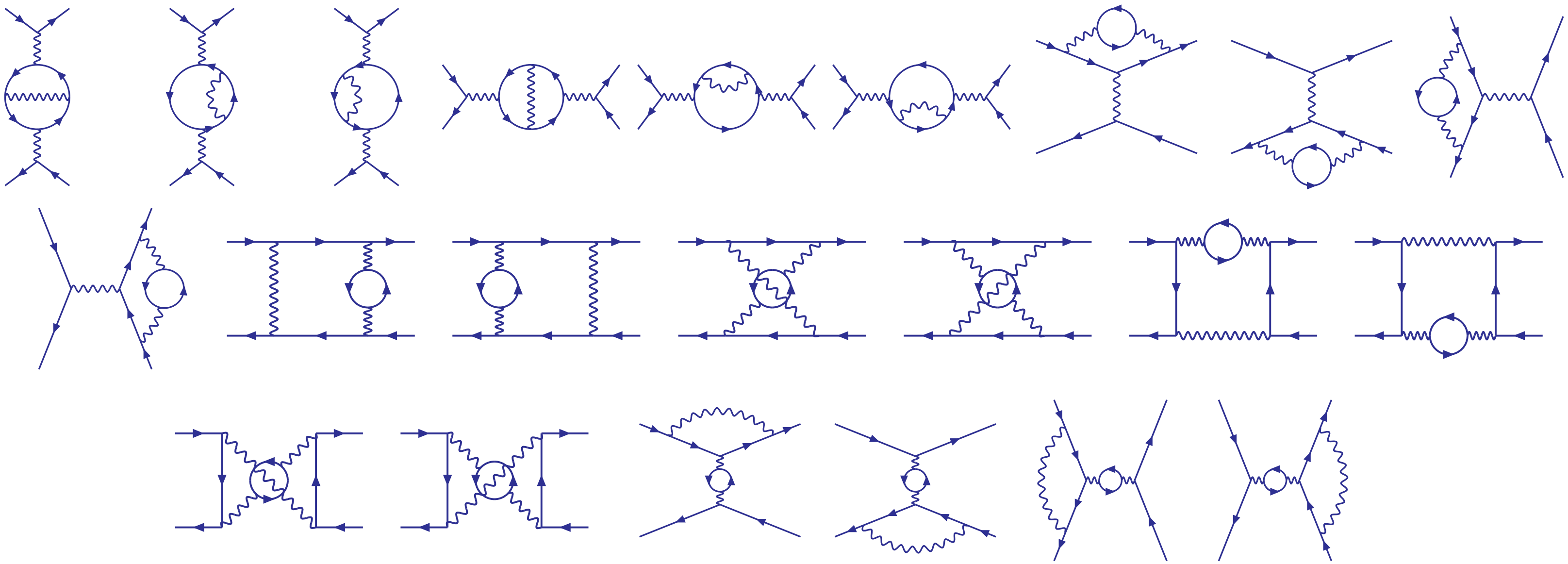
A. Mitov and S. Moch, JHEP 0705 (2007) 001.
T. Becher and K. Melnikov, JHEP 0706 (2007) 084.

The Electron-Loop Contribution

The Electron-Loop Contribution

The Cross Section is known with the full dependence on m_e

$N_F = 1$



- We calculated with standard methods the above set of Feynman diagrams
- UV and IR divergences regularized in dim reg, appear as poles in $(D-4)$
- After UV-renormalization and inclusion of the real soft radiation, the set is finite
- Expanding in m_e ....

R. B., A. Ferroglia, P. Mastrolia, E. Remiddi and J. J. van der Bij, Nucl. Phys. B716 (2005) 280

The Electron-Loop Contribution

$$\frac{d\sigma_{N_F=1}}{d\sigma_0} = \delta^{(2)}_{N_F=1,3} \ln^3\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F=1,2} \ln^2\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F=1,1} \ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F=1,0}$$

The Electron-Loop Contribution

$$\frac{d\sigma_{N_F=1}}{d\sigma_0} = \delta^{(2)}_{N_F=1,3}\ln^3\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F=1,2}\ln^2\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F=1,1}\ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F=1,0}$$

where:

$$\delta^{(2)}_{N_F=1,3} = \frac{1}{(1-\xi+\xi^2)^2}\left[-\frac{1}{9}+\frac{2}{9}\xi-\frac{1}{3}\xi^2+\frac{2}{9}\xi^3-\frac{1}{9}\xi^4\right]$$

$$\delta^{(2)}_{N_F=1,2} = \frac{1}{(1-\xi+\xi^2)^2}\left[\ln\left(\frac{4w^2}{s}\right)\left[\left(-\frac{4}{3}+\frac{8}{3}\xi-4\xi^2+\frac{8}{3}\xi^3-\frac{4}{3}\xi^4\right)-\left(\frac{17}{18}-\frac{17}{9}\xi+\frac{17}{6}\xi^2-\frac{17}{9}\xi^3+\frac{17}{18}\xi^4\right)\right.\right.$$
$$\left.-\left(\frac{1}{3}-\frac{2}{3}\xi+\xi^2-\frac{2}{3}\xi^3+\frac{1}{3}\xi^4\right)\ln(1-\xi)-\left(\frac{1}{3}-\frac{1}{3}\xi+\frac{1}{3}\xi^3-\frac{1}{3}\xi^4\right)\ln(\xi)\right]$$

$$\delta^{(2)}_{N_F=1,1} = \frac{1}{(1-\xi+\xi^2)^2}\left[\ln\left(\frac{4w^2}{s}\right)\left[\left(\frac{4}{3}-\frac{8}{3}\xi+4\xi^2-\frac{8}{3}\xi^3+\frac{4}{3}\xi^4\right)\ln(1-\xi)+\left(\frac{32}{9}-\frac{64}{9}\xi+\frac{32}{3}\xi^2-\frac{64}{9}\xi^3\right.\right.\right.$$
$$\left.\left.\left.+\frac{32}{9}\xi^4\right)-\left(\frac{8}{3}-\frac{14}{3}\xi+6\xi^2-\frac{10}{3}\xi^3+\frac{4}{3}\xi^4\right)\ln(\xi)\right]\cdots\right]$$

$$\delta^{(2)}_{N_F=1,0} = \frac{1}{(1-\xi+\xi^2)^2}\left[\ln\left(\frac{4w^2}{s}\right)\left[\left(-\frac{20}{9}+\frac{40}{9}\xi-\frac{20}{3}\xi^2+\frac{40}{9}\xi^3-\frac{20}{9}\xi^4\right)\ln(1-\xi)-\left(\frac{20}{9}-\frac{40}{9}\xi+\frac{20}{3}\xi^2\right.\right.\right.$$
$$\left.\left.\left.-\frac{40}{9}\xi^3+\frac{20}{9}\xi^4\right)\cdots\right]\right.$$

In agreement with Becher-Melnikov '07 and Actis-Czakon-Gluza-Riemann '07

A. B. Arbuzov, E. A. Kuraev, N. P. Merenkov and L. Trentadue, Phys. Atom. Nucl. 60 (1997) 591 [Yad. Fiz. 60N4 (1997) 673]

R. B., A. Ferroglia, P. Mastrolia, E. Remiddi and J. J. van der Bij, Nucl. Phys. B716 (2005) 280

Soft-Pair Production

$$\frac{d\sigma_{Pair}}{d\sigma_0} = \delta^{(2)}_{Pair,3} \ln^3\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{Pair,2} \ln^2\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{Pair,1} \ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{Pair,0}$$

where:

$$\delta^{(2)}_{Pair,3} = \frac{1}{(1-\xi+\xi^2)^2}\left(\frac{1}{9} - \frac{2}{9}\xi + \frac{1}{3}\xi^2 - \frac{2}{9}\xi^3 + \frac{1}{9}\xi^4\right)$$

$$\delta^{(2)}_{Pair,2} = \frac{1}{(1-\xi+\xi^2)^2}\left[\ln\left(\frac{4w^2}{s}\right)\left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right) - \left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right)\ln(1-\xi)\right.$$
$$\left. -\left(\frac{5}{9} - \frac{10}{9}\xi + \frac{5}{3}\xi^2 - \frac{10}{9}\xi^3 + \frac{5}{9}\xi^4\right) + \left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right)\ln(\xi)\right]$$

$$\delta^{(2)}_{Pair,1} = \frac{1}{(1-\xi+\xi^2)^2}\left[\ln^2\left(\frac{4w^2}{s}\right)\left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right) + \ln\left(\frac{4w^2}{s}\right)\left[\left(-\frac{2}{3} + \frac{4}{3}\xi - 2\xi^2 + \frac{4}{3}\xi^3\right.\right.\right.$$
$$\left.\left.\left. -\frac{2}{3}\xi^4\right)\ln(1-\xi) - \left(\frac{10}{9} - \frac{20}{9}\xi + \frac{10}{3}\xi^2 - \frac{20}{9}\xi^3 + \frac{10}{9}\xi^4\right) + \left(\frac{2}{3} - \frac{4}{3}\xi + \cdots\right)\right]\right]$$

$$\delta^{(2)}_{Pair,0} = \frac{1}{(1-\xi+\xi^2)^2}\left[\ln^2\left(\frac{4w^2}{s}\right)\left[\left(-\frac{1}{3} + \frac{2}{3}\xi - \xi^2 + \frac{2}{3}\xi^3 - \frac{1}{3}\xi^4\right)\ln(1-\xi) + \left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2\right.\right.\right.$$
$$\left.\left.\left. -\frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right)\ln(\xi)\right]\cdots\right]$$

A. B. Arbuzov, E. A. Kuraev, N. P. Merenkov and L. Trentadue, Phys. Atom. Nucl. 60 (1997) 591 [Yad. Fiz. 60N4 (1997) 673]; Nucl. Phys. B474 (1996) 271.

Heavy-Fermion Contribution

Heavy-Fermion Contribution

The calculation of the two-loop heavy-fermion contribution to the Bhabha scattering differential cross section is, in principle, a four-scales problem !

Heavy-Fermion Contribution

The calculation of the two-loop heavy-fermion contribution to the Bhabha scattering differential cross section is, in principle, a **four-scales problem** !

$\Longrightarrow$ it is REALLY VERY difficult to solve in a standard way (diagrammatically), keeping the full dependence on the heavy and electron mass.

Heavy-Fermion Contribution

The calculation of the two-loop heavy-fermion contribution to the Bhabha scattering differential cross section is, in principle, a four-scales problem !

$\Longrightarrow$ it is REALLY VERY difficult to solve in a standard way (diagrammatically), keeping the full dependence on the heavy and electron mass.

Nevertheless, it can be afforded if one relaxes some constraints.

Heavy-Fermion Contribution

The calculation of the two-loop heavy-fermion contribution to the Bhabha scattering differential cross section is, in principle, a four-scales problem !

$\Longrightarrow$ it is REALLY VERY difficult to solve in a standard way (diagrammatically), keeping the full dependence on the heavy and electron mass.

Nevertheless, it can be afforded if one relaxes some constraints.

- One can get the solution in the "small-m_f" limit

$$m_e^2 \ll m_f^2 \ll s, t, u$$

T. Becher and K. Melnikov, JHEP 0706 (2007) 084.
S. Actis, M. Czakon, J. Gluza and T. Riemann, Nucl. Phys. B 786 (2007) 26.

Heavy-Fermion Contribution

The calculation of the two-loop heavy-fermion contribution to the Bhabha scattering differential cross section is, in principle, a four-scales problem !

$\Longrightarrow$ it is REALLY VERY difficult to solve in a standard way (diagrammatically), keeping the full dependence on the heavy and electron mass.

Nevertheless, it can be afforded if one relaxes some constraints.

- One can get the solution in the "small-m_f" limit

$$m_e^2 \ll m_f^2 \ll s, t, u$$

T. Becher and K. Melnikov, JHEP 0706 (2007) 084.
S. Actis, M. Czakon, J. Gluza and T. Riemann, Nucl. Phys. B 786 (2007) 26.

- but it is also possible to keep the full dependence on the heavy-fermion mass

$$\boxed{m_e^2 \ll m_f^2 \sim s, t, u}$$

R. B., A. Ferroglia, A. A. Penin, Phys. Rev. Lett. 100 (2008) 131601; JHEP 0802 (2008) 080.

Heavy-Fermion Contribution: small-m_f

Heavy-Fermion Contribution: small-m_f

- Reconstruction from massless CS

If we include closed fermion loops, the formula changes a bit

$$\mathcal{M}^{(m\neq 0)} = Z^2(m,\epsilon)\mathcal{M}^{(m=0)}S(s,t,u,m_f,\epsilon)$$

where Z is the ratio between the massive and massless Dirac form factor and S is the "soft" function, calculated in SCET.

Again, from the totally massless result of Bern-Dixon-Ghinkulov '00, one can reconstruct the N_F part of the CS, in the limit $m_e \ll m_f \ll s, t, u$

T. Becher and K. Melnikov, JHEP 0706 (2007) 084.

Heavy-Fermion Contribution: small-m_f

- **Reconstruction from massless CS**

  If we include closed fermion loops, the formula changes a bit

  $$\mathcal{M}^{(m\neq 0)} = Z^2(m,\epsilon)\mathcal{M}^{(m=0)}S(s,t,u,m_f,\epsilon)$$

  where Z is the ratio between the massive and massless Dirac form factor and S is the "soft" function, calculated in SCET.

  Again, from the totally massless result of Bern-Dixon-Ghinkulov '00, one can reconstruct the N_F part of the CS, in the limit $m_e \ll m_f \ll s,t,u$

  T. Becher and K. Melnikov, JHEP 0706 (2007) 084.

- **Diagrammatic Calculation**

  - reduction to the MIs with the Laporta algorithm
  - calculation of the MIs directly in the $m_e/s \to 0$ limit with Mellin-Barnes

  S. Actis, M. Czakon, J. Gluza and T. Riemann, Nucl. Phys. B 786 (2007) 26.

Heavy-Fermion Contribution: small-m_f

The constraint $\boxed{m_e^2 \ll m_f^2 \ll s, t, u}$ is well verified for instance for leptons at high-energy accelerators (ILC) and for the muon at meson factories.

Heavy-Fermion Contribution: small-m_f

The constraint $\boxed{m_e^2 \ll m_f^2 \ll s, t, u}$ is well verified for instance for leptons at high-energy accelerators (ILC) and for the muon at meson factories.

However it is no longer satisfied in the following cases

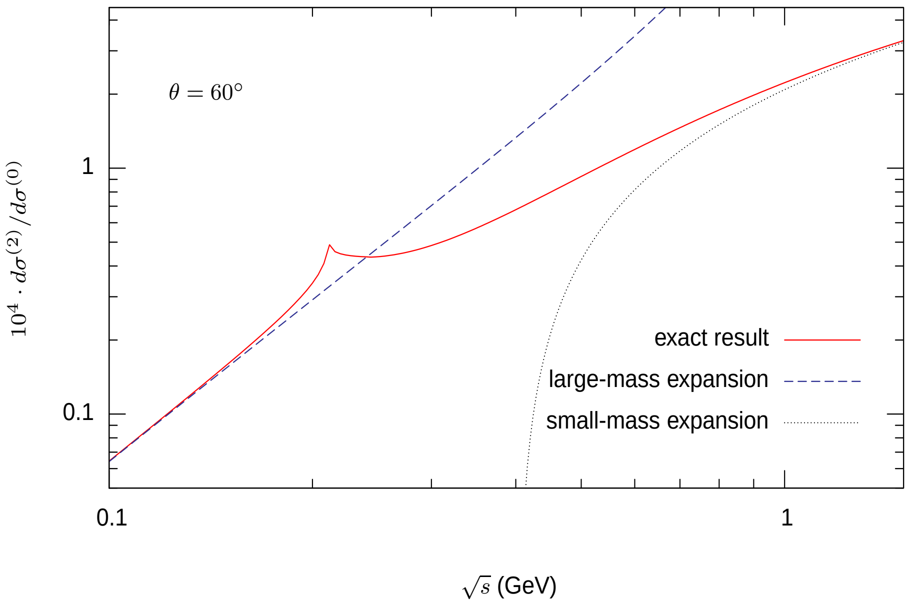
- Contributions coming from a tau loop at KLOE energies ($m_\tau \sim \sqrt{s}$)
- Contributions coming from a top loop at high energies (ILC)

Heavy-Fermion Contribution: small-m_f

The constraint $m_e^2 \ll m_f^2 \ll s, t, u$ is well verified for instance for leptons at high-energy accelerators (ILC) and for the muon at meson factories.

However it is no longer satisfied in the following cases

- Contributions coming from a tau loop at KLOE energies ($m_\tau \sim \sqrt{s}$)
- Contributions coming from a top loop at high energies (ILC)



Two-loop corrections to the Bhabha scattering differential cross section at $\theta = 60^\circ$ due to a closed loop of muon. The solid line represents the exact result. The dashed and dotted lines represent the results of the large-mass expansion and small-mass expansion, respectively.

Heavy-Fermion Contribution: small-m_f

The constraint $\boxed{m_e^2 \ll m_f^2 \ll s, t, u}$ is well verified for instance for leptons at high-energy accelerators (ILC) and for the muon at meson factories.

However it is no longer satisfied in the following cases

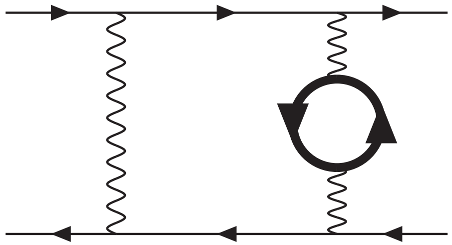
- Contributions coming from a tau loop at KLOE energies ($m_\tau \sim \sqrt{s}$)
- Contributions coming from a top loop at high energies (ILC)

⇒ A solution with the full dependence on m_f is desirable

Heavy-Fermion Contribution: exact m_f

Heavy-Fermion Contribution: exact m_f

- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor ’76)

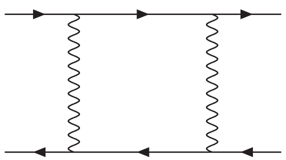
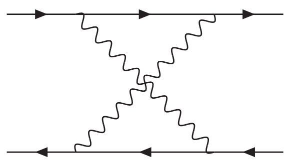


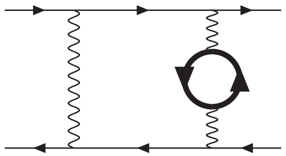
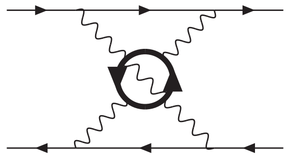
Heavy-Fermion Contribution: exact m_f

- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor ’76)
- This is in general not true in any gauge, but, since the boxes (planar + crossed) constitute a gauge-independent set (actually each pair planar+crossed is gauge independent), in any gauge their sum is collinear safe!

Heavy-Fermion Contribution: exact m_f

- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor '76)
- This is in general not true in any gauge, but, since the boxes (planar + crossed) constitute a gauge-independent set (actually each pair planar+crossed is gauge independent), in any gauge their sum is collinear safe!

 $+$  $=$ Free of collinear poles

 $+$  $=$ Free of collinear poles

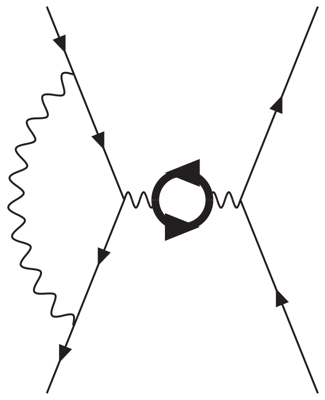
Heavy-Fermion Contribution: exact m_f

- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor ’76)
- This is in general not true in any gauge, but, since the boxes (planar + crossed) constitute a gauge-independent set (actually each pair planar+crossed is gauge independent), in any gauge their sum is collinear safe!

$\Longrightarrow$ we can choose from the beginning $m_e = 0$ in the calculation, reducing, effectively, the number of scales in the game from 4 to 3.
Moreover, we can evaluate the boxes in Feynman gauge.

Heavy-Fermion Contribution: exact m_f

- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor '76)
- This is in general not true in any gauge, but, since the boxes (planar + crossed) constitute a gauge-independent set (actually each pair planar+crossed is gauge independent), in any gauge their sum is collinear safe!
  $\Longrightarrow$ we can choose from the beginning $m_e = 0$ in the calculation, reducing, effectively, the number of scales in the game from 4 to 3, and we can evaluate the boxes in Feynman gauge.
- The collinear divergence comes from the other sets of graphs. In particular it is possible to show that it comes from the reducible ones!



Heavy-Fermion Contribution: exact m_f

- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor '76)
- This is in general not true in any gauge, but, since the boxes (planar + crossed) constitute a gauge-independent set (actually each pair planar+crossed is gauge independent), in any gauge their sum is collinear safe!
  $\Longrightarrow$ we can choose from the beginning $m_e = 0$ in the calculation, reducing, effectively, the number of scales in the game from 4 to 3, and we can evaluate the boxes in Feynman gauge.
- The collinear divergence comes from the other sets of graphs. In particular it is possible to show that it comes from the reducible ones!

  $\Longrightarrow$ in these trivial diagrams we can keep the electrom mass and the heavy-fermion mass different from zero.

Heavy-Fermion Contribution: exact m_f

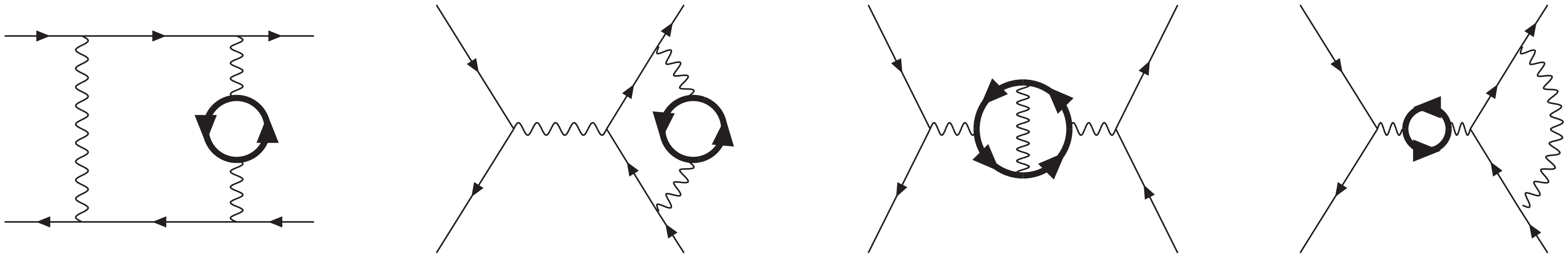
- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor '76)
- This is in general not true in any gauge, but, since the boxes (planar + crossed) constitute a gauge-independent set (actually each pair planar+crossed is gauge independent), in any gauge their sum is collinear safe!
  $\Longrightarrow$ we can choose from the beginning $m_e = 0$ in the calculation, reducing, effectively, the number of scales in the game from 4 to 3, and we can evaluate the boxes in Feynman gauge.
- The collinear divergence comes from the other sets of graphs. In particular it is possible to show that it comes from the reducible ones!
  $\Longrightarrow$ in these trivial diagrams we can keep the electrom mass and the heavy-fermion mass different from zero.

The collinear structure of the cross section is

$$\frac{d\sigma_{N_F>1}}{d\sigma_0} = \delta^{(2)}_{N_F>1,1}(s,t,m_f^2) \ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F>1,0}(s,t,m_f^2)$$

Boxes and two-loop vertices contribute to $\delta^{(2)}_{N_F>1,0}(s,t,m_f^2)$ while the reducible diagrams contribute to $\delta^{(2)}_{N_F>1,1}(s,t,m_f^2)$

Heavy-Fermion Contribution: exact m_f

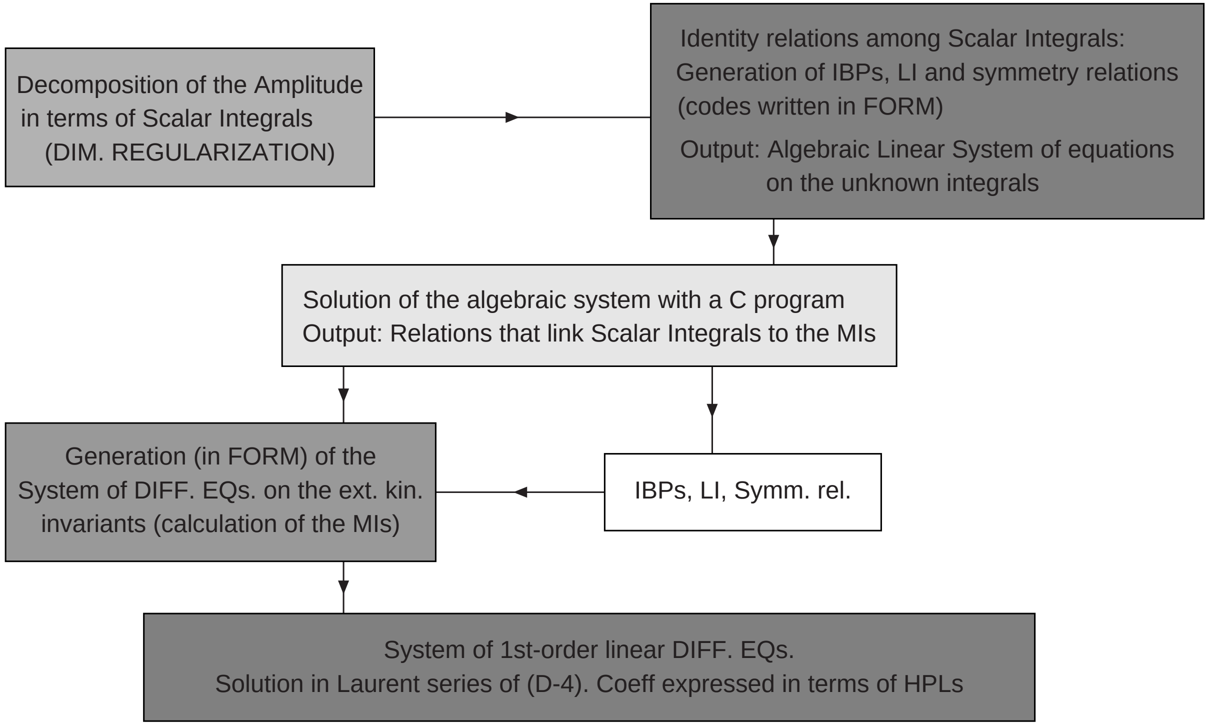


Laporta Algorithm

- Reduction to the MIs

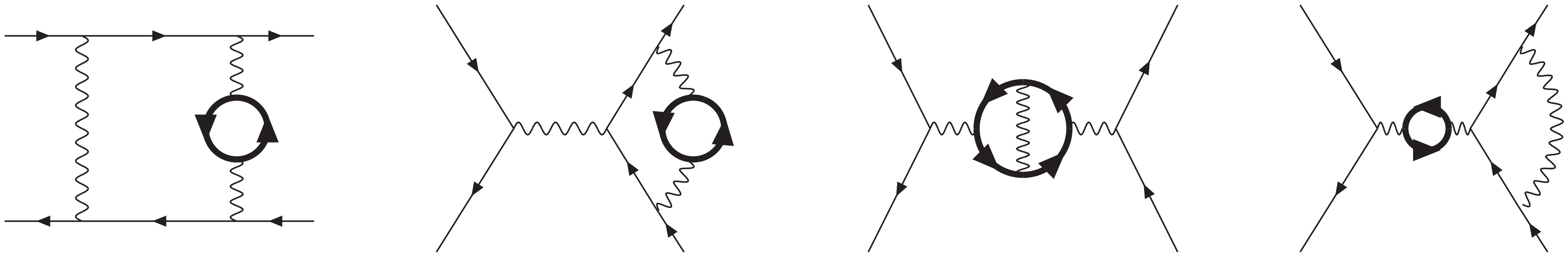
Differential Equations

- Analytic evaluation of the MIs



(B-Ferroglia-Penin ’07-’08)

Heavy-Fermion Contribution: exact m_f

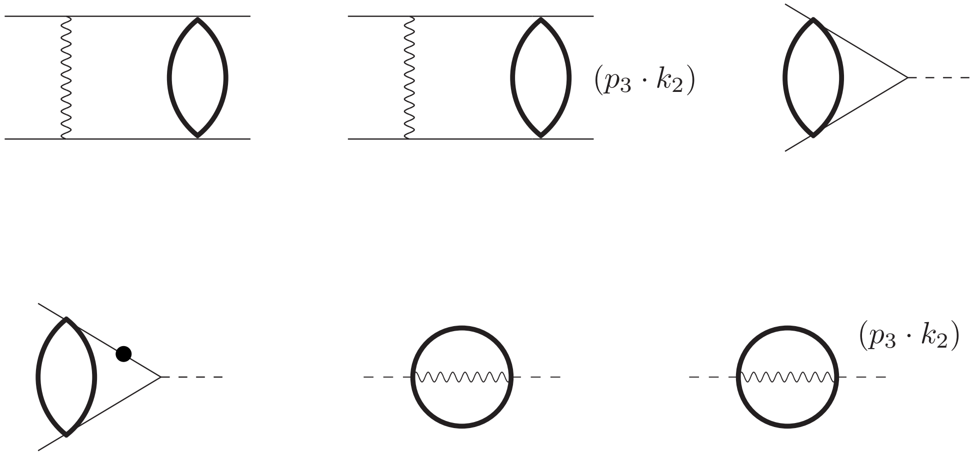


Laporta Algorithm

- Reduction to the MIs

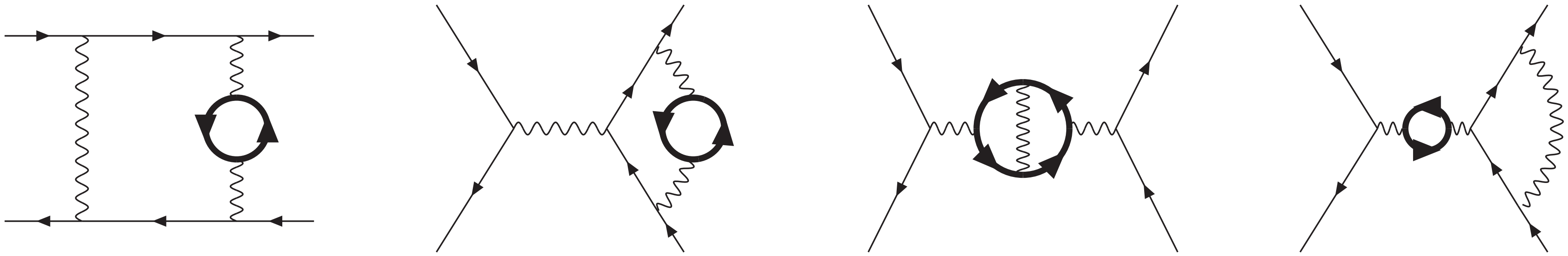
Differential Equations

- Analytic evaluation of the MIs



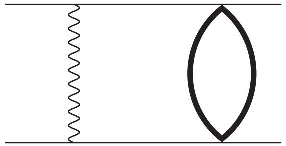
(B-Ferroglia-Penin ’07-’08)

Heavy-Fermion Contribution: exact m_f



Laporta Algorithm

- Reduction to the MIs



$$= \sum_{i=-3}^{0} M_{1,i}(m_f^2, x; y)\,(D-4)^i + \mathcal{O}(D-4)$$

Differential Equations

- Analytic evaluation of the MIs

$$M_{1,-3} = -\frac{1}{2m_f^2 x}$$

$$M_{1,-2} = \frac{1}{4m_f^2 x}\left[2 - G(0;x) - \frac{y+4}{\sqrt{y(y+4)}}G(-\mu;y)\right]$$

$$M_{1,-1} = \frac{1}{8m_f^2 x}\Big[-4 + \zeta(2) + 2G(0;x) - G(0,0;x) + 2G(-\mu,-\mu;y)$$

$$+ \frac{y+4}{\sqrt{y(y+4)}}\left[2G(-\mu;y) - 3G(-4,-\mu;y) - G(0;x)G(-\mu;y)\right]\Big]$$

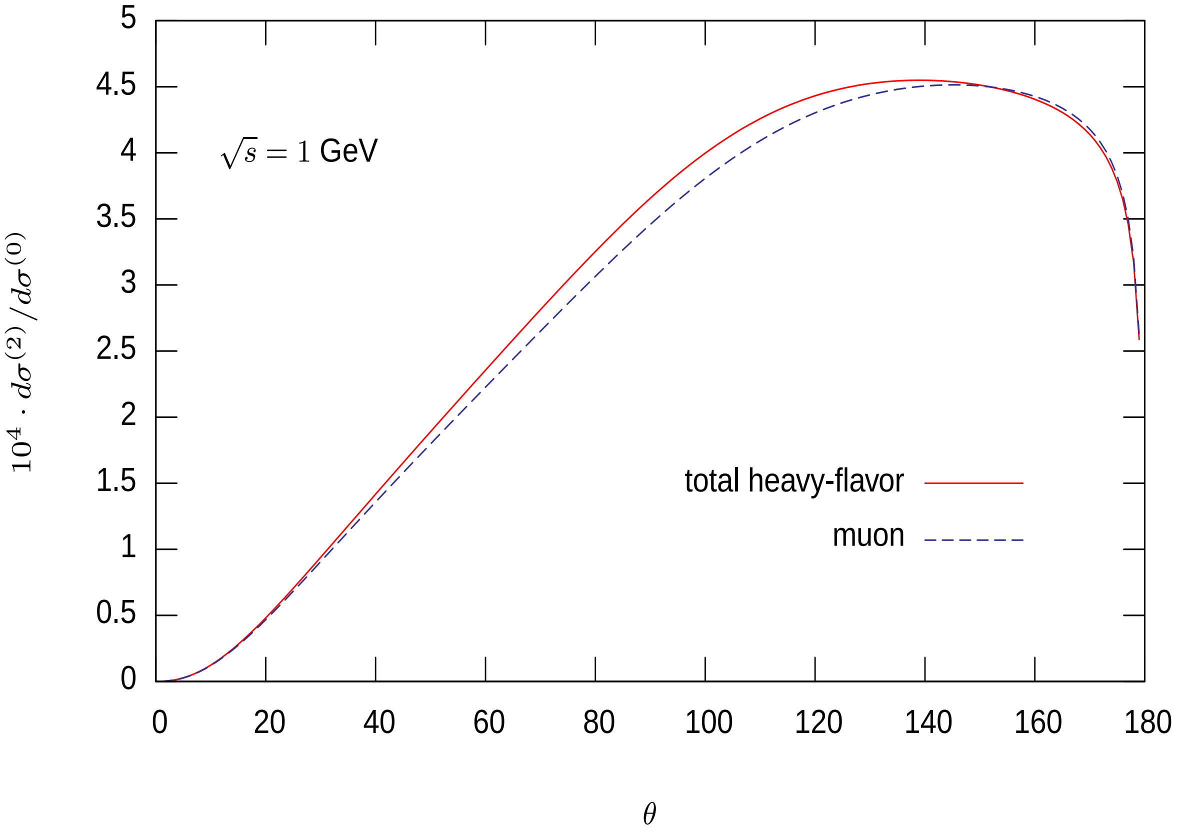
$$M_{1,0} = \cdots$$

(B-Ferroglia-Penin '07-'08)

Numerical Analysis

Numerical Analysis

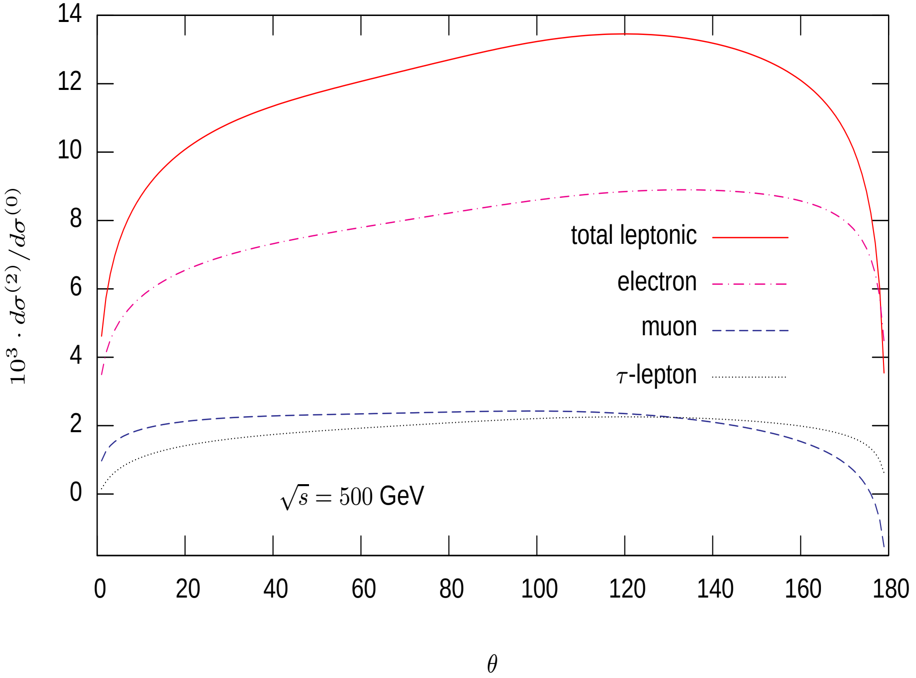
$\sqrt{s} = 1$ GeV QED Corrections



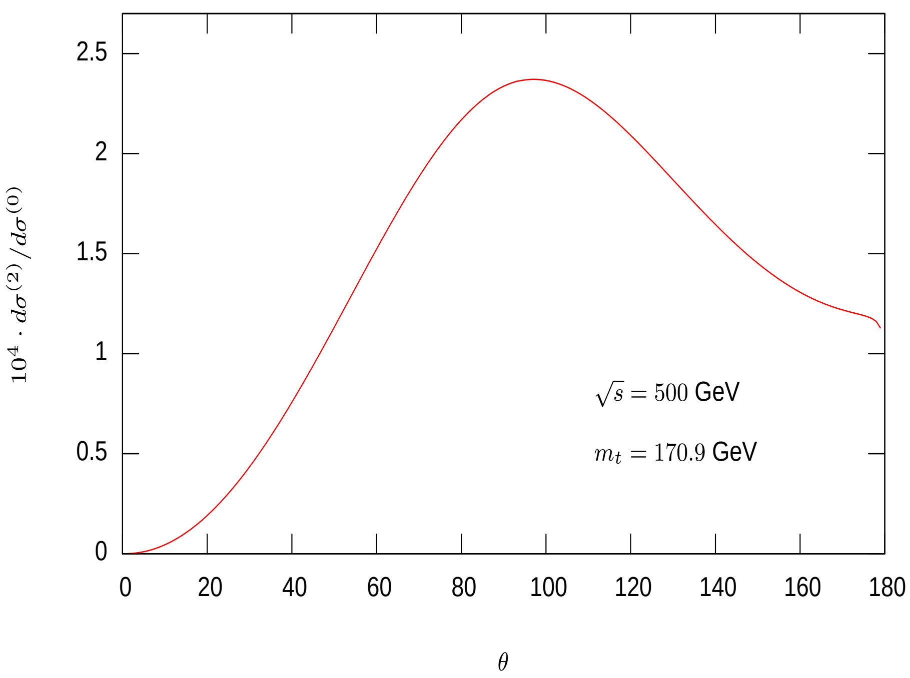
Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV due to a closed loop of muon (dashed line). The solid line represents the sum of the contributions of the muon, τ-lepton, and heavy-quarks. The τ is already two orders of magnutude suppressed.

Numerical Analysis

$\sqrt{s} = 500$ GeV   QED Corrections



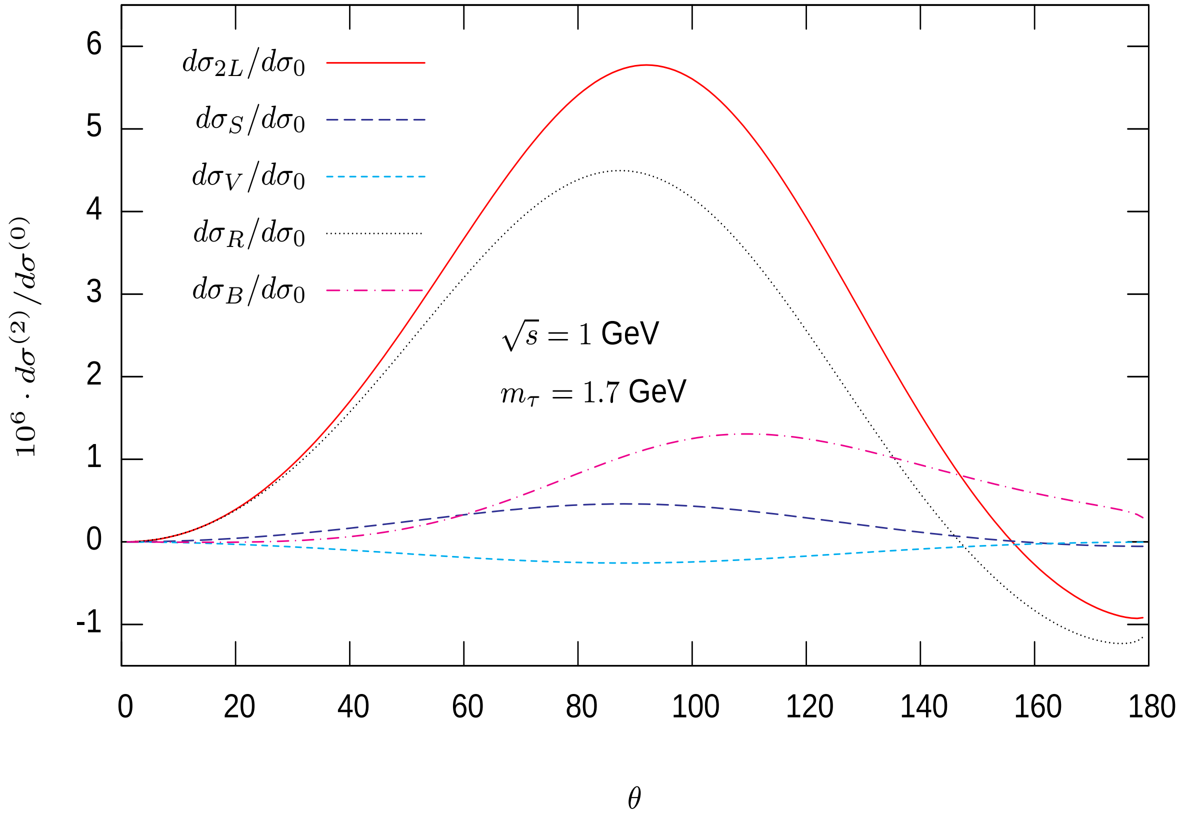
Two-loop leptonic corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 500$ GeV. The dash-dotted line represents the electron contribution including the soft-pair radiation. The dashed and dotted lines represent the contributions of muon and τ-lepton. The solid line is the sum of the three.



Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 500$ GeV due to a closed loop of top quark for $m_t = 170.9$ GeV.

Numerical Analysis

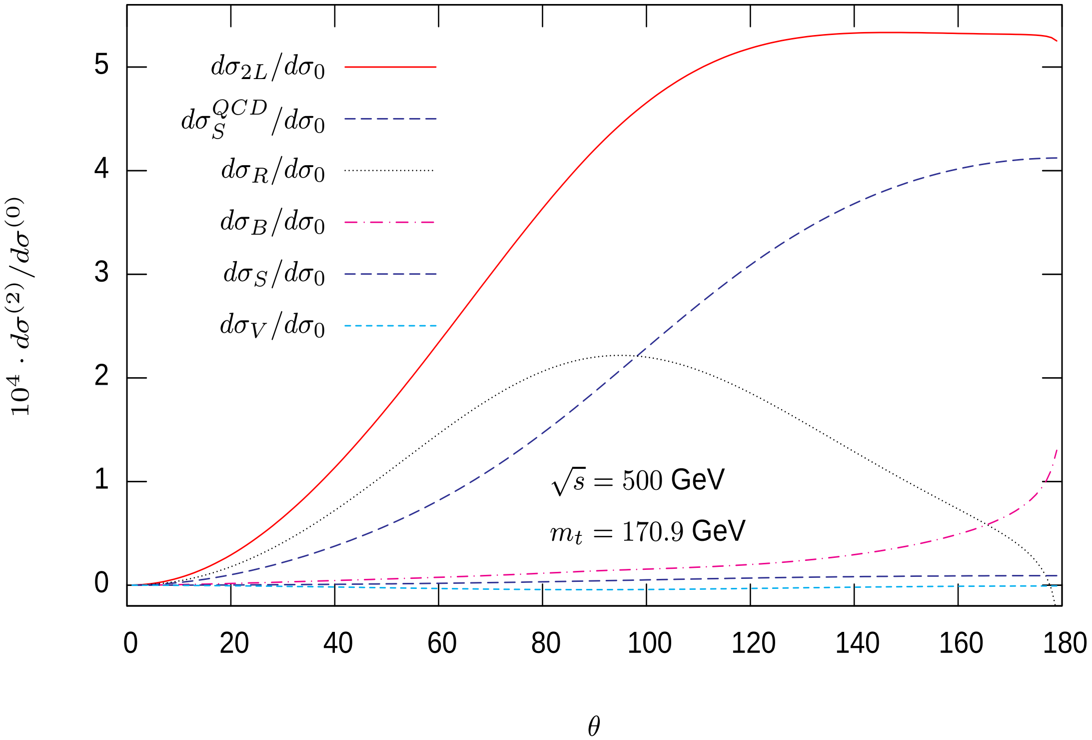
$\sqrt{s} = 1$ GeV Structure of the QED Corrections



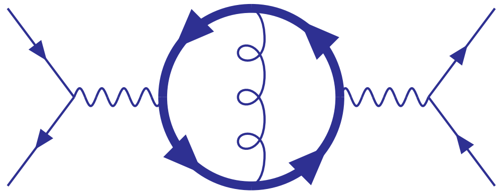
Self-energy ("S"), vertex ("V"), reducible plus one-loop times one-loop ("R"), and box ("B") contributions to the two-loop τ-lepton correction to the differential cross section of Bhabha scattering at $\sqrt{s} = 1$ GeV.

Numerical Analysis

$\sqrt{s} = 500$ GeV Including QCD Corrections



QED and QCD self-energy ("S"), vertex ("V"), reducible plus one-loop times one-loop ("R") and box ("B") contribution to the two-loop top-quark corrections to the differential cross section of Bhabha scattering at $\sqrt{s} = 500$ GeV.



$$\left.\frac{d\sigma_2^V}{d\Omega}\right|^{QCD}_{(2l,S)} = \frac{C_F}{Q_f^2}\frac{\alpha_s(m_f^2)}{\alpha}\left.\frac{d\sigma_2^V}{d\Omega}\right|_{(2l,S)}$$

$C_F = (N_c^2 - 1)/(2N_c)$ is the Casimir operator of the fundamental representation of the $SU(N_c)$ color group, and the strong coupling constant is evaluated at the scale $\mu = m_f$, using the NLO RG equation with the appropriate number of active quarks, starting from the input value $\alpha_S(M_Z) = 0.118$.

Summary

- Bhabha scattering is among the “easiest” precesses to be studied in perturbation theory (it is basically “only” QED). This is the reason why its CS is known at the level of NNLO quantum corrections
- In the past years, several groups contributed to the calculation of the CS. The state of the art includes
  - The complete NLO in the full Electroweak Standard Model
  - The full set of NNLO QED corrections ($\mathcal{O}(\alpha^4)$ and $\mathcal{O}(\alpha^3\alpha_S)$) and hadronic effects for the process $e^+e^- \to e^+e^-$
- These corrections have been included already in several Monte Carlos that provide, at the moment, a very good precision. In the case of LABS at DAΦNE energies the CS is known at the level of better than 0.1%
- In order to complete the knowledge of the Bhabha scattering CS at the level of NNLO perturbative corrections (mostly for esthetic reasons), some pieces are still missing:
  - the soft-pair production contribution is known at the logarithmic level
  - the process $e^+e^- \to e^+e^- + \gamma$ (hard photon) enters in the MCs at the LO
  - the two-loop electroweak logarithmic corrections in four-fermion processes are studied (for instance for $e^+e^- \to \mu^+\mu^-$), but not yet included in the analysis