Irrational constants in positronium decay

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LOOPS & LEGS 2008

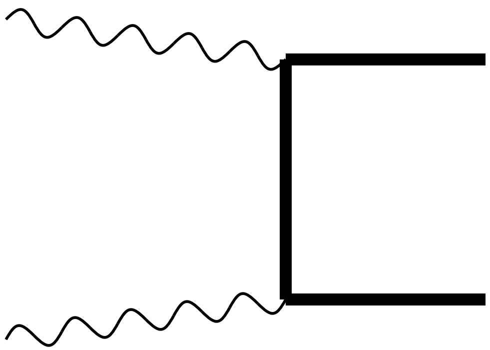
- intro
- ortho-positronium decay
- para-positonium decay
- conclusions

positronium

- accurate experiments
  - positronium lifetime (few ppm)
  - spectroscopy (few pppm)
- pure QED system $\longrightarrow$ full theoretical control is possible
- bound state, excellent laboratory to test NRQFT effective theory
- search for new physics
- applications in nanotechnology
- hope for the future diminishing of experimental errors

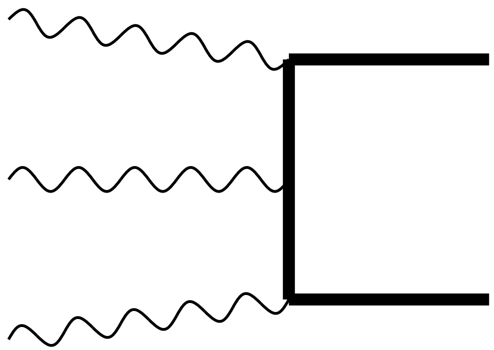
positronium decay

p-Ps



$$\Gamma_p = 7990.9(1.7)\mu s^{-1}$$

o-Ps



$$\Gamma_o = 7.0404(10)\mu s^{-1}$$

theory: $\Gamma_p = 7989.6178(2)\mu s^{-1}$ $\Gamma_p = 7.03998(1)\mu s^{-1}$

nonrelativistic QED (NRQED)

$$L_{\mathrm{NRQED}} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \psi^\dagger \Big[iD_t + \frac{\mathbf{D^2}}{2m} + \frac{\mathbf{D^4}}{8m^3} + \ldots + \frac{c_F e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}$$
$$+ \frac{c_D e}{8m^2}(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D}) + \frac{c_S e}{8m^2} i\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}) + \ldots \Big]\psi + \ldots$$

theoretical prediction for o-Ps decay width

lowest order:

$$\Gamma_0 = \frac{2}{9}(\pi^2 - 9)\frac{m\alpha^6}{\pi}$$

radiative corrections within NRQED:

$$\Gamma = \Gamma_0\left(1 + A\frac{\alpha}{\pi} + \frac{\alpha^2}{3}\ln\alpha + B\frac{\alpha^2}{\pi^2} - \frac{3\alpha^3}{2\pi}\ln^2\alpha + C\frac{\alpha^3}{\pi}\ln\alpha + \ldots\right)$$

W. Caswell, G. Lepage, J. Sapirstein 1977, I. Harris, L. Brown 1957 (1-loop)
B. Kniehl, A. Penin 2000, R. Hill, G. Lepage 2000, K. Melnikov, A. Yelkhovski 2000 (log-terms)
G. Adkins, R. Fell, J. Sapirstein 2002 (2-loop)

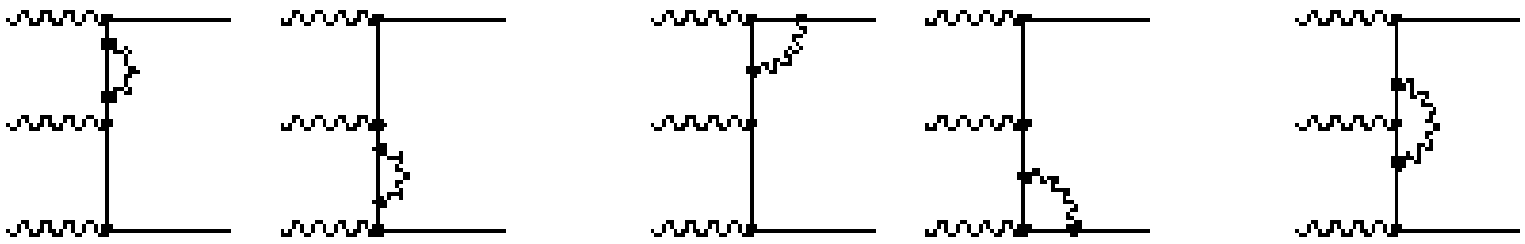
coefficients A , B and C are known numerically

$$A = -10.286606(10)$$

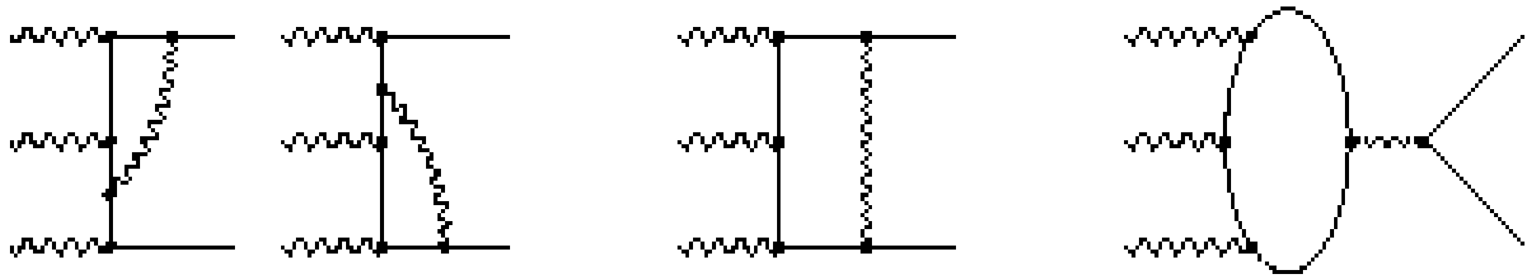
$$B = 45.06(26)$$

$$C = -5.51702455(23)$$

1-loop diagrams contributing to A



(a) (b) (c)



(d) (e) (f)

1-loop correction for o-Ps $\rightarrow 3\gamma$ decay width

phase-space integral can be parametrised as follows

$$\Gamma_1 = \frac{m\alpha^7}{36\pi^2} \int_0^1 \int_0^1 \int_0^1 \frac{F(x_1,x_2,x_3)}{x_1 x_2 x_3} \delta(2 - x_1 - x_2 - x_3)\, dx_1\, dx_2\, dx_3$$

function $F(x_1,x_2,x_3)$ includes subexpressions, like

$$\frac{\text{polynom}(x_1,x_2,x_3)}{\text{polynom}(x_1,x_2,x_3)} \int_0^1 \frac{\ln(x_1 + (1-x_1)\, y^2)}{(1-x_1)x_3 - x_1(1-x_3)\, y^2}\, dy$$

$$\frac{\text{polynom}(x_1,x_2,x_3)}{\text{polynom}(x_1,x_2,x_3)} \int_0^1 \frac{\ln(x_1 + (1-x_1)\, y^2)}{x_1 x_3 - (1-x_1)(1-x_3)\, y^2}\, dy$$

and many others ...

single scale diagrams

- multiple ζ-values, (RG-calculations), $\zeta(a), \zeta(a,b), \ldots$

Broadhurst, Kreimer

- Euler–Zagier sums or alternative sums ($g-2$, pole masses in QED/QCD etc.), $\mathrm{Li}_a(1/2), \ln 2, \ldots$

Broadhurst

- ”binomial” sums, (pole masses in SM, μ-decay etc.), $\mathrm{Ls}_j^{(k)}(\pi/3)$

Davydychev, Fleischer, Kalmykov, Veretin

- multiple polylogarithms $\sum_{n_1 \ldots n_k} \frac{z^{n_1} \ldots z^{n_k}}{n_1^{m_1} \ldots n_k^{m_k}}$

Goncharov, Kummer, Poincare

- others , elliptic integrals and (?)

constants that appear in o-Ps $\to 3\gamma$

"usual"-basis, including e.g.:

$$\ln 2\,, \qquad \zeta(n)\,, \qquad \mathrm{Li}_4\left(\frac{1}{2}\right)\,, \quad \text{etc}$$

additional constants:

$$\ln(R)\,, \quad \text{where} \quad R = \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

up to weight 4

$$\mathrm{Li}_2\left(\frac{1}{3}\right)\,, \qquad \mathrm{Li}_4\left(\frac{1}{3}\right)\,, \qquad \mathrm{Li}_4\left(-\frac{1}{3}\right)\,,$$

$$\mathrm{Li}_3\left(\frac{1}{\sqrt{2}}\right)\,, \qquad \mathrm{Li}_3\,(R)\,, \qquad \mathrm{S}_{1,2}\,(R)\,,$$

$$\mathrm{Li}_4\,(\pm R)\,, \qquad \mathrm{S}_{1,3}\,(\pm R)\,, \qquad \mathrm{S}_{2,2}\,(\pm R)$$

final result for o-Ps decay width

$$\frac{2}{9}\ (\pi^2 - 9) A =$$

$$+\frac{56}{27} - \frac{901}{1296}\pi^2 - \frac{12983}{17280}\pi^4 + \frac{19}{6}\ln 2 + \ldots$$

$$\ldots - \frac{2701}{648}\pi^2 \ln^2 2 - \frac{7\sqrt{2}}{4}\mathrm{Li}_3(R) + \frac{7\sqrt{2}}{2}\mathrm{S}_{1,2}(R)$$

$$-\frac{63}{4}\mathrm{Li}_4(R) + \frac{63}{4}\mathrm{S}_{2,2}(R) + 7\mathrm{Li}_4(-R) - 7\mathrm{S}_{2,2}(-R)$$

and similar for C , where

$$R = \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

numerically

$$A = -10.286614808628262240150169210991253179644007490 2\ldots$$

$$C = +5.5170274917298582713788660986650051819440014218 6\ldots$$

theoretical prediction for o-Ps decay width

$$\Gamma = \Gamma_0\left(1 + A\frac{\alpha}{\pi} + \frac{\alpha^2}{3}\ln\alpha - \frac{3\alpha^3}{2\pi}\ln^2\alpha + C\frac{\alpha^3}{\pi}\ln\alpha + B\frac{\alpha^2}{\pi^2}\right)$$

$$B = 45.06(26)$$

$$\Gamma^{\text{o-Ps}}(\text{theory}) = 7.039979(11)\mu s^{-1}$$

theoretical prediction for p-Ps decay width

radiative corrections within NRQED:

$$\Gamma_p = \frac{\alpha^5 m_e}{2}\Big\{1+\frac{\alpha}{\pi}\left(\frac{\pi^2-20}{4}\right)+\frac{\alpha^2}{\pi^2}\left(-2\pi^2\ln\alpha+A_p\right)$$
$$+\frac{\alpha^3}{\pi}\left(-\frac{3}{2}\ln^2\alpha+\left(\frac{533}{90}-\frac{\pi^2}{2}+10\ln 2\right)\ln\alpha\right)\Big\}$$

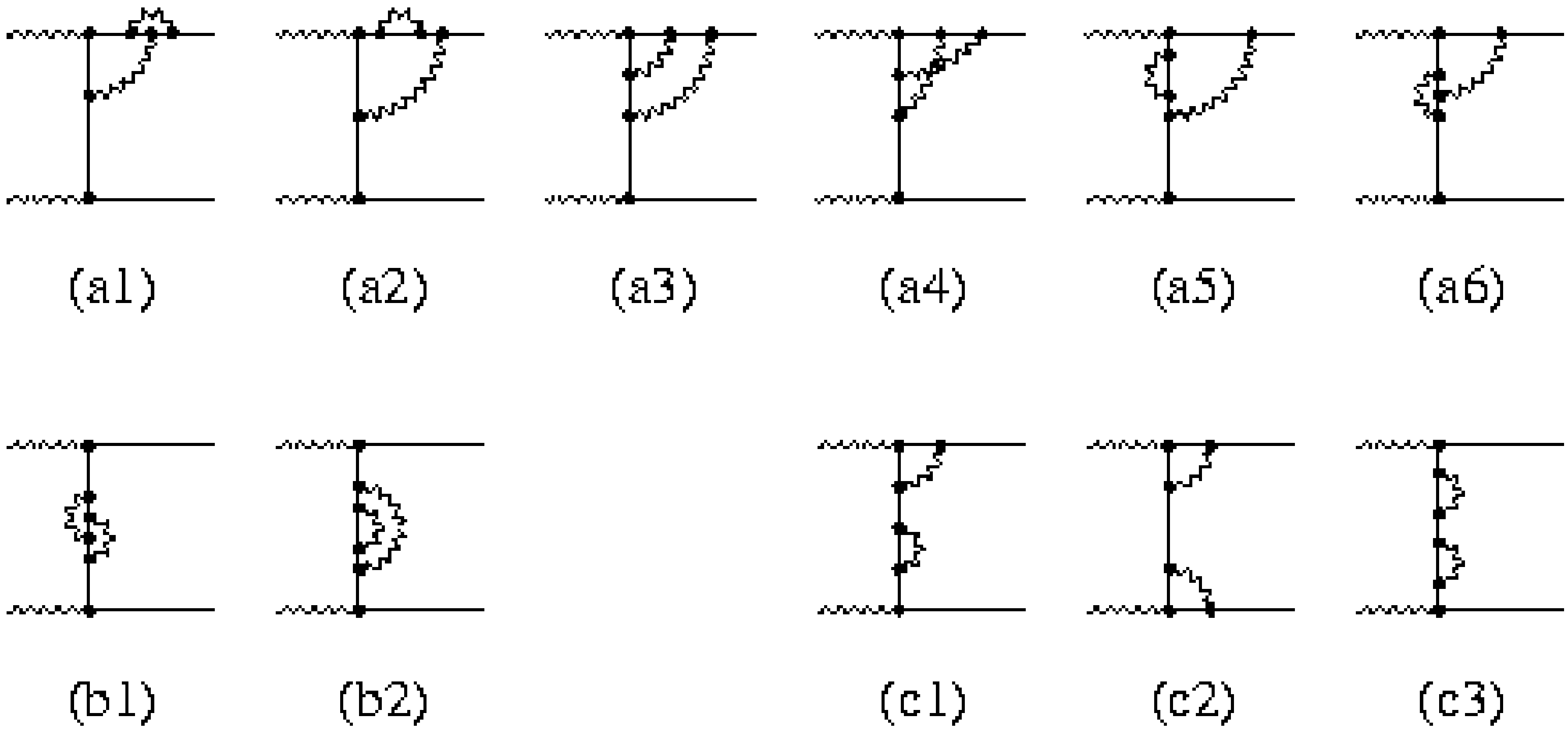
W. Caswell, G. Lepage 1979
A. Czarnecki, K. Melnikov, A. Yelkhovski 1999 (2-loop)
B. Kniehl, A. Penin 2000, (log-terms)

coefficient A_p is known numerically

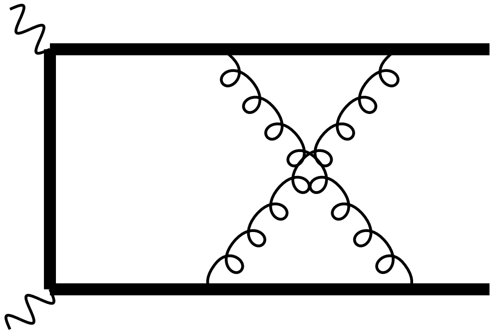
$$A_p = 5.12443(33)$$

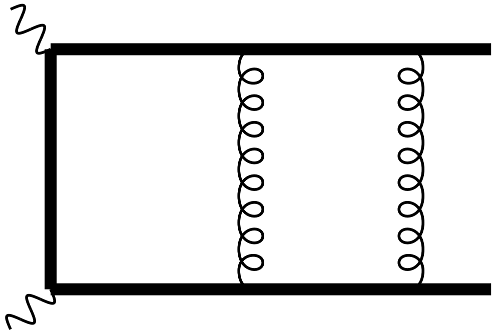
$$\Gamma_p(\text{theory}) = 7989.6178(2)\mu s^{-1}, \qquad \Gamma_p(\text{exp}) = 7990.9(1)\mu s^{-1}$$

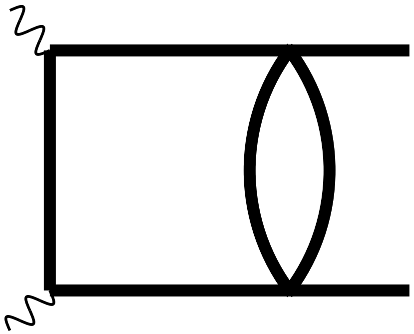
some diagrams contributing to p-PS lifetime

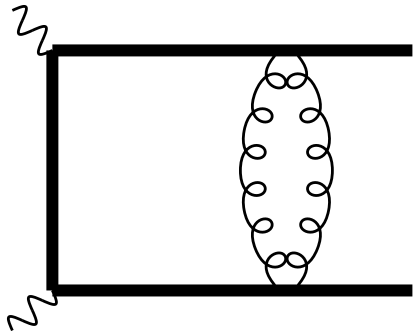


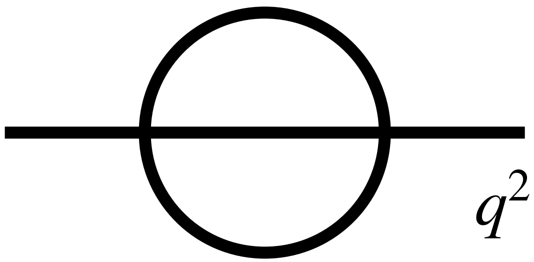
generic (not masterintegrals)











$q^2 = -m^2$

result for the sunset diagrams

e.g. for our sunset we obtain (PSLQ)

$$J = -\frac{3}{2\varepsilon^2} - \frac{19}{4\varepsilon} - \frac{215}{24} + \frac{9}{4}\zeta(2) + \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left(-\frac{5}{2}\phi + \frac{15\phi}{4n} + \frac{15}{8n^2} \right) + \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{2n}{n}\binom{4n}{2n}} \left(\frac{25}{3} - \frac{25}{6n} \right),$$

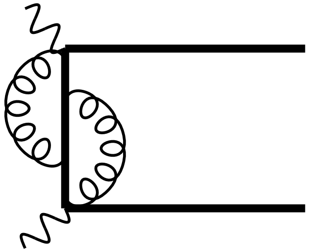
where

$$\phi = S_1(n-1) - 3S_1(2n-1) + 2S_1(4n-1),$$

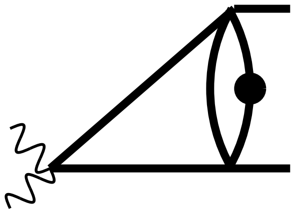
a and b sums

$$a_{\ldots} \leftrightarrow \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} (\ldots), \qquad b_{\ldots} \leftrightarrow \frac{(-16)^n}{\binom{2n}{n}\binom{4n}{2n}} (\ldots)$$

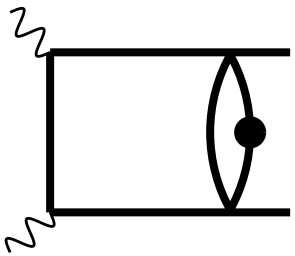
others masterintegrals ...



$$= \frac{83}{48}\zeta_3 - \frac{1}{2}\zeta_2 \log 2 + \frac{5}{24}a_3 + \frac{5}{24}a_{\phi 2} + \frac{5}{96}b_3$$



$$= \frac{9}{16}\zeta_3 - \frac{1}{8}a_3 - \frac{1}{8}a_{\phi 2} - \frac{1}{32}b_3$$



$$= \frac{9}{32}\zeta_4 + \frac{1}{4}a_4 - \frac{1}{8}a_{\phi 3} + \frac{1}{16}b_4$$

relations for sums (1)

there are relations between a and b sums!

example:

$$\sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{2n}{n}\binom{4n}{2n}} \frac{1}{n} = -6\log^2 2 + 3\zeta(2) + 2\log 2$$

$$+ \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{3n-3}{2n^2} - \frac{2n+3}{2n} \Big(7(S_1(n-1) - S_1(2n-1)) \right.$$

$$\left. + 4(S_1(2n-1) - S_1(4n-1)) + 6\log 2 \Big) \right\}$$

more complicated relations $\longrightarrow$

$$a\,\zeta(2) \longleftrightarrow b\,\log^2 2 \qquad \text{etc.}$$

relations for sums (2)

- MZV and binomial sums have alwais definite weights

$$\sum_{n_1 \ldots n_k} \frac{z^{n_1} \ldots z^{n_k}}{n_1^{m_1} \ldots n_k^{m_k}} \quad \longrightarrow \quad \text{weight} = m_1 + \ldots + m_k$$

- only relations between terms with the same weight

e.g.: $\zeta(4,1,3) = -\zeta(5,3) + \frac{71}{36}\zeta(8) - \frac{5}{2}\zeta(5)\zeta(3) + \frac{1}{2}\zeta^2(3)\zeta(2)$

- what is the ”weight” here???

$$\sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{2n}{n}\binom{4n}{2n}} \left(20 - \frac{16}{n} + \frac{3}{n^2}\right) = -4$$

introduce

$$\varphi = \arctan \sqrt[4]{5}$$

and elliptic integrals of the 1st and 2nd kind

$$\begin{aligned} F(\phi,k) &= \int_0^{\phi} \mathrm{d}\theta\,(1-k\sin^2\theta)^{-1/2} \\ E(\phi,k) &= \int_0^{\phi} \mathrm{d}\theta\,(1-k\sin^2\theta)^{1/2} \end{aligned}$$

then new constants

$$\begin{aligned} f_1 &= \frac{1}{\sqrt[4]{5}} F\left(2\varphi, \frac{1}{2\sin^2\varphi}\right) = 1.8829167613\ldots \\ e_1 &= \frac{1}{\sqrt[4]{5}} E\left(2\varphi, \frac{1}{2\sin^2\varphi}\right) = 0.9671227369\ldots \end{aligned}$$

connection to elliptic integrals (2)

we have for two lowest sums

$$a_0 \equiv \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \quad = \quad -\frac{1}{2} + \frac{1}{2\sqrt{5}} + \frac{1}{\sqrt{5}} e_1 - \frac{1}{2\sqrt{5}} f_1$$

$$a_1 \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\binom{2n}{n}}{\binom{4n}{2n}} \quad = \quad -\frac{4}{3} \ln 2 + \frac{1}{3} f_1$$

Conclusions

- corrections to the ortho-positronium lifetime are obtained in analytical form in terms of polylogarithmic functions
- in the para-positronium case the structure of the analytical answer is established
- the new irrational constants are introduced in terms of infinite sums
- it is shown that they are related to the elliptic integrals