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# Integrability in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory


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## References:


- ES ht/0603157, BES ht/0610251.
- BBKS ht/0611135, AABEK ht/0702028.
- BKK ht/07083933, KSV ht/08012542.
- KMMZ ht/0402207, BDS ht/0405001, AFS ht/0406256, BK ht/0510124.
- B ht/0307015, S ht/0412188, BS ht/0504190.
- MVV hp/0403192, KLV ht/0301021, KLOV ht/0404092.
- BT ht/0509084, J ht/0603038, HL ht/0603204, BHL ht/0609044.
- BDS ht/0505205, BCDKS ht/0610248.
- CVS ht/0612309, BMcLR ht/07050321.
- GKP ht/0204051, FT ht/0204226, RT ht/07043638.

# 0 Introduction

## $\mathcal{N} = 4$ SYM

- The **AdS/CFT duality** relates $\mathcal{N} = 4$ SYM to IIB string theory on $AdS_5 \times S_5$. It is a weak/strong-coupling duality.
- The large N limit of the SYM theory can be described by **spin chains**.

## Derivative operators

- Built from scalar fields X and **covariant derivatives**.
- The derivatives act as **magnons** moving on the chain of scalars.

## Large spin all-loops anomalous dimension

- We start from an **all-loops conjecture** for the Bethe ansatz. A **large spin continuum limit** yields an **integral equation** for the density of Bethe roots.
- The energy grows logarithmically with the spin. It is given by sums of zeta values respecting a principle of **maximal transcendentality**.
- We discuss **dressing phases** (integrable modifications of the Bethe ansatz) that do not violate transcendentality.
- A **kernel from string theory** reverses the sign of certain contributions to the energy. At four loops, **agreement with field theory** is obtained.

## Outloook

- Limitations of the model — **Wrapping**

# 1 Spin Chain Picture for BMN Operators

Composite operators are characterised by

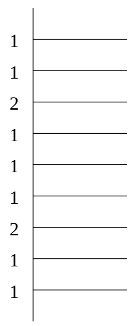
- Lorentz spin,
- $SU(4)$ Dynkin labels,
- dimension $\Delta(g^2, N)$.

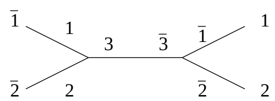
Two-point functions of scalar singlets $\mathcal{O}_1$, $\mathcal{O}_2$ obey

$$< \mathcal{O}_1(1)\,\bar{\mathcal{O}}_2(2) > = 0\,, \qquad \Delta_1 \neq \Delta_2\,,$$
$$< \mathcal{O}_1(1)\,\bar{\mathcal{O}}_1(2) > = \frac{c(g^2, N)}{(x_{12}^2)^{\Delta(g^2,N)}}\,, \qquad x_{12} = x_1 - x_2\,.$$

## $SU(2)$-sector BMN operators

$$\begin{aligned}
\mathcal{O}_I(\Pi_X,\, k,\, p) &= \Pi_i\, Tr(X^{l_i})\, Tr(\Phi_2 X^{(k-p)} \Phi_2 X^p)\,,\\
\mathcal{O}_{II}(\Pi_X,\, k,\, p) &= \Pi_i\, Tr(X^{l_i})\, Tr(\Phi_2 X^{(k-p)})(\Phi_2 X^p)\,.
\end{aligned}$$




$$v = ([\bar{X}, \bar{\Phi}^2][\Phi_2, X])$$

- Large $N$: Spin chain picture, $X, \Phi_2$ as up and down spins. The one-loop interaction defines a Hamiltonian MZ ht/0212208, BKS ht/0303060.
- Higher order Feynman diagrams give a perturbation.
- **Wrapping:** The interaction length becomes equal to or greater than the number of fields in an operator.

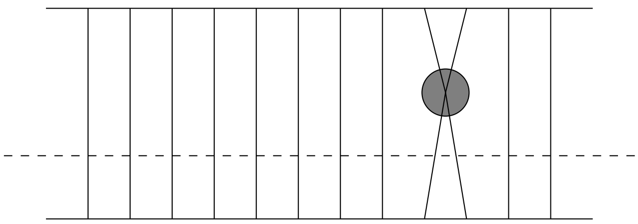
## 2 Spin Chain Picture for Twist Operators

**Derivative sector:**

$$\{s_1, s_2, s_3, \ldots\} = Tr\Big((\mathcal{D}_z^{s_1}X)(\mathcal{D}_z^{s_2}X)(\mathcal{D}_z^{s_3}X)\ldots\Big)$$

- $X$ is a complex scalar field of the $\mathcal{N}=4$ SYM theory with $SU(N)$ gauge group. $\mathcal{D}_\mu = \partial_\mu + i\, g_{YM}\, A_\mu$.
- The operators carry traceless symmetric Lorentz representation of spin $s = s_1 + s_2 + s_3 + \ldots$; project $z = x_1 + ix_2$.

- Loop diagrams define a Hamiltonian that can transfer derivatives from one site to another. Free lines do not (as long as we look at a certain tensor component).
- In the large $N$ limit this defines a nearest neighbour interaction.



Two-site Hamiltonian.

We may view the derivatives as "magnons" moving on the sites of a spin chain.

At one loop (B):

$$\mathcal{H}^{(0)} = \sum_{i=1}^{L} \mathcal{H}_i^{(0)}$$

$$\mathcal{H}_i^{(0)}(\{s_1,\, s_2\} \to \{s_1,\, s_2\}) = h(s_1) + h(s_2)\,,$$

$$\mathcal{H}_i^{(0)}(\{s_1,\, s_2\} \to \{s_1 - d,\, s_2 + d\}) = -\frac{1}{|d|}$$

# 3 Bethe Equations

- The one-loop Hamiltonian above defines the Heisenberg XXX chain with spin $-\frac{1}{2}$.

The dynamics of the system is captured by the **Bethe ansatz**

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j \neq k} \left(\frac{u_k - u_j - i}{u_k - u_j + i}\right), \quad j, k \in \{1, \ldots, s\},$$

$$\prod_{k=1}^{s} \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right) = 1, \qquad E = \sum_{k=1}^{s} \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}}\right).$$

**All-loops conjecture** (S,BS):

$$u \pm \frac{i}{2} = x^{\pm} + \frac{g^2}{2x^{\pm}}, \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

The deformed system is

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j \neq k} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+},$$

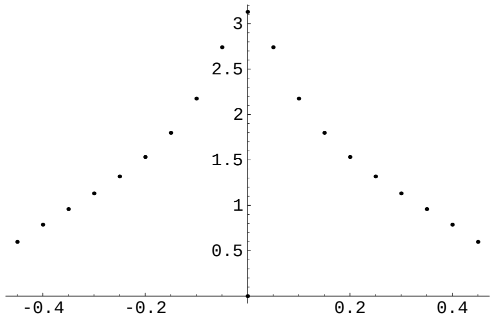
$$\prod_{k=1}^{s} \left(\frac{x_k^+}{x_k^-}\right) = 1, \qquad E(g) = \sum_{k=1}^{s} \left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right).$$

- Valid only for **infinite** spin chain **length**!

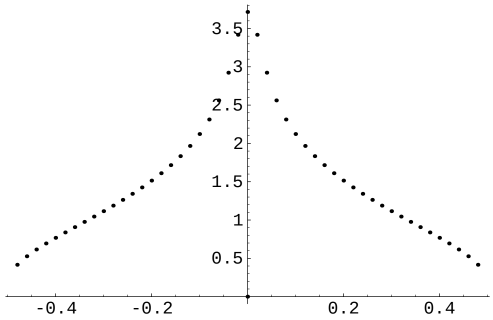
# 4 Some One-Loop Solutions for $L = 2$

The results of KLOV may be reproduced from the Bethe ansatz. The large spin limit of the universal anomalous dimension should connect via the AdS/CFT duality to the prediction by GKP.

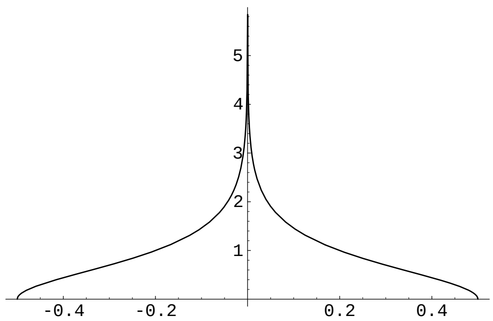
- There are only solutions for even spin $s$. The roots are all real and symmetrically distributed around zero. We label them as $u_{-s/2}, \ldots, u_{-1}, u_1, \ldots, u_{s/2}$.
- For $k > 0$ we plot $\rho_k = \frac{1}{u_k - u_{k-1}}$ against $(k-1)/s$, similarly for $k < 0$.




$s = 20$




$s = 50$




$s \longrightarrow \infty$

## 5 One-Loop Large Spin Limit

- The $L = 2$ case is exactly solvable for any (even) spin; the $u_k$ are the zeroes of certain Hahn polynomials DKM ht/0210216.
- The roots are real and symmetrically distributed around zero. The density peaks at the origin, there is no gap.
- The outermost roots grow as $\max\{|u_k|\} \to s/2$.
- The mode numbers are $\mp 1$ for negative/positive roots.
- For $L > 2$ there is more than one state. However, for the lowest state the root distribution is again real and symmetric with $n = \mathrm{sign}(u)$.

We take the logarithm of the Bethe equations

$$-i\,L\,\log\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right) = 2\,\pi\,n_k - i\sum_{j\neq k}\log\frac{u_k - u_j - i}{u_k - u_j + i}\,,$$

rescale $u \longrightarrow s\,\bar{u}$, expand in 1/s, and take a continuum limit:

$$0 = 2\,\pi\,\epsilon(\bar{u}) - 2\,⨍_{-1/2}^{1/2} d\bar{u}'\,\frac{\bar{\rho}_0(\bar{u}')}{\bar{u} - \bar{u}'}$$

One may solve by an inverse Hilbert transform:

$$\bar{\rho}_0(\bar{u}) = \frac{1}{\pi}\log\frac{1 + \sqrt{1 - 4\,\bar{u}^2}}{1 - \sqrt{1 - 4\,\bar{u}^2}} = \frac{2}{\pi}\,\mathrm{arctanh}\left(\sqrt{1 - 4\,\bar{u}^2}\right)$$

The one-loop energy is:

$$E_0 = \frac{1}{s}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{u}\,\frac{\bar{\rho}_0(\bar{u})}{\bar{u}^2 + \frac{1}{4\,s^2}} = 4\,\log(s) + \mathcal{O}(s^0)$$

# 6 Asymptotic All-Loops Large Spin Limit

Split

$$\rho(u) = \rho_0(u) - g^2 \, \frac{E_0}{s} \, \sigma(u)$$

and integrate out the one-loop density.

Large spin limit:

$$\boxed{\begin{aligned} 0 \;=\;& 2\,\pi\,\sigma(u) \\ & -2\int_{-\infty}^{\infty} du' \frac{\sigma(u')}{(u-u')^2+1} \\ & -\left(\frac{1}{2}\frac{d}{du}\right)\left[\frac{1}{x^+(u)}+\frac{1}{x^-(u)}\right] \\ & +2i\int_{-\infty}^{\infty} du'\sigma(u')\frac{d}{du}\log\left(\frac{1-g^2/2x^+(u)x^-(u')}{1-g^2/2x^-(u)x^+(u')}\right) \end{aligned}}$$

- This is an asymptotic result, because $L$ needs to grow with the order in $g^2$ to avoid "wrapping".
- The final formula is $L$ independent. "Wrapping" is thus absent.

# 7 Weak Coupling and Transcendentality

We introduce the Fourier transform $\hat{\sigma}(t)$ of the fluctuation density $\sigma(u)$

$$\hat{\sigma}(t) = e^{-\frac{t}{2}} \int_{-\infty}^{\infty} du\, e^{-itu}\, \sigma(u)\,.$$

The integral equation becomes

$$\boxed{\begin{aligned} \hat{\sigma}(t) \;=\; \frac{t}{e^{\,t}-1} \Big[ & \frac{J_1(2\,g\,t)}{2\,g\,t} - \\ & - 4\,g^2 \int_0^{\infty} dt'\; \hat{K}(2\,g\,t, 2\,g\,t')\; \hat{\sigma}(t') \Big]\,, \end{aligned}}$$

with the non-singular kernel

$$\hat{K}(t,\, t') \;=\; \frac{J_1(t)\, J_0(t') \;-\; J_0(t)\, J_1(t')}{t \;-\; t'}\,.$$

The energy is

$$f(g) \;=\; \frac{E(g)}{\log(s)} \;=\; 8\,g^2 \;-\; 64\,g^4 \int_0^{\infty} dt\, \hat{\sigma}(t)\, \frac{J_1(2\,g\,t)}{2\,g\,t}\,.$$

The integral equation is of Fredholm II type. One may solve by iteration:

$$\hat{\sigma}(t) = \frac{1}{2}\,\frac{t}{e^t \,-\, 1} - g^2\, \left( \frac{1}{4}\,\frac{t^3}{e^t \,-\, 1} \,+\, \zeta(2)\,\frac{t}{e^t \,-\, 1} \right) + \ldots\,,$$

where we have used

$$\zeta(n+1) = \frac{1}{n!} \int_0^{\infty} \frac{dt\; t^n}{e^t \,-\, 1}\,.$$

We find

$$f(g) = 8\,g^2 - 16\,\zeta(2)\,g^4 + \Big(4\,\zeta(2)^2 + 12\,\zeta(4)\Big)\,8\,g^6$$
$$- \Big(4\,\zeta(2)^3 + 24\,\zeta(2)\zeta(4) - 4\,\zeta(3)^2 + 50\,\zeta(6)\Big)\,16\,g^8 + \dots$$

or, alternatively:

$$\boxed{f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - \left(\frac{73}{630}\pi^6 - 4\zeta(3)^2\right)16g^8 + \dots}$$

- Agrees with KLOV up to three loops (in the large spin limit their harmonic sums become zeta functions).

The result obeys a principle of **uniform transcendentality**:

| |
|---|
| The $l$-loop contributions have degree of transcendentality 2 l - 2. |

## 8 Dressing Kernels

The higher-loop Bethe equations receive corrections KMMZ, BDS, AFS, BK:

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1-g^2/x_k^+x_j^-}{1-g^2/x_k^-x_j^+}\,\exp\bigl(2i\theta(u_k,u_j)\bigr),$$

For perturbative string theory write the dressing phase as

$$\theta(u_k,u_j) = \sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} c_{r,s}(g)\bigl(\tilde{q}_r(u_k)\,\tilde{q}_s(u_j) - \tilde{q}_s(u_k)\,\tilde{q}_r(u_j)\bigr)\,.$$

The $\tilde{q}_r(u)$ are the higher conserved charges. The strong-coupling expansion of $c_{r,s}$ within string theory is

$$c_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}.$$

Proposal for the all-order strong-coupling expansion:

$$\begin{aligned} c_{r,s}^{(n)} \;&=\; \frac{\bigl(1-(-1)^{r+s}\bigr)\zeta(n)}{2(-2\pi)^n\,\Gamma(n-1)}\,(r-1)(s-1)\;*\\ &\qquad\qquad * \;\frac{\Gamma[\frac{1}{2}(s+r+n-3)]\,\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s+r-n+1)]\,\Gamma[\frac{1}{2}(s-r-n+3)]}\,, \end{aligned}$$

Singular for $n=0,1$, when

$$c_{r,s}^{(0)} = \delta_{r+1,s}\,, \qquad c_{r,s}^{(1)} = -\frac{\bigl(1-(-1)^{r+s}\bigr)}{\pi}\,\frac{(r-1)(s-1)}{(s+r-2)(s-r)}\,.$$

(The latter are the AFS and BT,HL terms, respectively.)

Based on:

- $n=0,1$: available data
- for even $n$: crossing symmetry (J,BHL)
- for odd $n$: natural choice

Can we **interpolate to weak coupling** in order to recompute $f(g)$ with this dressing kernel?

$\varPsi(z)=\partial_z \log \varGamma(z)$ has the asymptotic expansion ($z >> 0$)

$$\Psi(1+z)=\log z+\sum_{n=1}^{\infty}\frac{c_n}{z^n}\,,\qquad c_n=-\frac{\mathrm{B}_n}{n}=(-1)^n\zeta(1-n)\,,$$

while the expansion around $z=0$ reads

$$\Psi(1+z)=-\gamma_{\mathrm{E}}+\sum_{k=1}^{\infty}\tilde{c}_k\,z^k,\qquad \tilde{c}_k=-(-1)^k\zeta(1+k)\,.$$

The expansion coefficients for large and small $z$ are almost the same!

$$c_n=-\tilde{c}_{-n}$$

**In our situation**: $c_{r,s}(g)$ has the weak coupling expansion

$$c_{r,s}(g)=-\sum_{n=1}^{\infty}c_{r,s}^{(-n)}g^{1+n}.$$

We use the identities

$$\zeta(1-z)=2(2\pi)^{-z}\cos(\tfrac{1}{2}\pi z)\,\Gamma(z)\,\zeta(z)\quad\text{and}\quad\Gamma(1-z)=\frac{\pi}{\sin(\pi z)\,\Gamma(z)}$$

to obtain

$$c_{r,s}^{(n)}=\frac{\big(1-(-1)^{r+s}\big)\cos(\frac{1}{2}\pi n)\,(-1)^{s-1-n}\,\zeta(1-n)}{\Gamma[\frac{1}{2}(5-n-r-s)]\,\Gamma[\frac{1}{2}(3-n+r-s)]}*$$
$$*\,\frac{\Gamma(2-n)\,\Gamma(1-n)\,(r-1)(s-1)}{\Gamma[\frac{1}{2}(3-n-r+s)]\,\Gamma[\frac{1}{2}(1-n+r+s)]}\,.$$

- Only even $n$ contribute.
- Strong argument in BES, v2 contains a proof for $c_{2,3}$.
  General proof in KL `ht/0611204`.

# 9 String Phase and Scaling Function

The weak coupling expansion of the string theory dressing phase yields the kernel

$$
\begin{aligned}
&c_{2,3}^{(-2)} = -4\,\zeta(3), \\
&c_{2,3}^{(-4)} = +40\,\zeta(5), \quad c_{3,4}^{(-4)} = -24\,\zeta(5), \quad c_{2,5}^{(-4)} = +8\,\zeta(5), \\
&c_{2,3}^{(-6)} = -420\,\zeta(7), \quad c_{3,4}^{(-6)} = +420\,\zeta(7), \quad c_{2,5}^{(-6)} = -168\,\zeta(7), \\
&c_{2,3}^{(-8)} = +4704\,\zeta(9), \qquad \ldots
\end{aligned}
$$

The scaling function becomes

$$
\begin{aligned}
f_+(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - 16\left(\frac{73}{630}\pi^6 + 4\,\zeta(3)^2\right) g^8 \\
+ 32\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\zeta(3)^2 + 40\,\zeta(3)\,\zeta(5)\right) g^{10} \\
- 64\left(\frac{136883}{3742200}\pi^{10} + \frac{8}{15}\pi^4\zeta(3)^2 + \frac{40}{3}\pi^2\zeta(3)\,\zeta(5)\right. \\
\left. + 210\,\zeta(3)\,\zeta(7) + 102\,\zeta(5)^2\right) g^{12} + \ldots .
\end{aligned}
$$

$f_+(g)$ is obtained from $f(g)$ (trivial dressing phase) by multiplying all odd zeta functions by the imaginary unit $i$.

# 10 Agreement with Field Theory

In parallel to our effort, BCDKS have completed a direct computation of the scaling function $f(g)$ at four loops. Their calculation uses unitarity methods and conformal invariance to predict a set of integrals which are evaluated with the help of the MB representation. The exponentiation of infrared singularities is a stringent check.

BCDKS find

$$\begin{aligned} f(g) &= \ldots - 64 \times (29.335 \pm 0.052)\, g^8 + \ldots \\ &= \ldots - \big(3.0192 \pm 0.0054\big) \times 10^{-6} \lambda^4 + \ldots . \end{aligned}$$

Recall our value:

$$\begin{aligned} f_+(g) &= \ldots - 16 \left( \frac{73}{630} \pi^6 + 4\,\zeta(3)^2 \right) g^8 + \ldots \\ &\approx \ldots - 3.01502 \times 10^{-6} \lambda^4 + \ldots . \end{aligned}$$

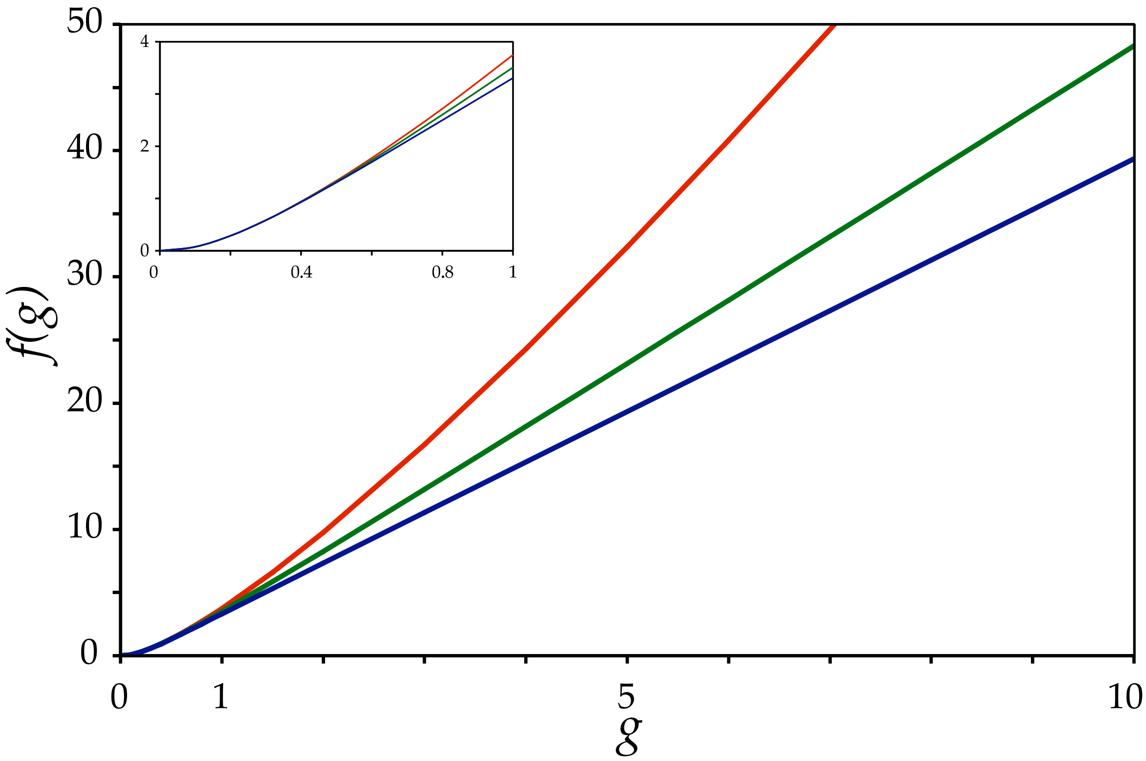
The four-loop value calculated by Bern, Czakon, Dixon, Kosower and Smirnov matches the fourth term in $f_+(g)$.

- BCDKS independently guessed the sign-flipped scaling function $f_+(g)$. They checked compatibility with the KLV approximation to rather high order.
- CSV `ht/0612309` have improved the error bar of the BCDKS result by three orders of magnitude.
- BMcLR constructed the four-loop Hamiltonian of the $su(2)$ sector from Feynman graphs. They confirm

$$\beta_{2,3}^{(3)} = 4\,\zeta(3)\,.$$

## 11 Numerics by BBKS

$f_0(g)$ arises by omitting the odd zeta values.




$f(g)$, $f_0(g)$, $f_+(g)$.

- The transition to the linear regime happens around $g \approx 1$. Extrapolation is well behaved.

Strong coupling behaviour of $f_+(g)$:

$$f_+(g) = 4.000000\, g - 0.661907 - 0.0232\, g^{-1} + \ldots$$

Error: $\pm\{1, 2, 1\}$ in the last digit displayed.

Exact result: GKP, FT, RT; BKK, KSV

$$f_+(g) = 4\, g - \frac{3\log(2)}{\pi} - \frac{K}{4\pi^2}\frac{1}{g} + \ldots$$

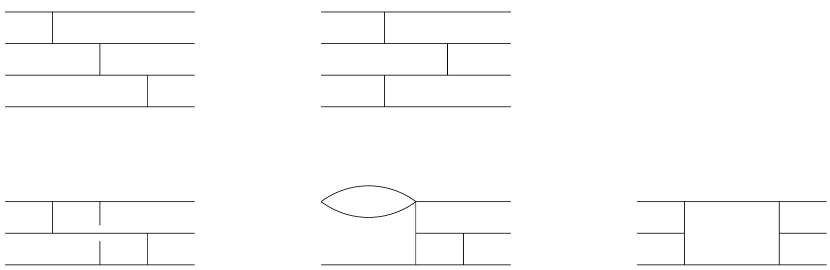
## 12 Conclusions

- In the planar limit, the operator spectrum of $\mathcal{N} = 4$ SYM is described by an integrable system. We have presented a quick review of the strategy for the so-called $su(2)$ and $sl(2)$ sectors (BMN and twist operators, respectively). The approach has been generalised to the full set of multiplets, and to higher loop orders.
- The weak coupling (gauge theory) Bethe ansatz is fixed up to four loops by current data. It contains a **dressing factor** which becomes relevant at four loops and beyond.
- The Bethe equations are valid in the asymptotic regime of infinite spin chain length.
- **Wrapping:** For a discussion of strong coupling behaviour one would need all orders in perturbation theory. In general, no such result can be obtained for operators of finite length, since the interaction range grows with the loop order.
- In string theory, there is an equivalent problem with **finite size corrections**.
- We have discussed the all-loops Bethe ansatz for the derivative operator sector. The energy of the lowest lying state scales logarithmically with the total spin $s$ as the number of derivatives becomes large. The coefficient of $\log(s)$ is the **scaling function** $f(g)$. The calculation is not affected by wrapping.
- At strong-coupling (string theory) the dressing phase had been conjectured on grounds of calculational data paired with crossing symmetry constraints. We have presented the weak coupling expansion of this string theory dressing phase and discussed its effect on the scaling function.
- The four-loop term of the result $f_+(g)$ agrees with field theory calculations!

# 13 Outlook

We must understand the wrapping regime/finite size corrections.

- $\mathcal{N} = 4$ version of the BFKL equation KLRSV ht/07043586
- thermodynamic Bethe ansatz AF ht/07101568
- quantum corrections to the "giant magnon" GSV ht/08013671
- In field theory, the first case of wrapping is the fourth anomalous dimension of the Konishi operator. Two calculations of the appropriate modification of the four-loop spin chain Hamiltonian have been presented, but they lead to results inconsistent with each other and with the BFKL prediction FSSZ ht/07123522, KM 08011661.
- My initiative ht/07123513 involves a relatively small ($O(100)$) number of numerator terms with six derivatives for the following two-point topologies:

  The numerators of the six-loop diagrams have at least one $p^2$.

  I am currently reducing the four-loop part by IBP. (M36 running now.) Higher diagrams perhaps first by MB?