One-loop Pentagons and Hexagons

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DESY, Zeuthen

Loops and Legs, Sondershausen, April 2008

- Melrose (1965)
- van Neerven, Vermaseren (1984)
- Bern, Dixon, Kosower (1993)
- Jegerlehner, Fleischer, Tarasov (2000)
- Denner, Dittmaier (2003, 2006)
- Binoth, Heinrich, Kauer (2003)
- Giele, Glover (2004)
- Hahn, Rauch (2006)
- . . .

A formula by Davydychev

An N-point tensor integral of rank R in d-dimensional space-time:

$$J^{(N)}_{\mu_1\ldots\mu_R}(d;\nu_1,\ldots,\nu_N) = \int \frac{d^d q}{i\pi^{d/2}} \frac{q_{\mu_1}\ldots q_{\mu_R}}{D_1^{\nu_1}\ldots D_N^{\nu_N}}$$

$$D_j = (q+p_j)^2 - m_j^2 + i\epsilon$$

$$J^{(N)}_{\mu_1\ldots\mu_R}(d;\nu_1,\ldots,\nu_N) = \sum_{\lambda,\kappa_1,\ldots,\kappa_N} (-\frac{1}{2})^{\lambda} \left\{[g]^{\lambda}[p_1]^{\kappa_1}\ldots[p_N]^{\kappa_N}\right\}_{\mu_1\ldots\mu_R}$$

$$\times(\nu_1)_{\kappa_1}\ldots(\nu_N)_{\kappa_N} J^{(N)}(d+2(R-\lambda);\nu_1+\kappa_1,\ldots,\nu_N+\kappa_N)$$

$$2\lambda+\kappa_1+\ldots+\kappa_N = R \qquad (\nu)_{\kappa} = \frac{\Gamma(\nu+\kappa)}{\Gamma(\nu)}$$

Phys. Lett. B263 (1991) 107

Determinants

Modified Cayley determinant:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix} \qquad Y_{ij} = -(p_i - p_j)^2 + m_i^2 + m_j^2$$

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Gram determinant:

$$\left(\right)_N = -2^{N-1} \times \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & \ldots & p_1 \cdot p_{N-1} \\ p_2 \cdot p_1 & p_2 \cdot p_2 & \ldots & p_2 \cdot p_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N-1} \cdot p_1 & p_{N-1} \cdot p_2 & \ldots & p_{N-1} \cdot p_{N-1} \end{vmatrix} \qquad (p_N = 0)$$

Signed Minors

Signed minor:

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv (-1)^{\sum_l (j_l + k_l)} \times \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right|$$

Antisymmetric in $j_1 \cdots j_m$ and in $k_1 \cdots k_m$.

Eg:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1N} \\ Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix}$$

$$\sum_{k=1}^{N} \binom{j}{k}_N \; = \; \delta_{0j} \, ()_N$$

$$\sum_{k=1}^{N} Y_{ik} \binom{j}{k}_N + \binom{j}{0}_N \; = \; \delta_{ij} \, ()_N$$

$$\binom{i}{j}_N \binom{k}{l}_N - \binom{i}{l}_N \binom{k}{j}_N \; = \; \binom{ik}{jl}_N \, ()_N$$

D. B. Melrose, Nuovo Cim. 40 (1965) 181.

Decomposition of metric tensor:

$$g^{\mu\nu} \; = \; 2 \sum_{i,j=1}^{N} \frac{\binom{i}{j}_N}{()_N} \, p_i^\mu p_j^\nu \; + \; g_\perp^{\mu\nu}$$

Recurrence relations

$$()_n \, \nu_j \mathbf{j}^+ I_n^{(d+2)} = \left[-\binom{j}{0}_n + \sum_{k=1}^{n} \binom{j}{k}_n \mathbf{k}^- \right] I_n^{(d)},$$

$$(d - \sum_{i=1}^{n} \nu_i + 1)\, ()_n \, I_n^{(d+2)} \; = \; \left[\binom{0}{0}_n - \sum_{k=1}^{n} \binom{0}{k}_n \mathbf{k}^- \right] I_n^{(d)}.$$

$$\binom{0}{0}_n \nu_j \mathbf{j}^+ I_n^{(d)} = \sum_{k=1}^{n} \binom{0j}{0k}_n \left[d - \sum_{i=1}^{n} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_n^{(d)}.$$

J. Fleischer, F. Jegerlehner and O. V. Tarasov, Nucl. Phys. B566 (2000) 423

Pentagons

- Main interest: $\nu_1 = \ldots = \nu_5 = 1$
- Scalar pentagon:

$$(d-4)\,()_5\, I_5^{[d+]} = \binom{0}{0}_5 I_5 - \sum_{s=1}^{5} \binom{0}{s}_5 I_4^s$$

- For $d \to 4$:

$$I_5 = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0}{s}_5 I_4^s$$

Second rank tensor

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} p_i^{\mu}\, p_j^{\nu} E_{ij} + g^{\mu\nu} E_{00},$$

$$E_{ij} = \sum_{s=1}^{5} S_{ij}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ij}^{3,st} I_3^{st}$$

$$S_{ij}^{4,s} = \frac{1}{\binom{0}{0}_5} \frac{1}{\binom{s}{s}_5} X_{ij}^{s0}, \quad S_{ij}^{3,st} = -\frac{1}{\binom{0}{0}_5} \frac{1}{\binom{s}{s}_5} X_{ij}^{st}$$

$$X_{ij}^{st} = -\binom{0s}{0j}_5 \binom{ts}{is}_5 + \binom{0i}{sj}_5 \binom{ts}{0s}_5 .$$

$$E_{00} = -\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\binom{s}{0}_5}{\binom{s}{s}_5} \left[\binom{0s}{0s}_5 I_4^s - \sum_{t=1}^{5} \binom{ts}{0s}_5 I_3^{st} \right]$$

Hexagons

- For 4-dimensional external momenta, $()_6 = 0$, and a linear relation between the propagators D_j exists:

$$1 \;=\; \sum_{j=1}^{6} \frac{\binom{0}{j}_6}{\binom{0}{0}_6} D_j$$

  Thus, any hexagon is immediately reduced to pentagons, eg:

$$I_6 = \sum_{r=1}^{6} \frac{\binom{0}{r}_6}{\binom{0}{0}_6} I_5^r$$

- For tensor hexagons in $d = 4$

$$I_6^{\mu_1\cdots\mu_R} \;=\; \sum_{r=1}^{6} v_r^{\mu_1} I_5^{\mu_2\cdots\mu_R\,,r}\,, \qquad v_r^{\mu} \;\equiv\; \frac{1}{\binom{0}{0}_6} \sum_{i=1}^{5} \binom{0i}{0r}_6 p_i^{\mu}$$

Conclusion

- Complete analytical reduction of one-loop tensor pentagons and hexagons down to master integrals A_0, B_0, C_0, D_0 with coefficients expressed in terms of signed minors
- Numerical implementations by Kajda (mathematica) and Diakonidis (fortran)
- Cross checks with independent code by Uwer