Loops for hot QCD

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Quantum Chromodynamics (QCD)

Reality check?!

- outraged claim: none of qu, gl ever seen!
  - ▷ *have to explain confinement*

how to check QCD vs Reality?

- (a) just solve its eqs ($\longrightarrow$ see next slide)
  - ▷ *by computer (lattice); tough; ''oracle''; understand?!*
- (b) consider models ''close to QCD''
  - ▷ *fewer dims; different sy groups; diff particle content*
- (c) consider circumstances in which eqs simplify
  - ▷ *remainder of this talk*

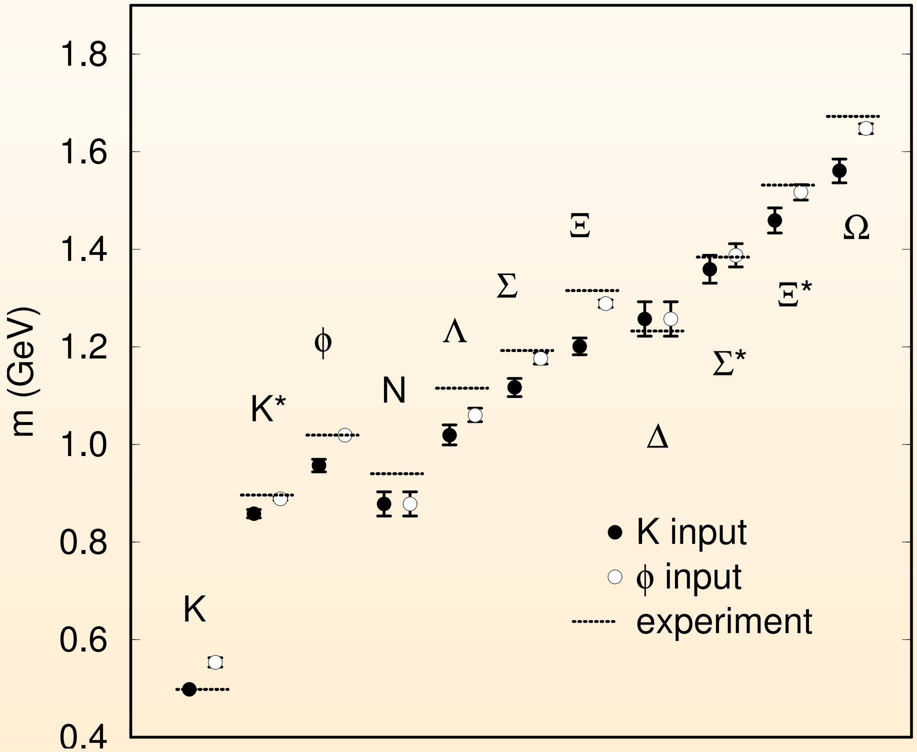
QCD reality check (a:computer)

look at hadron spectrum

- solve QCD eqs by computer
  [e.g. S. Aoki et.al., CP-PACS 1999]

- what does not come out:
  ▷ *gluons*
  ▷ *fractional charges*

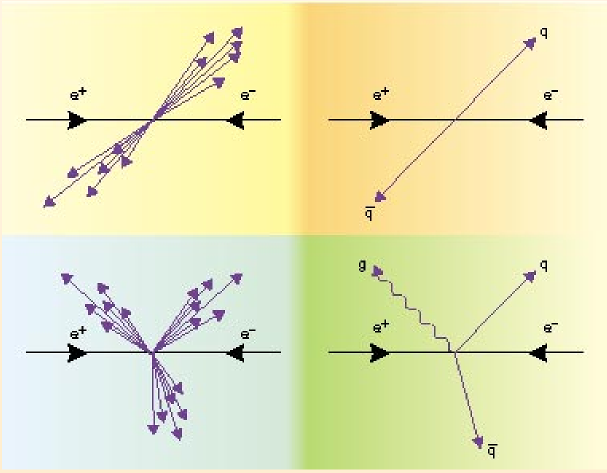
- what one gets:
  ▷ *just the observed particles + masses*
  ▷ *no more, no less!*



- punchline: obtain amazingly realistic spectrum, with 10% error
  ▷ *QCD lite; need to add remaining quark effects + quark masses*
  ▷ *much development here; teraflop speeds, worldwide effort*

QCD reality check (c:collider)

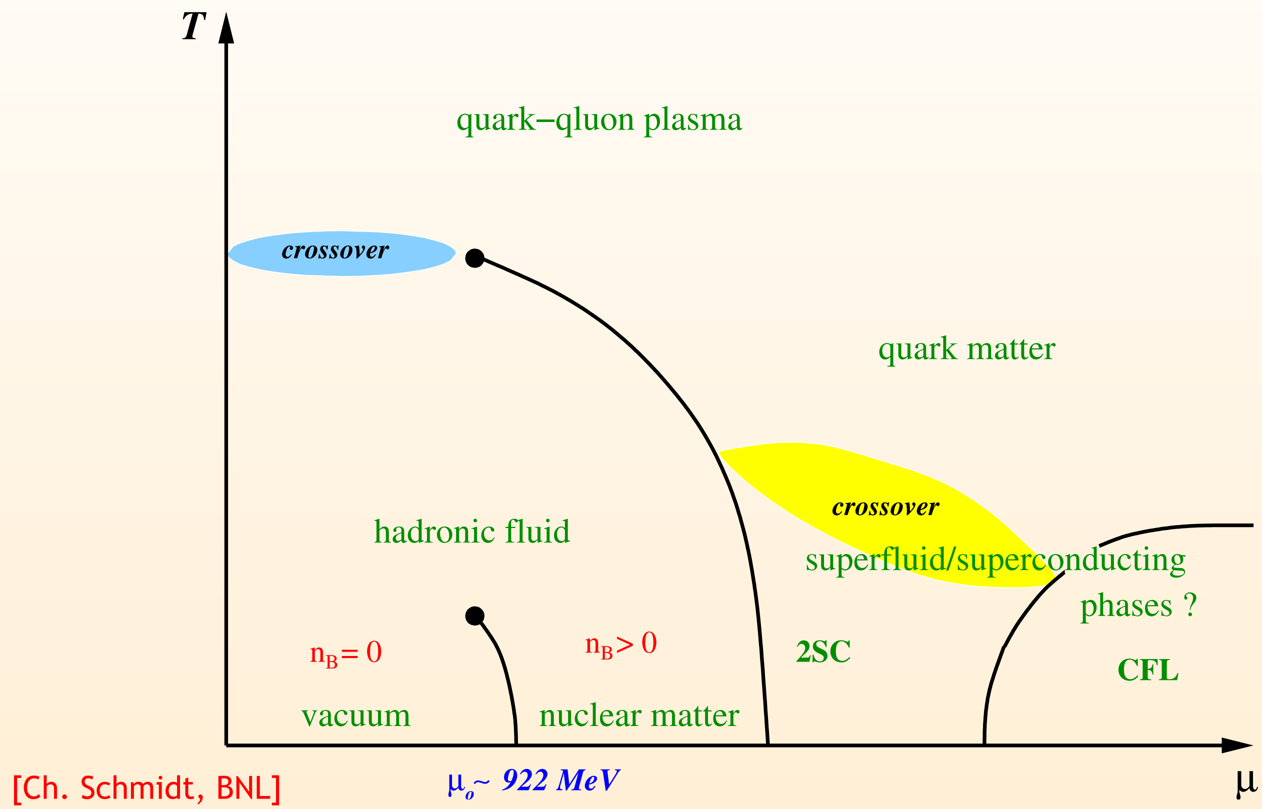
- e.g. LEP, $e^+e^- \longrightarrow X$ (stuff hitting detector): find 2 broad classes of events (QM!)
- (1) $X = e^+e^-$ or $\tau^+\tau^-$ or … l^+l^-
  - ▷ *leptons: no color charge $\longrightarrow$ mainly QED interactions*
  - ▷ *simple final state: coupling small ($\alpha = e^2/(4\pi) \approx 1/137$) most of the time (99%) nothing happens*
  - ▷ $e^+e^-\gamma \sim$ *1% $\longrightarrow$ check details of QED*
  - ▷ $e^+e^-\gamma\gamma \sim$ *0.01% $\longrightarrow$ …*
- (2) $X > 10$ particles: $\pi,\ \rho,\ p,\ \bar{p},\ \ldots$
  - ▷ *"greek+latin soup" constructed from qu+gl*
  - ▷ *pattern: flow of E+momentum in "jets"*
  - ▷ *2 jets $\sim$ 90%; 3 jets $\sim$ 9%; 4 jets $\sim$ 0.9%*
  - ▷ *direct confirmation of asy. freedom!*
  - ▷ *hard radiation is rare $\longrightarrow$ # of jets*
  - ▷ *soft radiation is common $\longrightarrow$ broadens jet*



- nowadays: "testing QCD" $\longrightarrow$ "calculating backgrounds" in search for new phenomena

QCD reality check (c:extremes)

naive questions: what happens when I heat or squeeze matter?



nature: early univ, μ tiny ($\sim \frac{\#baryons}{entropy}$), $T_c \sim 170 MeV \sim 10 \mu s$
neutron/quark stars

lab expt.: SPS / RHIC $\mu_B \sim \frac{\#baryons}{pions} \sim 45 MeV$ / LHC / GSI

basic thermodynamic observable: pressure $p(T)$

$p(T)$ important for cosmology:

- cooling rate of the universe

$$\partial_t T = -\frac{\sqrt{24\pi}}{m_{\mathsf{Pl}}} \frac{\sqrt{e(T)}}{\partial_T \ln s(T)}$$

- with entropy $s = \partial_T\, p$ and energy density $e = T s - p$
- $\Rightarrow$ cosmol. relics (dark matter, background radiation etc.) originate when an interaction rate $\tau(T)$ gets larger than the age of the universe $t(T)$.
  - ▷ *Ex.: ''sterile'' ν_R with $m_\nu \sim$ keV can be warm dark matter, and decouple around $T \sim 150$ MeV* *[Abazajian, Fuller 02; Asaka, Shaposhnikov 05]*

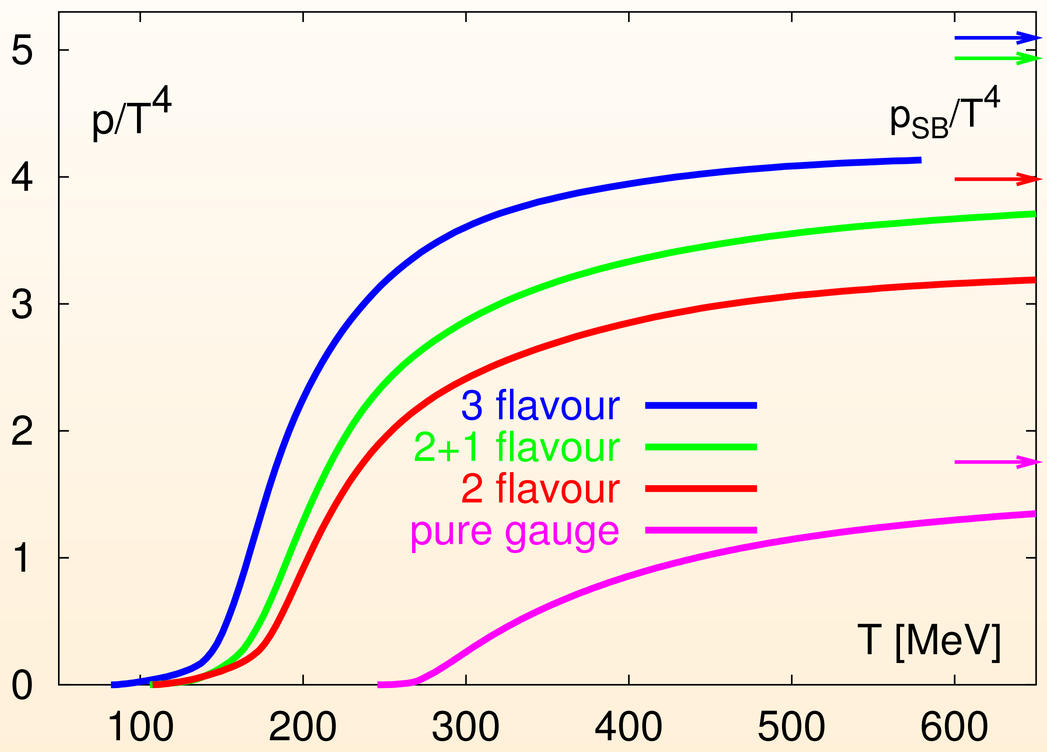
$p(T)$ in heavy ion collisions:

- expansion rate (after thermalization) given by

$$\partial_\mu T^{\mu\nu} = 0 \quad , \quad T^{\mu\nu} = [p(T) + e(T)]\, u^\mu u^\nu - p(T) g^{\mu\nu}$$

- with flow velocity $u^\mu(t, x)$
  - ▷ *hydrodynamic expansion: hadronization at $T \sim 100 - 150$ MeV $\Rightarrow$ observed hadron spectrum depends (indirectly) on $p(T)$*

$p(T)$ via (large) computer ($\mu_B = 0$)

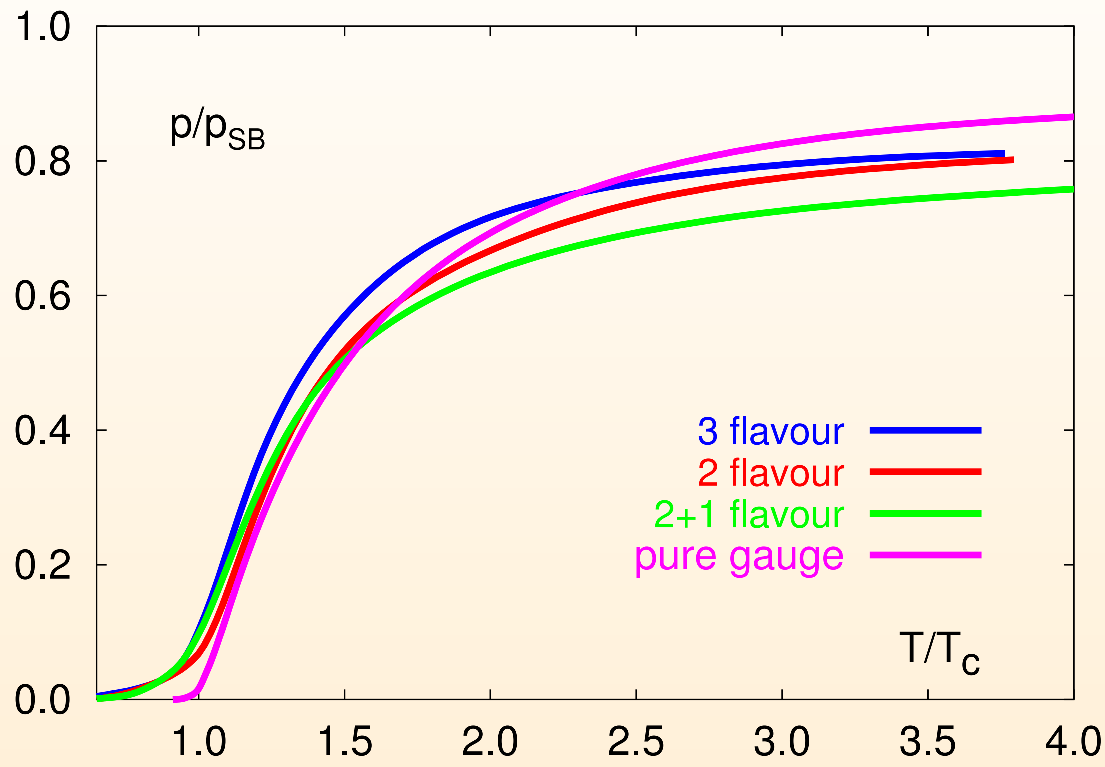


[lattice data from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

confirms simplicity: 3 dofs (π) $\to$ 52 ($3 \times 3 \times 2 \times 2$ qu + 8×2 gl)

$p(T)$ via (large) computer ($\mu_B = 0$)



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$p(T)$ via weak-coupling expansion

need to explain 20% deviation from ideal gas at $T \sim 4T_c$

structure of pert series is non-trivial !

- Ex.: $$\begin{aligned} p(T) &\equiv \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}[A^a_\mu, \psi, \bar\psi] \exp\left(-\frac{1}{\hbar}\int_0^{\hbar/T} d\tau \int d^{3-2\epsilon}x\, \mathcal{L}_{\text{QCD}}\right) \\ &= 1 + g^2 + g^3 + g^4 \ln g + g^4 + g^5 + g^6 \ln g + g^6 + \ldots \end{aligned}$$

reason: interactions make QCD a multiscale system

dynamically generated scales ($|k| \sim \pi T$ is called ''hard''):
color-electric screening at $|k| \sim m_{\text{E}} \sim gT$ (''soft'')
color-magnetic screening at $|k| \sim g^2 T$ (''ultrasoft'')

expansion parameter

$$g^2\, n_b(|k|) = \frac{g^2}{e^{|k|/T} - 1} \overset{|k| \lesssim T}{\approx} \frac{g^2 T}{|k|}$$

treatment of a multiscale system: effective field theory !

Effective theory setup: QCD $\to$ EQCD

high T: QCD dynamics contained in 3d EQCD

integrate out $|p| \gtrsim 2\pi T$: ψ, $A_\mu(n \neq 0)$

$$p_{\rm QCD}(T) \equiv p_{\rm E}(T) + \frac{T}{V} \ln \int \mathcal{D}[A_k^a, A_0^a] \exp\Big(- \int d^d x \, \mathcal{L}_{\rm E} \Big)$$

$$\mathcal{L}_{\rm E} = \frac{1}{2} Tr\, F_{kl}^2 + Tr\, [D_k, A_0]^2 + m_{\rm E}^2 Tr\, A_0^2 + \lambda_{\rm E}^{(1)} (Tr\, A_0^2)^2 + \lambda_{\rm E}^{(2)} Tr\, A_0^4 + \ldots$$

five matching coefficients [E. Braaten, A. Nieto, 95; KLRS 02; M. Laine, YS, 05]

$p_{\rm E} = T^4 \left[\# + \# g^2 + \# g^4 + \# g^6 + \ldots \right]$, $m_{\rm E}^2 = T^2 \left[\# g^2 + \# g^4 + \ldots \right]$,

$g_{\rm E}^2 = T \left[g^2 + \# g^4 + \# g^6 + \ldots \right]$, $\lambda_{\rm E}^{(1),(2)} = T \left[\# g^4 + \ldots \right]$.

higher order operators do not (yet) contribute [S. Chapman, 94; Kajantie et al, 97, 02]

$$\frac{\delta p_{\rm QCD}(T)}{T} \sim \delta \mathcal{L}_{\rm E} \sim g^2 \frac{D_k D_l}{(2\pi T)^2} \mathcal{L}_{\rm E} \sim g^2 \frac{(gT)^2}{(2\pi T)^2} (gT)^3 \sim g^7 T^3$$

Effective theory setup: QCD $\to$ EQCD $\to$ MQCD

the IR of 3d EQCD is contained in 3d MQCD

integrate out $|p| \gtrsim gT$: A_0

$$p_{\mathsf{QCD}}(T) \equiv p_{\mathsf{E}}(T) + p_{\mathsf{M}}(T) + \frac{T}{V} \ln \int \mathcal{D}[A_k^a] \exp\Big(-\int d^d x\, \mathcal{L}_{\mathsf{M}}\Big)$$

$$\mathcal{L}_{\mathsf{M}} = \frac{1}{2} Tr\, F_{kl}^2 + \ldots$$

two matching coefficients [KLRS 03; P. Giovannangeli 04, M. Laine/YS 05]

$$p_{\mathsf{M}} = T m_{\mathsf{E}}^3 \left[\# + \# \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \# \frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \# \frac{g_{\mathsf{E}}^6}{m_{\mathsf{E}}^3} + \ldots\right],\; g_{\mathsf{M}}^2 = g_{\mathsf{E}}^2 \left[1 + \# \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \# \frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \ldots\right].$$

higher order operators could contribute

$$\frac{\delta p_{\mathsf{QCD}}(T)}{T} \sim \delta \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{D_k D_l}{m_{\mathsf{E}}^3} \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{(g^2 T)^2}{m_{\mathsf{E}}^3} (g^2 T)^3 \sim g^9 T^3$$

Effective theory prediction for $p(T)$

- collect contributions to $p(T)$ from all physical scales
  - ▷ *weak coupling, effective field theory setup*
  - ▷ *faithfully adding up all Feynman diagrams*
  - ▷ *get long-distance input from clean 3d lattice observable:*

$$p_{\mathsf{G}}(T) \quad \equiv \quad \frac{T}{V} \ln \int \mathcal{D}[A_k^a] \exp\Big(-S_{\mathsf{M}}\Big) = T\# \, g_{\mathsf{M}}^6$$

only one non-perturbative (but computable!) coeff needed

$$\frac{p_{\mathsf{QCD}}(T)}{p_{\mathsf{SB}}} = \frac{p_{\mathsf{E}}(T)}{p_{\mathsf{SB}}} + \frac{p_{\mathsf{M}}(T)}{p_{\mathsf{SB}}} + \frac{p_{\mathsf{G}}(T)}{p_{\mathsf{SB}}} \quad , \quad p_{\mathsf{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$$

$$= 1 + g^2 \quad + g^4 \quad + g^6 \quad + \dots \qquad \Leftarrow \text{4d QCD}$$

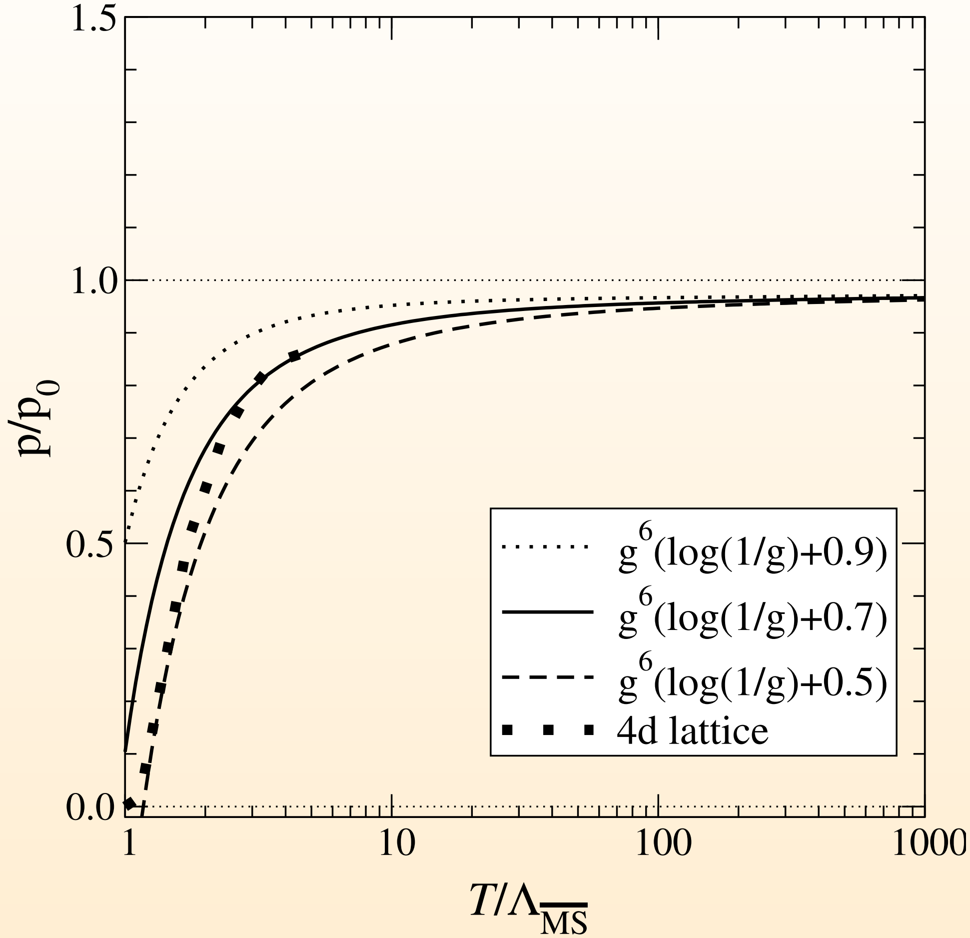
$$+ g^3 + g^4 + g^5 + g^6 + \dots \qquad \Leftarrow \text{3d adj H}$$

$$+ \frac{1}{p_{\mathsf{SB}}} \frac{T}{V} \int \mathcal{D}[A_k^a] \exp\Big(-S_{\mathsf{M}}\Big) \qquad \Leftarrow \text{3d YM}$$

$$= c_0 + c_2 g^2 + c_3 g^3 + (c_4' \ln g + c_4) g^4 + c_5 g^5 + (c_6' \ln g + c_6) g^6 + \mathcal{O}(g^7)$$

[c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold/Zhai 94, c_5 Zhai/Kastening 95, Braaten/Nieto 96, c_6' KLRS 03]

Thermal pressure $p(T)$: 4d vs 3d $(N_f = 0)$



depencence on g^6 constant

g^6 constant not yet fully known, but computable!

Outlook: 08 → 10 → 12

$$\frac{p_\mathsf{G}}{p_\mathsf{SB}} = \#_{(6)}\left(\frac{g_\mathsf{M}^2}{T}\right)^3 + [\delta\mathcal{L}_\mathsf{M}]_{(9)}$$

$$g_\mathsf{M}^2 = g_\mathsf{E}^2\left[1 + \#_{(7)}\frac{g_\mathsf{E}^2}{m_\mathsf{E}} + \left(\frac{g_\mathsf{E}^2}{m_\mathsf{E}}\right)^2\left(\#_{(8)} + \#_{(10)}\frac{\lambda_\mathsf{E}}{g_\mathsf{E}^2}\right) + \dots_{(9)}\right]$$

$$\frac{p_\mathsf{M}}{p_\mathsf{SB}} = \frac{m_\mathsf{E}^3}{T^3}\left[\#_{(3)} + \frac{g_\mathsf{E}^2}{m_\mathsf{E}}\left(\#_{(4)} + \#_{(6)}\frac{\lambda_\mathsf{E}}{g_\mathsf{E}^2}\right) + \left(\frac{g_\mathsf{E}^2}{m_\mathsf{E}}\right)^2\left(\#_{(5)} + \#_{(7)}\frac{\lambda_\mathsf{E}}{g_\mathsf{E}^2} + \#_{(9)}\left(\frac{\lambda_\mathsf{E}}{g_\mathsf{E}^2}\right)^2\right)\right.$$

$$+ \left(\frac{g_\mathsf{E}^2}{m_\mathsf{E}}\right)^3\left(\#_{(6)} + \#_{(8)}\frac{\lambda_\mathsf{E}}{g_\mathsf{E}^2} + \#_{(10)}\left(\frac{\lambda_\mathsf{E}}{g_\mathsf{E}^2}\right)^2 + \#_{(12)}\left(\frac{\lambda_\mathsf{E}}{g_\mathsf{E}^2}\right)^3\right)$$

$$\left. + [\text{3d 5loop 0pt}]_{(7)} + [\delta\mathcal{L}_\mathsf{E}]_{(7)} + [\text{3d 6loop 0pt}]_{(8)} + \dots_{(9)}\right]$$

$$m_\mathsf{E}^2 = T^2\left[\#_{(3)}g^2 + \#_{(5)}g^4 + [\text{4d 3loop 2pt}]_{(7)} + \dots_{(9)}\right]$$

$$\lambda_\mathsf{E} = T\left[\#_{(6)}g^4 + \#_{(8)}g^6 + \dots_{(10)}\right]$$

$$g_\mathsf{E}^2 = T\left[g^2 + \#_{(6)}g^4 + \#_{(8)}g^6 + \dots_{(10)}\right]$$

$$\frac{p_\mathsf{E}}{p_\mathsf{SB}} = \#_{(0)} + \#_{(2)}g^2 + \#_{(4)}g^4 + \#_{(6)}g^6 + [\text{4d 5loop 0pt}]_{(8)} + \dots_{(10)}$$

notation: $\#_{(n)}$ enters p_QCD at g^n

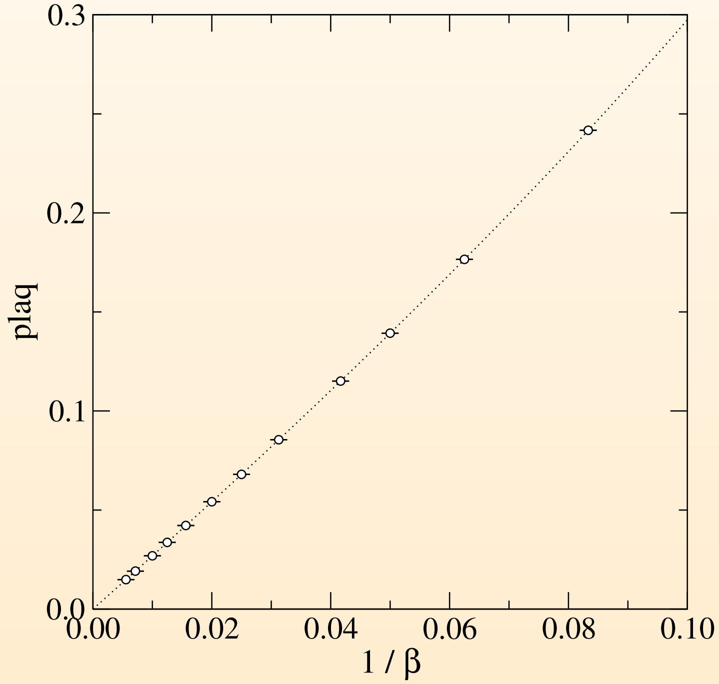
[cave: no $\frac{1}{\epsilon} + 1 + \epsilon$, no IR/UV, and no logs shown above]

3d lattice MC simulation

3d, finite box $(aL)^3$. infinite-volume ($L \to \infty$) and continuum ($\frac{1}{\beta} \equiv \frac{g_{\mathsf{M}}^2 a}{2N_c} \to 0$) limits

$$\frac{1}{2g_{\mathsf{M}}^2}\left\langle Tr\,[F_{kl}^2]\right\rangle_{\overline{\mathsf{MS}}} \equiv g_{\mathsf{M}}^2 \frac{\partial}{\partial g_{\mathsf{M}}^2} p_{\mathsf{G},\overline{\mathsf{MS}}} = 3g_{\mathsf{M}}^6 \frac{d_A C_A^3}{(4\pi)^4}\left[\alpha_{\mathsf{G}}\left(\ln\frac{\bar{\mu}}{2C_A g_{\mathsf{M}}^2} - \frac{1}{3}\right) + B_{\mathsf{G}} + \mathcal{O}(\epsilon)\right]$$

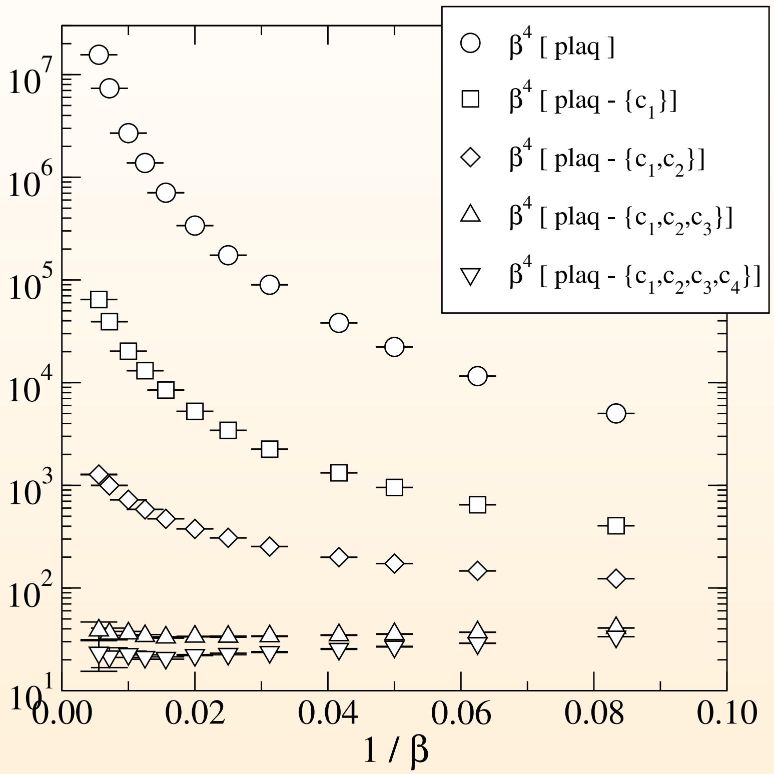
$$8\frac{d_A C_A^6}{(4\pi)^4} B_{\mathsf{G}} = \lim_{\beta\to\infty} \beta^4 \left\{\left\langle 1 - \frac{1}{C_A} Tr\,[P_{12}]\right\rangle_a - \left[\frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4}\left(\ln\beta + c_4'\right)\right]\right\}$$

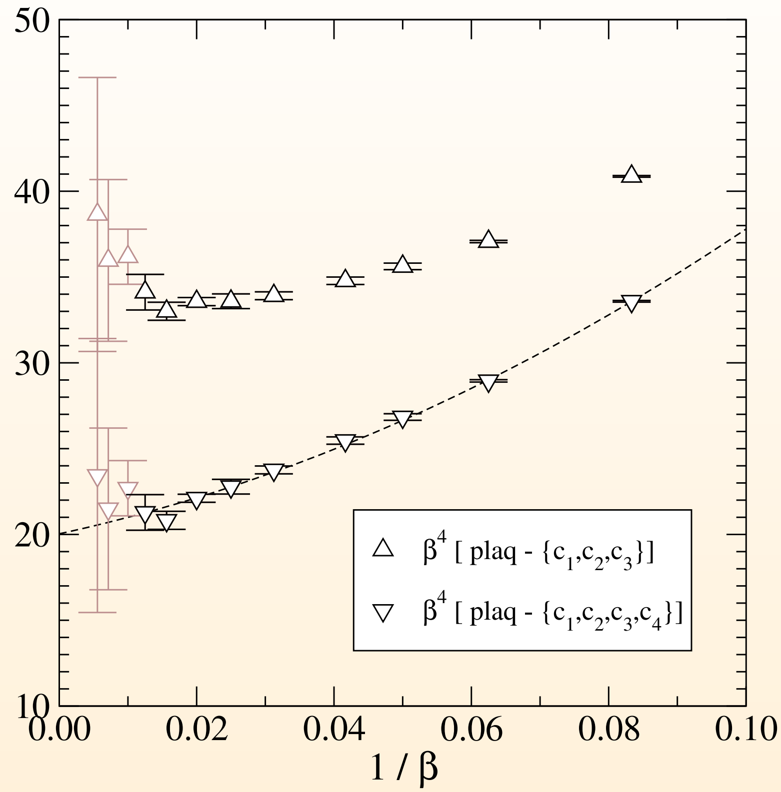


statistical errors are (much) smaller than the symbol sizes

Fit: $c_1/\beta + c_2/\beta^2 + c_3/\beta^3 + c_4 \ln\beta/\beta^4 + c_4'/\beta^4 + c_5/\beta^5 + c_6/\beta^6$

3d lattice MC simulation





significance loss due to the UV subtractions

continuum limit of infinite-volume extrapolated data

$$B_{\mathsf{G}} + \left(\frac{43}{12} - \frac{157}{768}\pi^2\right) c_4' \quad = \quad 10.7 \pm 0.4 \qquad (N_c = 3) \quad \text{after } 5 \times 10^{16} \text{ flops}$$

3d lattice perturbation theory

- amusing: 1loop tadpole has elliptic integral in 3d [M.Shaposhnikov] [in 4d as well: Laporta 08]

$$a^{2-d}\int_{-\pi}^{\pi}\frac{d^d\hat{k}}{(2\pi)^d}\frac{1}{\sum_{\mu=0}^{d-1}4\sin^2(\hat{k}_\mu/2)+\hat{m}^2}=\frac{1}{a}\sum_{n\geq 0}\hat{m}^{2n}\left(\{1,\Sigma,\xi\}+\{1\}\hat{m}\right)$$

where $\Sigma=4\pi G(0)=\frac{8}{\pi}(18+2\sqrt{2}-10\sqrt{3}-7\sqrt{6})K^2((2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2)$

- 2loop example:

$$\kappa_5=\frac{1}{\pi^4}\int_{-\pi/2}^{\pi/2}d^3x\,d^3y\frac{\sum_i\sin^2 x_i\sin^2(x_i+y_i)\sin^2 y_i}{\sum_i\sin^2 x_i\sum_i\sin^2(x_i+y_i)\sum_i\sin^2 y_i}=1.013041(1)$$

- $\longrightarrow$ classification? *very* little is known systematically.

- 1loop IBP + coordinate-space method [Lüscher/Weisz] [Becher/Melnikov]

- or Numerical Stochastic Perturbation Theory [with F. Di Renzo, V. Miccio, C. Torrero]
  - ▷ *stochastic quantization [Parisi/Wu, 81]; no diagrams!*
  - ▷ *fields* $\phi(x)\longrightarrow\phi(x,\tau)$ *evolve in stochastic time* τ *according to Langevin eq.* $\partial_\tau\phi=-\partial_\phi S[\phi]+\eta$
  - ▷ $\frac{1}{Z}\int[\mathcal{D}\phi]\,\mathcal{O}[\phi]\,e^{-S[\phi]}=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau d\tau'\,\langle\mathcal{O}[\phi_\eta]\rangle_\eta$
  - ▷ $\phi\longrightarrow\sum_k g^k\,\phi^{(k)}\Rightarrow$ *Langevin tower; numerically integrate!*
  - ▷ $c_4'=7.0\pm0.3\ (N_c=3)$ *after* 4×10^{17} *flops*

3d vacuum bubbles

- reported at L+L previously: masses $\in (0, m) \Rightarrow$ 0-scale problem
- reduction via IBP (in d dim)
- 18 (fully massive) + 10 (QED type) master integrals up to 4-loop
- integration of masters via various methods, e.g.
  - ▷ *explicit integration in x-space*
  - ▷ *explicit solution of low-order difference equations: ${}_PF_{P-1}$ etc.*
  - ▷ *numerical solution of difference equations via factorial series*
  - ▷ *differential eqs (in mass ratio); solve iteratively with HPLs*
- just to show something new: 4-loop 7-lines

```
[VB5/J1^4.4d] =
       + ep^0 * ( - 1/6 )
       + ep^1 * ( - 5/6 )
       + ep^2 * ( - 11/3 - z3 )
       + ep^3 * ( - 44/3 - 3/5*z2^2 + 2/3*z3 )
       + ep^4 * ( - 166/3 + 53*z5 - 6*z2^2 + 31/3*z3 )
       + ...
       + ep^8 * ( [uptoweight9consts] );
[VB5/J1^4.3d] = ( 1 - 2*ep )^3/( 1 + 4*ep )*(
       + ep^0 * ( 1/4*z2 - 1/2*ln2^2 )
       + ep^1 * ( - 4*z3 + 9/2*ln2*z2 + ln2^3 )
       + ep^2 * ( 30*a4 - 1/4*z2^2 - 21/4*ln2*z3 - 23/2*ln2^2*z2 + 1/12*ln2^4 )
       + ...
       + ep^7 * ( [weight9consts] ) );
```

4d vacuum bubbles: sum-integrals

- notation: $\sum\!\!\!\!\!\!\int_P = T \sum_{n=-\infty}^{\infty} (4\pi T^2)^\epsilon \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}$; $P^2 = P_0^2 + p^2$ with $P_0 = 2\pi n T$ (bos)
- already ∞ many 1-loop masters

$$\sum\!\!\!\!\!\!\int_P \frac{(P_0)^m}{(P^2)^n} = \frac{2\pi^2 T^4}{(2\pi T)^{2n-m}} \frac{4^\epsilon\,\Gamma(n-\frac{3}{2}+\epsilon)}{\Gamma(\frac{1}{2})\Gamma(n)} \zeta(2n-m-3+2\epsilon)$$

- 2-loop sunset vanishes (proof e.g. by IBP): $\sum\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2\, Q^2\, (P+Q)^2} = 0$
- relevant 3-loop masters

$$\sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2\, Q^2\, R^2\, (P+Q+R)^2} = \frac{T^4}{(4\pi)^2} \frac{1}{24\epsilon} \left[1 + \epsilon b_{11} + \epsilon^2 b_{12} + \ldots\right]$$

$$\sum\!\!\!\!\!\!\int_{PQR} \frac{1}{(P^2)^2\, Q^2\, R^2\, (P+Q+R)^2} = \frac{T^2}{(4\pi)^4} \frac{1}{8\epsilon^2} \left[1 + \epsilon s_{11} + \epsilon^2 s_{12} + \ldots\right]$$

$$\sum\!\!\!\!\!\!\int_{PQR} \frac{((Q-R)^2)^2}{(P^2)^2\, Q^2\, R^2\, (P+Q)^2\, (P+R)^2} = \frac{T^4}{(4\pi)^2} \frac{11}{216\epsilon} \left[1 + \epsilon m_{11} + \epsilon^2 m_{12} + \ldots\right]$$

with $b_{11} = \frac{91}{15} - 3\gamma_E + 8\frac{\zeta'(-1)}{\zeta(-1)} - 2\zeta(3)$,

$s_{11} = \frac{17}{6} + \gamma_E + 2\frac{\zeta'(-1)}{\zeta(-1)}$, $s_{12} = 48.79..$,

$m_{11} = \frac{73}{22} - \frac{21}{11}\gamma_E + \frac{64}{11}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{10}{11}\zeta(3)$

4d vacuum bubbles: sum-integrals

- only one 4-loop master known at present

$$\sum\!\!\!\!\!\!\!\int_{PQRS} \frac{1}{P^2(P+S)^2\, Q^2(Q+S)^2\, R^2(R+S)^2} = \frac{T^4}{(4\pi)^4}\frac{1}{16\epsilon^2}\left[1+\epsilon t_{11}+\epsilon^2 t_{12}+\ldots\right]$$

with $t_{11} = \frac{44}{5} - 4\gamma_E + 12\frac{\zeta'(-1)}{\zeta(-1)} - 4\zeta(3) - \zeta(2)$,

$t_{12} = 2b_{12} + 25.70.. - 3\zeta(2)(28.92..)$

- IBP works in 3d piece

- integration of masters: brute force; not elegant

Conclusions

- QCD contains an extremely rich structure
- thermodynamic quantities of QCD are relevant for cosmology and heavy ion collisons
- these quantities can be determined numerically at $T \sim 200$ MeV,
  and analytically at $T \gg 200$ MeV; multi-loop sports, eff. theories convenient
- effective field theory opens up tremendous opportunities: analytic treatment of fermions, universality, superrenormalizabilty
- for precise results, sometimes need very involved mathematical tools
- toolbox is well equipped for dealing with (3d) lattice, lattice pert, and cont pert
- do (some of) these methods work for (4d) sum-integrals as well?