Evaluating the three-loop quark static potential

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- Introduction. Our calculational scheme.
- Reduction to master integrals by FIRE (an algorithm to solve reduction problems for Feynman integrals).
- Evaluating master integrals by Mellin–Barnes representation.
- Preliminary results and perspectives.

The QCD potential between a heavy static quark and its antiquark in an expansion in α_s and heavy-quark velocity $v \sim \alpha_s$.

Two alternative definitions of the potential:
in terms of a Wilson loop,
in the framework of an effective field theory.

The dynamics of a nonrelativistic quark-antiquark pair involves three well separated scales:
hard $\sim m_q$; soft $\sim m_q v$; ultrasoft $\sim m_q v^2$

Expansion by regions [M. Beneke & V.A. Smirnov '98]

Four regions are relevant:

hard (energy and momentum $\sim m_q$);
soft (energy and momentum $\sim m_q v$);
potential (energy $\sim m_q v^2$, momentum $\sim m_q v$);
ultrasoft (energy and momentum $\sim m_q v^2$)

'Integrating out' the hard modes: QCD $\rightarrow$ NRQCD
'Integrating out' the soft modes and potential gluons:
NRQCD $\rightarrow$ pNRQCD.

The static potential as an operator in pNRQCD Lagrangian.

The static colour-singlet potential

$$V(|\boldsymbol{q}|) \;=\; -\frac{4\pi C_F \alpha_s(|\boldsymbol{q}|)}{\boldsymbol{q}^2}\left[1+\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}a_1+\left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^2 a_2 \right.$$
$$\left. +\left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^3\left(a_3+8\pi^2 C_A^3\ln\frac{\mu^2}{\boldsymbol{q}^2}\right)+\cdots\right]$$

with $C_A = N$ and $C_F = (N^2-1)/(2N)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N)$ colour gauge group, respectively, $T_F = 1/2$ is the index of the fundamental representation, and n_l is the number of light-quark flavours.

One loop

[W. Fischler '77; A. Billoire '80]

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_l$$

Two loops

[M. Peter'97; Y. Schröder'99]

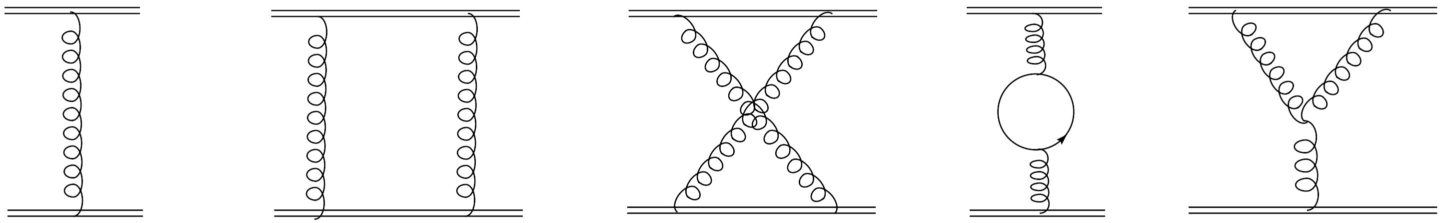
$$a_2 = \left[\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right] C_A^2 - \left[\frac{1798}{81} + \frac{56}{3}\zeta(3)\right] C_A T_F n_l$$
$$- \left[\frac{55}{3} - 16\zeta(3)\right] C_F T_F n_l + \left(\frac{20}{9} T_F n_l\right)^2$$

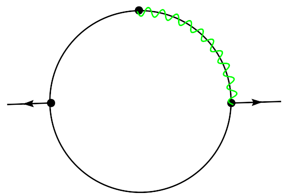
$1/m_q$-correction [B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser'02]

Three loops: $a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} = ?$

a_3 is one of a few missing NNNLO ingredients of the non-relativistic dynamics near threshold.

Tree and one-loop approximations:

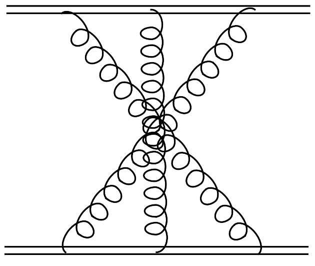


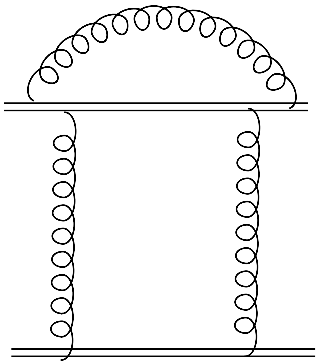


$$F(a_1, a_2, a_3) = \int \frac{\mathrm{d}^d k}{(-k^2 - i0)^{a_1}(-(q-k)^2 - i0)^{a_2}(-v \cdot k - i0)^{a_3}}$$

with $q \cdot v = 0, \quad v = (1, \vec{0})$

Two loops:





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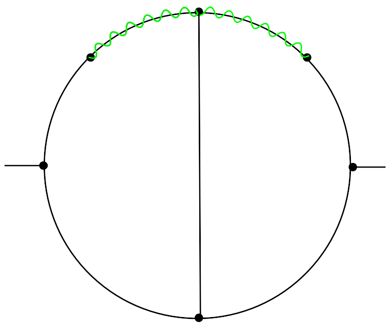
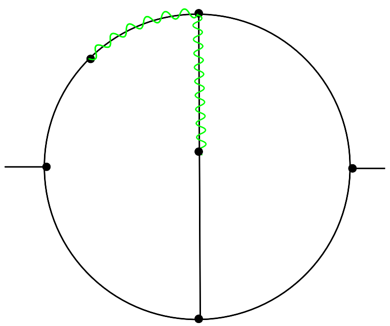
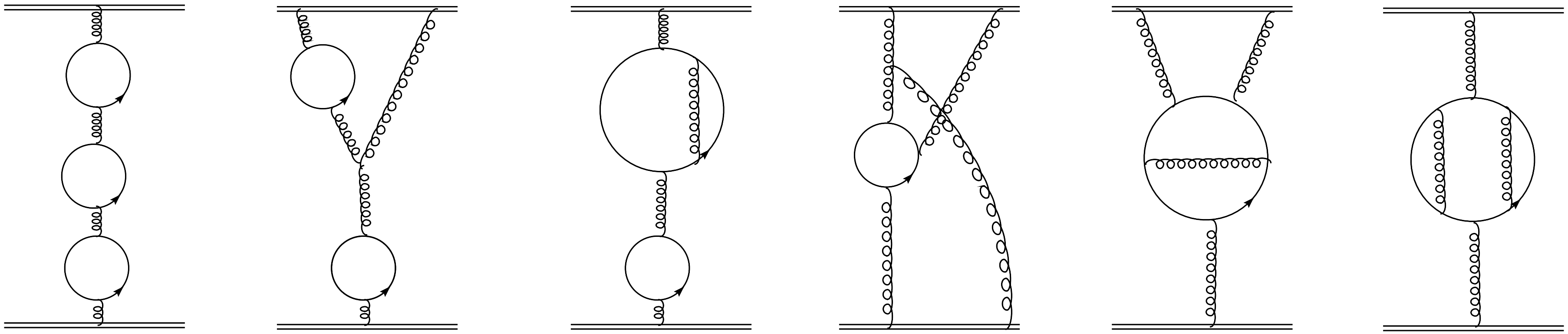




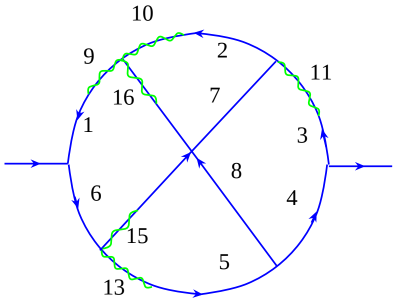
Generation of diagrams by `QGRAPH`

Typical diagrams in three loops:

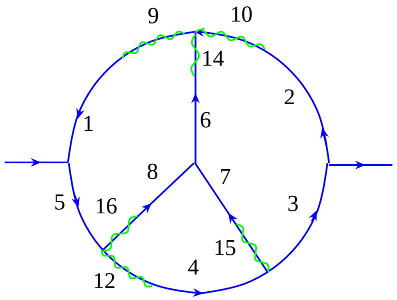


$C_F(n_lT_F)^3$ $C_FC_A(n_lT_F)^2$ $C_F^2(n_lT_F)^2$ $C_FC_A^2n_lT_F$ $C_F^2C_An_lT_F$ $C_F^3n_lT_F$

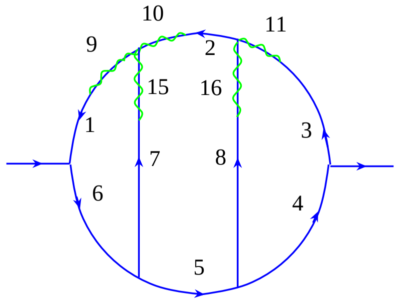
J11a



J11b



J11c



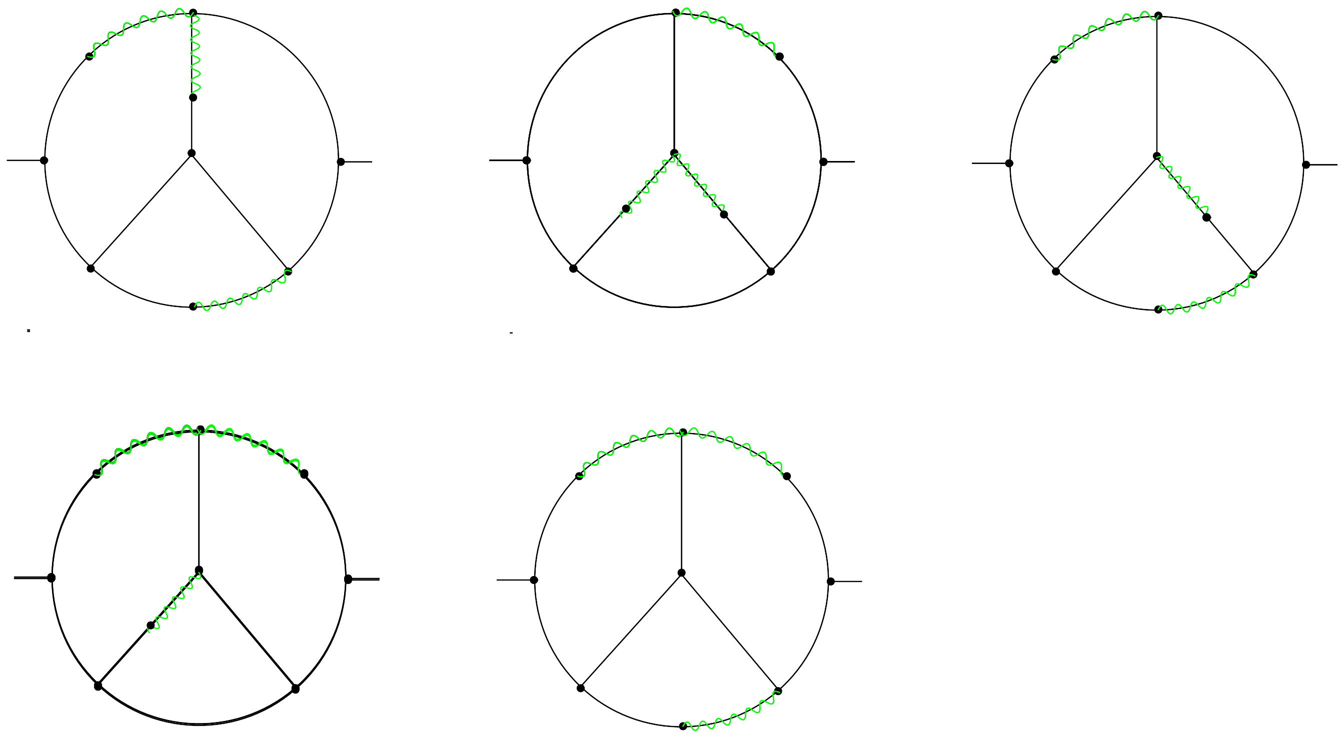
$$
\begin{aligned}
&J_{11}^{a}(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0;n_9,n_{10},n_{11},n_{13},n_{15},n_{16};s_9,s_{10},s_{11},s_{13},s_{15},s_{16})\\
&\quad=\int\!\!\int\!\!\int\frac{(-p_0^2)^{-n_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{n_1}(-p_2^2)^{n_2}(-p_3^2)^{n_3}(-p_4^2)^{n_4}(-p_5^2)^{n_5}(-p_6^2)^{n_6}(-p_7^2)^{n_7}(-p_8^2)^{n_8}}\\
&\qquad\times\frac{1}{(-v\cdot p_1-s_9 i0)^{n_9}(-v\cdot p_2-s_{10}i0)^{n_{10}}(-v\cdot p_3-s_{11}i0)^{n_{11}}(-v\cdot p_5-s_{13}i0)^{n_{13}}(-v\cdot p_7-s_{15}i0)^{n_{15}}(-v\cdot p_8-s_{16}i0)^{n_{16}}}\\
&\qquad p_0=k-r,p_1=k,p_2=l,p_3=r,p_4=q+r,p_5=q+k-l+r,p_6=q+k,p_7=l-r,p_8=k-l
\end{aligned}
$$

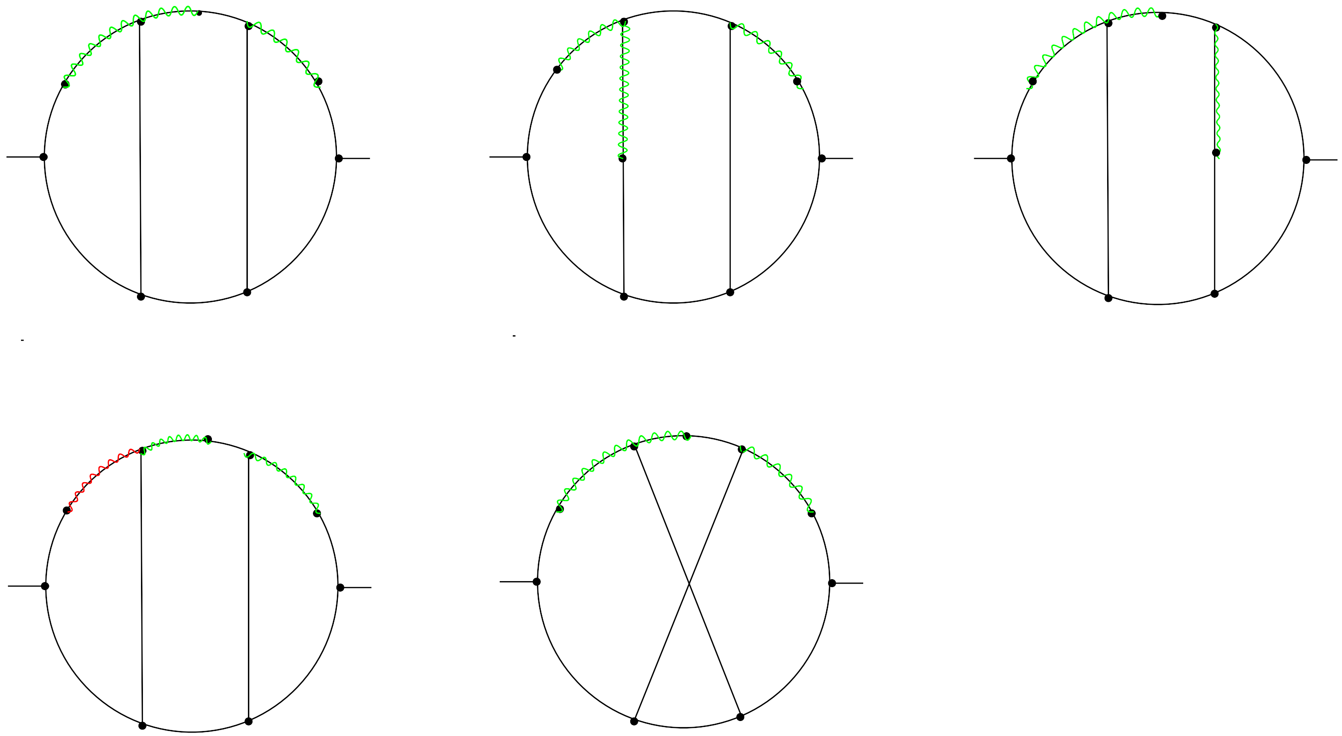
$$
\begin{aligned}
&J_{11}^{b}(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0;n_9,n_{10},n_{12},n_{14},n_{15},n_{16};s_9,s_{10},s_{12},s_{14},s_{15},s_{16})\\
&\quad=\int\!\!\int\!\!\int\frac{(-p_0^2)^{-n_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{n_1}(-p_2^2)^{n_2}(-p_3^2)^{n_3}(-p_4^2)^{n_4}(-p_5^2)^{n_5}(-p_6^2)^{n_6}(-p_7^2)^{n_7}(-p_8^2)^{n_8}}\\
&\qquad\times\frac{1}{(-v\cdot p_1-s_9 i0)^{n_9}(-v\cdot p_2-s_{10}i0)^{n_{10}}(-v\cdot p_4-s_{12}i0)^{n_{12}}(-v\cdot p_6-s_{14}i0)^{n_{14}}(-v\cdot p_7-s_{15}i0)^{n_{15}}(-v\cdot p_8-s_{16}i0)^{n_{16}}}\\
&\qquad p_0=l-q,p_1=k-q,p_2=r-q,p_3=r,p_4=l,p_5=k,p_6=k-r,p_7=l-r,p_8=k-l
\end{aligned}
$$

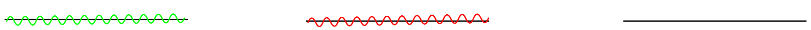
$$
\begin{aligned}
&J_{11}^{c}(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0;n_9,n_{10},n_{11},n_{15},n_{16};s_9,s_{10},s_{11},s_{15},s_{16})\\
&\quad=\int\!\!\int\!\!\int\frac{(-p_0^2)^{-n_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{n_1}(-p_2^2)^{n_2}(-p_3^2)^{n_3}(-p_4^2)^{n_4}(-p_5^2)^{n_5}(-p_6^2)^{n_6}(-p_7^2)^{n_7}(-p_8^2)^{n_8}}\\
&\qquad\times\frac{1}{(-v\cdot p_1-s_9 i0)^{n_9}(-v\cdot p_2-s_{10}i0)^{n_{10}}(-v\cdot p_3-s_{11}i0)^{n_{11}}(-v\cdot p_7-s_{15}i0)^{n_{15}}(-v\cdot p_8-s_{16}i0)^{n_{16}}}\\
&\qquad p_0=k-r,p_1=k,p_2=l,p_3=r,p_4=q+r,p_5=q+l,p_6=q+k,p_7=k-l,p_8=l-r
\end{aligned}
$$

Here $s_i = 1$ or -1. The casual -$i0$ is omitted in all the propagators with quadratic momentum dependence, i.e. we have $-p_1^2 - i0$, etc.

Various classes of Feynman integrals with 12 indices:





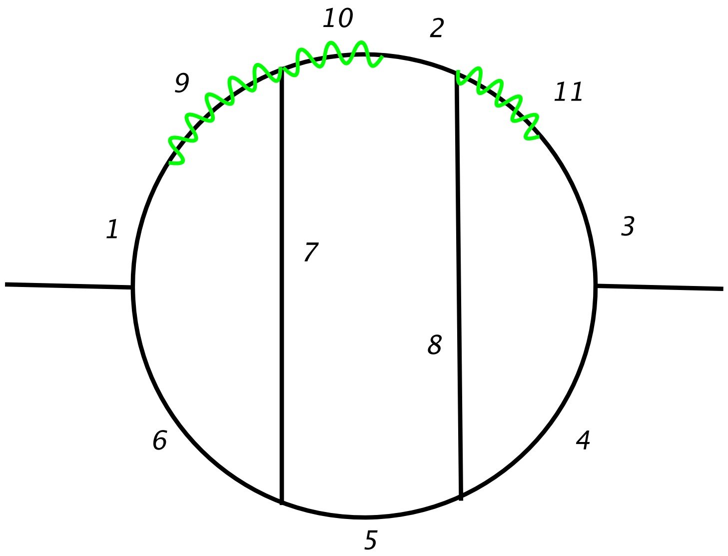


$$\frac{1}{(-v\cdot k - i0)^a} \qquad \frac{1}{(-v\cdot k + i0)^a} \qquad \frac{1}{(-k^2 - i0)^a}$$

And 15 types of two-loop Feynman integrals with 7 indices, where one index can be shifted: $a_i \to a_i + \epsilon$ or $a_i \to a_i + 2\epsilon$

For n_l^1 contribution: ~ 70000 integrals of different types in the general ξ-gauge

For example,



with the numerator chosen as $(-(k-r)^2)^{-a_{12}}$

IBP [K.G. Chetyrkin & F.V. Tkachov'81]

The whole problem of the evaluation of a given family of Feynman integrals $\rightarrow$

- constructing a reduction procedure using IBP
- evaluating master integrals

Solving reduction problems algorithmically:

- ‘Laporta’s algorithm’

  [S. Laporta and E. Remiddi’96; S. Laporta’00; T. Gehrmann and E. Remiddi’01]

  One public version AIR [C. Anastasiou and A. Lazopoulos’04]

  Private versions

  [T. Gehrmann and E. Remiddi, M. Czakon, Y. Schröder, C. Sturm, P. Marquard and D. Seidel, . . .]

- Baikov’s method

- Gröbner bases. Suggested by O.V. Tarasov [O.V. Tarasov’98]

An alternative approach:

[A.V. Smirnov & V.A. Smirnov’05–07;

A.G. Grozin, A.V. Smirnov and V.A. Smirnov’06

A.V. Smirnov, V.A. Smirnov, and M. Steinhauser’08]

FIRE = Feynman Integrals REduction [A.V. Smirnov]
(implemented in `Mathematica`)

Sectors

2^n regions labelled by subsets $\nu \subseteq \{1, \ldots, n\}$:

$\sigma_\nu = \{(a_1, \ldots, a_n) : a_i > 0$ if $i \in \nu, \; a_i \leq 0$ if $i \notin \nu\}$

Natural ordering.

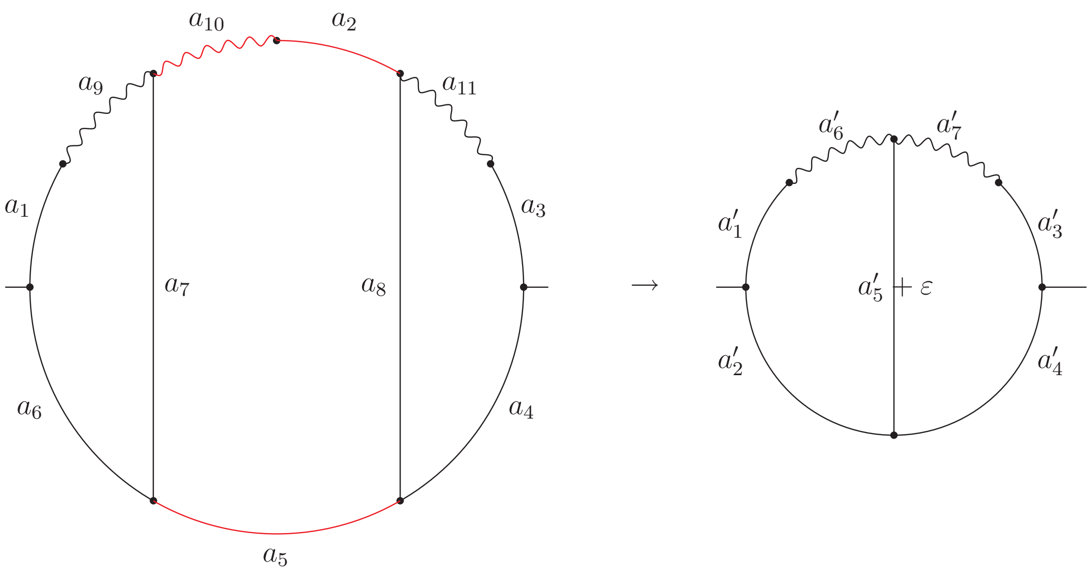
The goal of reduction: to make more non-positive indices.

Three different strategies in FIRE.

1. In sectors with a small number of non-positive indices, apply s-bases (generalizations of Gröbner bases). Constructing them automatically by a kind of Buchberger algorithm.

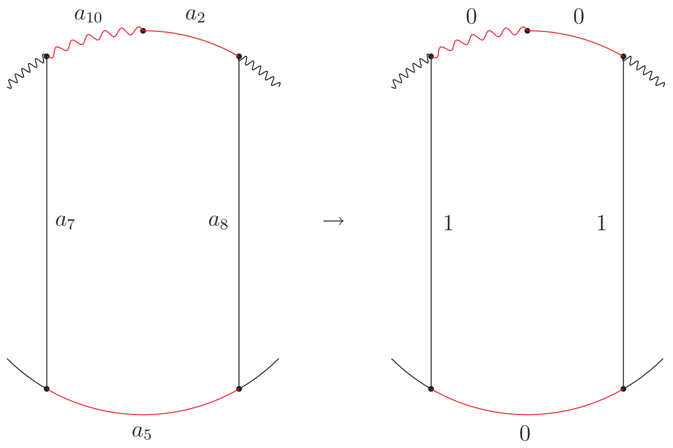
2. In sectors with a large number of non-positive indices, integrate over a loop momentum explicitly and reduce the problem to a family of two-loop integrals where the index of one propagator is, possibly, shifted by ϵ or 2ϵ.

Consider the region $a_2, a_5, a_{10} \leq 0 \, , \quad a_7, a_8 > 0$



Apply s-bases (within FIRE) to such 2loop reduction problems with 7 indices.

Reduce indices $a_2, a_5, a_{10}, a_7, a_8$ to their boundary values, i.e. $a_2, a_5, a_{10} = 0, \;\; a_7, a_8 = 1$



At these values, the transition to the 2loop problem because very simple (without multiple summations).

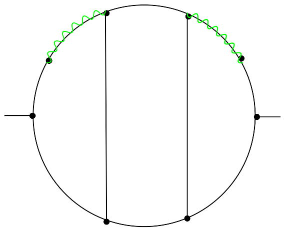
3. In ‘intermediate sectors’, the Laporta’s algorithm (implemented within FIRE) is applied.

Evaluating master integrals by MB [V.A. Smirnov'99, J.B Tausk'99]

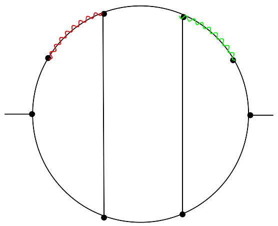
We could not use `AMBRE` [J. Gluza, K. Kajda & T. Riemann'07]
and used MB.m [M. Czakon'05]

~ 100 master integrals in the whole problem

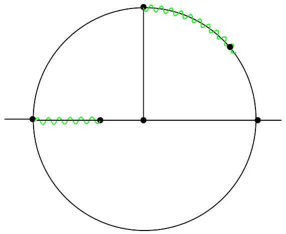
For example,



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2}\left[\frac{56\pi^4}{135\epsilon}+\frac{112\pi^4}{135}+\frac{16\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}+O(\epsilon)\right]$$



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2}\left[-\frac{64\pi^4}{135\epsilon}-\frac{128\pi^4}{135}-\frac{32\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}+O(\epsilon)\right]$$



$$\frac{(i\pi^{d/2})^3}{q^2v^2}\left[\frac{32\pi^4}{135\epsilon}-\frac{128\pi^4}{135}+\frac{88\pi^2\zeta(3)}{9}+\frac{188\zeta(5)}{3}+O(\epsilon)\right]$$

Transcendentality level 5 (and, sometimes, 6) is needed.
The constants of level 5 that we encounter:

$$\zeta(5),\ \zeta(4)\ln 2,\ \zeta(2)\zeta(3),\ \zeta(3)\ln^2 2,\ \zeta(2)\ln^3 2,$$
$$\ln^5 2,\ \mathrm{Li}_5\left(\tfrac{1}{2}\right),\ \mathrm{Li}_4\left(\tfrac{1}{2}\right)\ln 2$$

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)}$$

$$a_3^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3 \,,$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416\zeta(3)}{3}\right) C_F T_F^2 \,.$$

$$a_3^{(1)}, a_3^{(0)} = ?$$

The problem is complicated (although it is only a three-loop problem) because there are two different types of lines,

The status: all the integrals contributing to n_l^1 part were reduced by FIRE.
Operative memory: several Gb.
Creating tables of results (which are not big)

Almost all the master integrals were calculated. (Several constants in higher ϵ terms are missing.)

Various checks:

ξ-independence,
the finiteness of the result after the renormalization

Numerical checks by sector decompositions.

[T. Binoth & G. Heinrich'00,04; C. Bogner & S. Weinzierl'07]

One more version of SD:
A. Smirnov and M. Tentyukov, to be published

It can be applied also in various cases where negative terms in the function F are present, e.g. $\alpha_1^2 - 2\alpha_1\alpha_2 + \alpha_2^2$.

All the results for the master integrals checked by this SD.

Priorities: to complete the n_l part analytically,
then the n_l^0 part.