Subtraction method of computing QCD jet cross sections at NNLO accuracy

Zoltán Trócsányi

University of Debrecen
and
Institute of Nuclear Research

in collaboration with
G. Somogyi

April 21, 2008

Steps of setting up a subtraction scheme (Outline)

1. **IR structure**
   - Universal structure of explicit ϵ poles in loop corrections
   - Universal structure of kinematical singularities in real radiation
2. **Disentangle singularities**
   - traditional approach
   - ...among doubly-unresolved limits
   - Pure-soft factorization in Coulomb gauge
   - between doubly- and singly-unresolved limits
3. **Phase-space factorization**
   - Collinear mapping
   - Soft mapping
4. **Numerical integrations**
5. **Integration of singular factors**
6. **Cancel ϵ poles**
7. **Summary**
8. **Extra slides**

Traditional steps, but all revisited

Identify the structure of infrared singularities (⇒ factorization formulae): well-known

- ϵ poles of one-loop amplitudes

$$|\mathcal{M}_m^{(1)}(\{p\})\rangle = -\frac{1}{2}\boldsymbol{I}^{(0)}(\epsilon;\{p\})|\mathcal{M}_m^{(0)}(\{p\})\rangle + \mathrm{O}(\varepsilon^0)$$

$$\boldsymbol{I}(\epsilon) = \frac{\alpha_s}{2\pi}\sum_i \left[\frac{1}{\epsilon}\gamma_i - \frac{1}{\epsilon^2}\sum_{k\neq i}\boldsymbol{T}_i\cdot\boldsymbol{T}_k\left(\frac{4\pi\mu^2}{s_{ik}}\right)^{\epsilon}\right]$$

Z. Kunszt, Z. T. 1994, S. Catani, M. H. Seymour 1996
S. Catani, S. Dittmaier, Z. T. 2000

- ϵ poles of two-loop amplitudes

$$|\mathcal{M}_m^{(2)}(\{p\})\rangle = -\frac{1}{2}\left(\boldsymbol{I}^{(0)}(\epsilon;\{p\})|\mathcal{M}_m^{(1)}(\{p\})\rangle + \boldsymbol{I}^{(1)}(\epsilon;\{p\})|\mathcal{M}_m^{(0)}(\{p\})\rangle\right) + \mathrm{O}(\varepsilon^0)$$

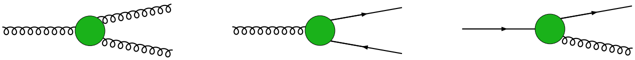
S. Catani 1998, G. Sterman, M. E. Tejeda-Yeomans 2003
S. Moch, M. Mitov 2007

Identify the structure of infrared singularities (⇒ factorization formulae): well-known

- universal soft and collinear factorization of QCD (squared) matrix elements at NLO:
  - $\mathbf{C}_{ir}$ is a symbolic operator that takes the collinear limit

$$\mathbf{C}_{ir}|\mathcal{M}_{m+1}^{(0)}(p_i,p_r,\ldots)|^2 \propto \frac{1}{s_{ir}}\langle\mathcal{M}_m^{(0)}(p_{ir},\ldots)|\hat{P}_{ir}^{(0)}|\mathcal{M}_m^{(0)}(p_{ir},\ldots)\rangle$$

$\hat{P}_{ir}^{(0)}$:

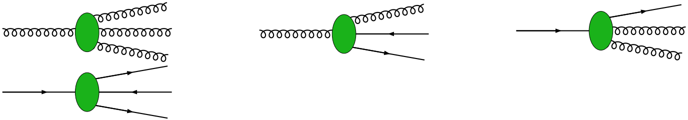


  - $\mathbf{S}_r$ is a symbolic operator that takes the soft limit

$$\mathbf{S}_r|\mathcal{M}_{m+1}^{(0)}(p_r,\ldots)|^2 \propto \sum_{\substack{i,k\\ i\neq k}} \frac{s_{ik}}{s_{ir}s_{kr}}\langle\mathcal{M}_m^{(0)}(\ldots)|\boldsymbol{T}_i\boldsymbol{T}_k|\mathcal{M}_m^{(0)}(\ldots)\rangle$$

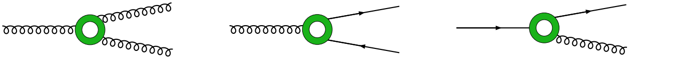
Identify the structure of infrared singularities ($\Rightarrow$ factorization formulae): well-known

- universal soft- and collinear factorization of QCD (squared) matrix elements at NNLO involves the
  - tree level 3-parton splitting functions and double soft gg and $q\bar{q}$ currents



J. M. Campbell, E. W. N. Glover 1997, S. Catani, M. Grazzini 1998
V. Del Duca, A. Frizzo, F. Maltoni, 1999, D. Kosower, 2002

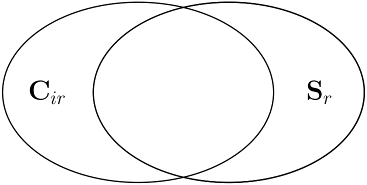
  - one-loop 2-parton splitting functions and soft gluon current



Z. Bern, L. J. Dixon, D. C. Dunbar, D. A. Kosower 1994
Z. Bern, V. Del Duca, W. B. Kilgore, C. R. Schmidt 1998-9
D. A. Kosower, P. Uwer 1999, S. Catani, M. Grazzini 2000
D. A. Kosower 2003

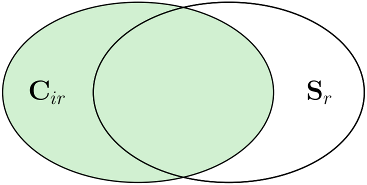
Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

- Simple at NLO:



Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

- Simple at NLO:

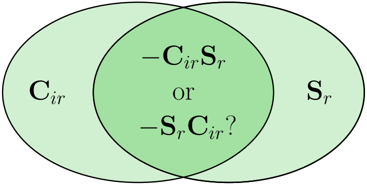


- The candidate subtraction term...

$$\mathbf{A}_1 |\mathcal{M}_{m+1}^{(0)}|^2 \stackrel{?}{=} \sum_r \left[\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} \right] |\mathcal{M}_{m+1}^{(0)}|^2$$

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

- Simple at NLO:



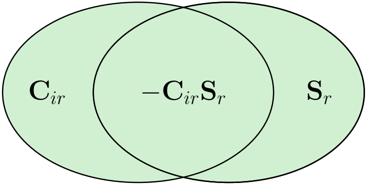
- The candidate subtraction term...

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+1}|^2 \stackrel{?}{=} \sum_r \left[\sum_{i\neq r} \frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r \right] |\mathcal{M}^{(0)}_{m+1}|^2$$

  - ...has the correct singularity structure but performs double subtraction in the regions of phase space where the limits overlap

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

- Simple at NLO:



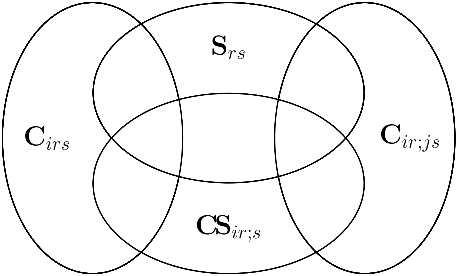
- The candidate subtraction term...

$$\mathbf{A}_1 |\mathcal{M}_{m+1}^{(0)}|^2 = \sum_r \left[\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} + \left(\mathbf{S}_r - \sum_{i \neq r} \mathbf{C}_{ir}\mathbf{S}_r \right) \right] |\mathcal{M}_{m+1}^{(0)}|^2$$

  - ...is now free of double subtractions
  - ...but only defined in the strict collinear and/or soft limits

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

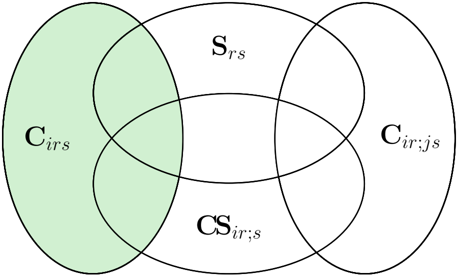
- Becomes rather involved at NNLO:



- possible (G. Somogyi, V. Del Duca, Z. T. 2006)

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

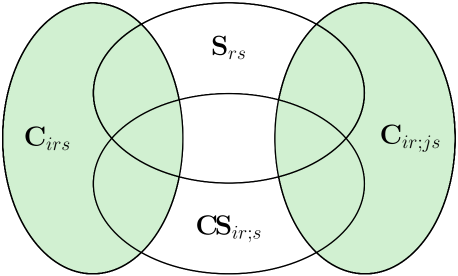
- Becomes rather involved at NNLO:



- possible (G. Somogyi, V. Del Duca, Z. T. 2006)

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

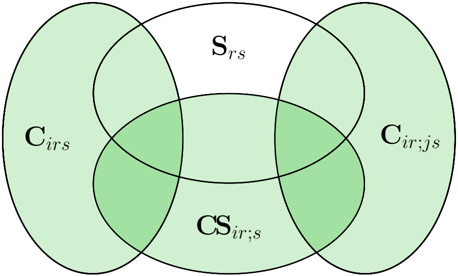
- Becomes rather involved at NNLO:



- possible (G. Somogyi, V. Del Duca, Z. T. 2006)

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

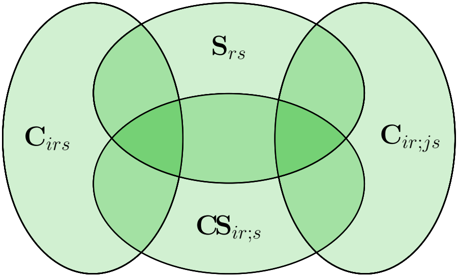
- Becomes rather involved at NNLO:



- possible (G. Somogyi, V. Del Duca, Z. T. 2006)

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

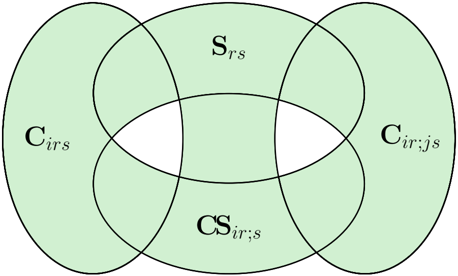
- Becomes rather involved at NNLO:



- possible (G. Somogyi, V. Del Duca, Z. T. 2006)

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

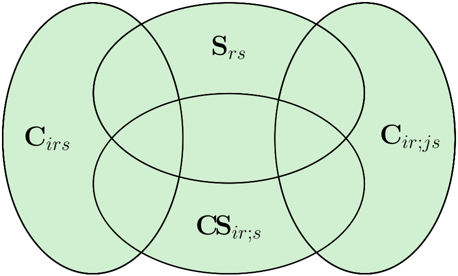
- Becomes rather involved at NNLO:



- possible (G. Somogyi, V. Del Duca, Z. T. 2006)

Disentangle overlapping singularities in real radiation (to avoid multiple subtraction)

- Becomes rather involved at NNLO:



- possible (G. Somogyi, V. Del Duca, Z. T. 2006)

but try to do better!

Pure soft limit of the squared matrix element

- Using the soft insertion rules one obtains

$$
\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r,\dots)|^2 \propto \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r)\frac{2p_i^\mu p_k^\nu}{s_{ir}s_{kr}}\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle
$$

$$
d^{\mu\nu}(p_r,n) = \sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r) = -g^{\mu\nu} + \frac{p_r^\mu n^\nu + p_r^\nu n^\mu}{p_r\cdot n}\,.
$$

Pure soft limit of the squared matrix element

- Using the soft insertion rules one obtains

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r,\dots)|^2 \propto \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r)\frac{2p_i^\mu p_k^\nu}{s_{ir}s_{kr}}\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle$$

$$d^{\mu\nu}(p_r,n) = \sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r) = -g^{\mu\nu} + \frac{p_r^\mu n^\nu + p_r^\nu n^\mu}{p_r\cdot n}\,.$$

- Soft-collinear contributions are given by the colour-diagonal terms,

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r,\dots)|^2 \propto \sum_{i=1}^{m}\left[\frac{1}{2}\sum_{k\neq i}^{m}\left(\frac{s_{ik}}{s_{ir}s_{rk}} - \frac{2s_{in}}{s_{rn}s_{ir}} - \frac{2s_{kn}}{s_{rn}s_{kr}}\right)\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle - \boldsymbol{T}_i^2\frac{2}{s_{ir}}\frac{s_{in}}{s_{rn}}|\mathcal{M}^{(0)}_m(\dots)|^2\right]$$

$$s_{in} = 2p_i\cdot n$$

Pure soft limit of the squared matrix element

- Using the soft insertion rules one obtains

$$\mathbf{S}_r|\mathcal{M}_{m+1}^{(0)}(p_r,\dots)|^2 \propto \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{\text{hel.}} \varepsilon_\mu(p_r)\varepsilon_\nu^*(p_r)\frac{2p_i^\mu p_k^\nu}{s_{ir}s_{kr}}\langle\mathcal{M}_m^{(0)}(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}_m^{(0)}(\dots)\rangle$$

$$d^{\mu\nu}(p_r,n) = \sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon_\nu^*(p_r) = -g^{\mu\nu} + \frac{p_r^\mu n^\nu + p_r^\nu n^\mu}{p_r\cdot n}\,.$$

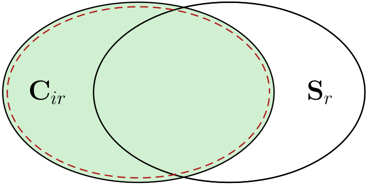
- choose Coulomb gauge and keep the pure soft only

$$\mathbf{S}_r|\mathcal{M}_{m+1}^{(0)}(p_r,\dots)|^2 \longrightarrow \sum_{i=1}^{m}\left[\frac{1}{2}\sum_{k\neq i}\left(\frac{s_{ik}}{s_{ir}s_{rk}} - \frac{2s_{iQ}}{s_{rQ}s_{ir}} - \frac{2s_{kQ}}{s_{rQ}s_{kr}}\right)\langle\mathcal{M}_m^{(0)}(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}_m^{(0)}(\dots)\rangle\right]$$

$$\text{with } n^\mu = Q^\mu - p_r^\mu\, Q^2/s_{rQ} \text{ and colour} - \text{conservation}$$

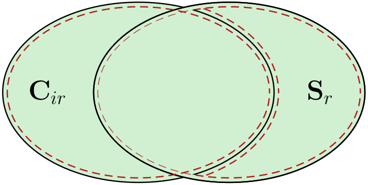
Collinear and soft limits are automatically disjoint

- at NLO:



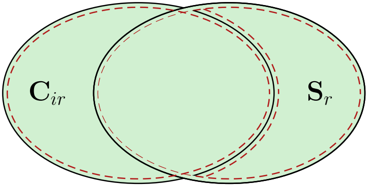
Collinear and soft limits are automatically disjoint

- at NLO:

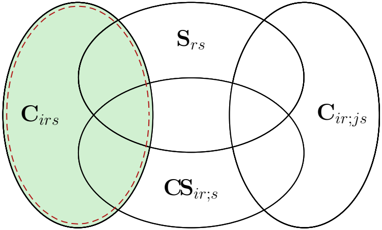


Collinear and soft limits are automatically disjoint

- at NLO:

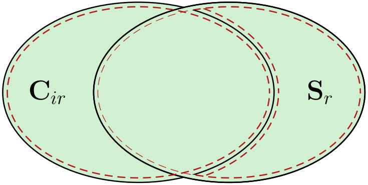


- and works at any order in PT, e.g. at NNLO:

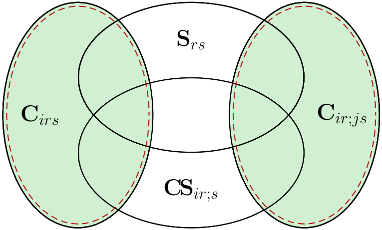


Collinear and soft limits are automatically disjoint

- at NLO:

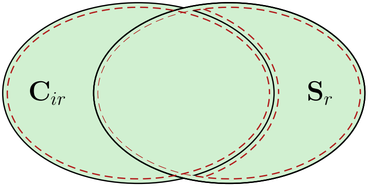


- and works at any order in PT, e.g. at NNLO:

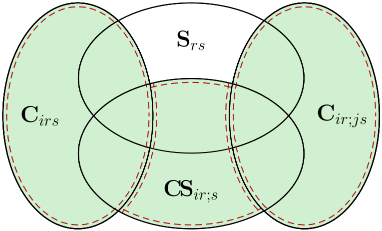


Collinear and soft limits are automatically disjoint

- at NLO:

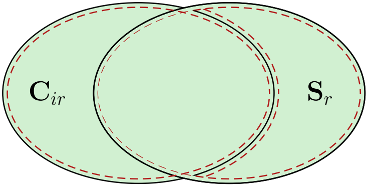


- and works at any order in PT, e.g. at NNLO:

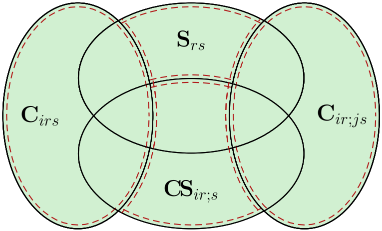


Collinear and soft limits are automatically disjoint

- at NLO:



- and works at any order in PT, e.g. at NNLO:

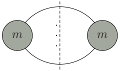


The perturbative expansion at NNLO accuracy

$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}} + \dots$$

The perturbative expansion at NNLO accuracy

$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}} + \dots$$

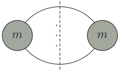
- Consider $e^+e^- \to m$ jet production
  - LO 

$$\sigma^{\mathrm{LO}} = \int_m \mathrm{d}\sigma_m^{\mathrm{B}} = \int \mathrm{d}\phi_m |\mathcal{M}_m^{(0)}|^2 J_m$$

The perturbative expansion at NNLO accuracy

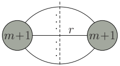
$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}} + \ldots$$

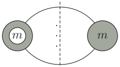
- Consider $e^+e^- \to m$ jet production
  - LO



$$\sigma^{\mathrm{LO}} = \int_m \mathrm{d}\sigma_m^{\mathrm{B}} = \int \mathrm{d}\phi_m |\mathcal{M}_m^{(0)}|^2 J_m$$

  - NLO



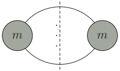


$$\sigma^{\mathrm{NLO}} = \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{R}} + \int_m \mathrm{d}\sigma_m^{\mathrm{V}}$$
$$= \int \mathrm{d}\phi_{m+1} |\mathcal{M}_{m+1}^{(0)}|^2 J_{m+1} + \int \mathrm{d}\phi_m 2\mathrm{Re}\langle \mathcal{M}_m^{(1)} | \mathcal{M}_m^{(0)} \rangle J_m$$

The perturbative expansion at NNLO accuracy

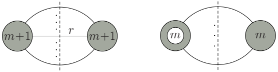
$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}} + \dots$$

- Consider $e^+e^- \to m$ jet production
  - LO



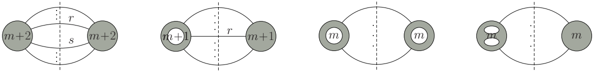
$$\sigma^{\mathrm{LO}} = \int_m \mathrm{d}\sigma^{\mathrm{B}}_m = \int \mathrm{d}\phi_m |\mathcal{M}^{(0)}_m|^2 J_m$$

  - NLO



$$\sigma^{\mathrm{NLO}} = \int_{m+1} \mathrm{d}\sigma^{\mathrm{R}}_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{V}}_m$$
$$= \int \mathrm{d}\phi_{m+1} |\mathcal{M}^{(0)}_{m+1}|^2 J_{m+1} + \int \mathrm{d}\phi_m 2\mathrm{Re}\langle \mathcal{M}^{(1)}_m | \mathcal{M}^{(0)}_m \rangle J_m$$

  - NNLO



$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{VV}}_m =$$

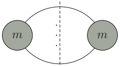
$$= \int \mathrm{d}\phi_{m+2} |\mathcal{M}^{(0)}_{m+2}|^2 J_{m+2} + \int \mathrm{d}\phi_{m+1} 2\mathrm{Re}\langle \mathcal{M}^{(1)}_{m+1} | \mathcal{M}^{(0)}_{m+1} \rangle J_{m+1} +$$

$$+ \int \mathrm{d}\phi_m \left[|\mathcal{M}^{(1)}_m|^2 + 2\mathrm{Re}\langle \mathcal{M}^{(2)}_m | \mathcal{M}^{(0)}_m \rangle \right] J_m$$

The perturbative expansion at NNLO accuracy

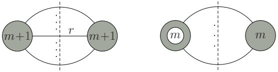
$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}} + \dots$$

- Consider $e^+e^- \to m$ jet production
  - LO



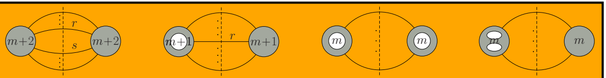
$$\sigma^{\mathrm{LO}} = \int_m \mathrm{d}\sigma^{\mathrm{B}}_m = \int \mathrm{d}\phi_m |\mathcal{M}^{(0)}_m|^2 J_m$$

  - NLO



$$\sigma^{\mathrm{NLO}} = \int_{m+1} \mathrm{d}\sigma^{\mathrm{R}}_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{V}}_m$$
$$= \int \mathrm{d}\phi_{m+1} |\mathcal{M}^{(0)}_{m+1}|^2 J_{m+1} + \int \mathrm{d}\phi_m 2\mathrm{Re}\langle \mathcal{M}^{(1)}_m | \mathcal{M}^{(0)}_m \rangle J_m$$

  - NNLO



$$\sigma^{\mathrm{NNLO}} = \int \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} + \int \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int \mathrm{d}\sigma^{\mathrm{VV}}_m =$$

$$= \int \mathrm{d}\phi_{m+2} |\mathcal{M}^{(0)}_{m+2}|^2 J_{m+2} + \int \mathrm{d}\phi_{m+1} 2\mathrm{Re}\langle \mathcal{M}^{(1)}_{m+1} | \mathcal{M}^{(0)}_{m+1} \rangle J_{m+1} +$$

$$+ \int \mathrm{d}\phi_m \left[|\mathcal{M}^{(1)}_m|^2 + 2\mathrm{Re}\langle \mathcal{M}^{(2)}_m | \mathcal{M}^{(0)}_m \rangle \right] J_m$$

Structure of subtraction is governed by the jet function

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left({\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left({\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right) J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$+ \int_{m} \left\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \left({\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right) + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

Structure of subtraction is governed by the jet function

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2} \left\{ d\sigma^{\rm RR}_{m+2} J_{m+2} - d\sigma^{\rm RR,A_2}_{m+2} J_m - \left(d\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - d\sigma^{\rm RR,A_{12}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left(d\sigma^{\rm RV}_{m+1} + \int_1 d\sigma^{\rm RR,A_1}_{m+2} \right) J_{m+1} - \left[d\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 d\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$+ \int_{m} \left\{ d\sigma^{\rm VV}_{m} + \int_2 \left(d\sigma^{\rm RR,A_2}_{m+2} - d\sigma^{\rm RR,A_{12}}_{m+2} \right) + \int_1 \left[d\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 d\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

- The approximate cross section $d\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of $d\sigma^{\rm RR}_{m+2}$

Structure of subtraction is governed by the jet function

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m \boxed{- \left({\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} \right.} \left. - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left({\rm d}\sigma^{\rm RV}_{m+1} \boxed{+ \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right) J_{m+1}} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$+ \int_m \left\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \left({\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right) + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

- The approximate cross section ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
- The approximate cross section ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$

Structure of subtraction is governed by the jet function

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left({\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left({\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right) J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$+ \int_m \left\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \left({\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right) + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

- The approximate cross section ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
- The approximate cross section ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
- The approximate cross section ${\rm d}\sigma^{\rm RR,A_{12}}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ **and** the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$

Structure of subtraction is governed by the jet function

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left({\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left({\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right) J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$+ \int_{m} \left\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \left({\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right) + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

- The approximate cross section ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
- The approximate cross section ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
- The approximate cross section ${\rm d}\sigma^{\rm RR,A_{12}}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ **and** the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$
- The approximate cross sections ${\rm d}\sigma^{\rm RV,A_1}_{m+1}$ and $\left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1}$ regularize the singly-unresolved limits of ${\rm d}\sigma^{\rm RV}_{m+1}$ and $\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}$

Phase-space factorization

- Approximate cross sections are constructed from the factorization formulae extended over the whole phase space

Phase-space factorization

- Approximate cross sections are constructed from the factorization formulae extended over the whole phase space
- This extension requires momentum mappings, $\{p\}_{n+s} \to \{\tilde{p}\}_n$ that
  - implement exact momentum conservation
  - lead to exact phase-space factorization
  - can be generalized to any number s of unresolved partons

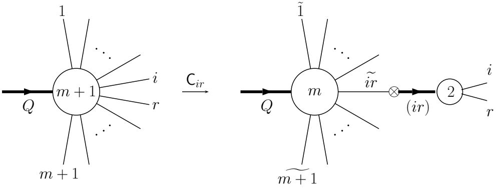
Phase-space factorization

- Approximate cross sections are constructed from the factorization formulae extended over the whole phase space
- This extension requires momentum mappings, $\{p\}_{n+s} \rightarrow \{\tilde{p}\}_n$ that
  - implement exact momentum conservation
  - lead to exact phase-space factorization
  - can be generalized to any number s of unresolved partons
- We use separate phase space mappings for the collinear and soft limits

Collinear mapping (for $s = 1$)

$$\tilde{p}_{ir}^{\mu} = \frac{1}{1-\alpha_{ir}}(p_i^{\mu} + p_r^{\mu} - \alpha_{ir} Q^{\mu}) \,, \qquad \tilde{p}_n^{\mu} = \frac{1}{1-\alpha_{ir}} p_n^{\mu} \,, \quad n \neq i, r$$

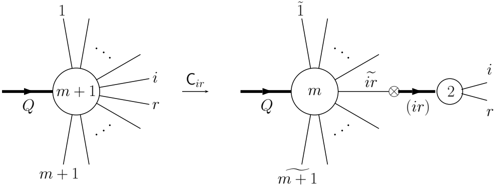
$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y_{(ir)Q}^2 - 4 y_{ir}}\right]$$



Collinear mapping (for $s = 1$)

$$\tilde{p}^{\mu}_{ir} = \frac{1}{1-\alpha_{ir}}(p^{\mu}_i + p^{\mu}_r - \alpha_{ir} Q^{\mu}) \, , \qquad \tilde{p}^{\mu}_n = \frac{1}{1-\alpha_{ir}} p^{\mu}_n \, , \quad n \neq i, r$$

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y^2_{(ir)Q} - 4 y_{ir}} \right]$$

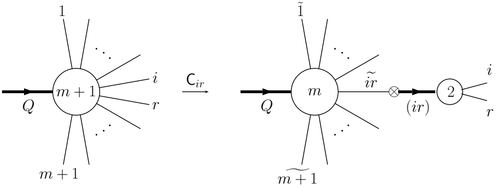


- momentum is conserved $\tilde{p}^{\mu}_{ir} + \sum_n \tilde{p}^{\mu}_n = p^{\mu}_i + p^{\mu}_r + \sum_n p^{\mu}_n$
- phase-space factorization is exact
  $\mathrm{d}\phi_{m+1}(p_1, \ldots; Q) = \mathrm{d}\phi_m(\tilde{p}_1, \ldots; Q) \otimes \mathrm{d}\phi_2(p_i, p_r; p_{(ir)})$

Collinear mapping (for $s = 1$)

$$\tilde{p}^{\mu}_{ir} = \frac{1}{1-\alpha_{ir}}(p^{\mu}_{i} + p^{\mu}_{r} - \alpha_{ir}Q^{\mu})\,, \qquad \tilde{p}^{\mu}_{n} = \frac{1}{1-\alpha_{ir}}p^{\mu}_{n}\,, \quad n \neq i, r$$

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y^{2}_{(ir)Q} - 4y_{ir}}\right]$$

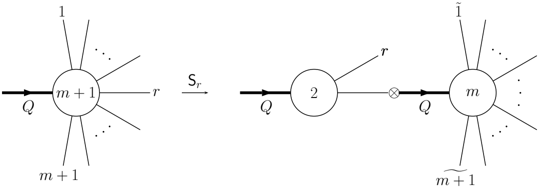


- momentum is conserved $\tilde{p}^{\mu}_{ir} + \sum_n \tilde{p}^{\mu}_{n} = p^{\mu}_{i} + p^{\mu}_{r} + \sum_n p^{\mu}_{n}$
- phase-space factorization is exact
  $\mathrm{d}\phi_{m+1}(p_1, \ldots ; Q) = \mathrm{d}\phi_m(\tilde{p}_1, \ldots ; Q) \otimes \mathrm{d}\phi_2(p_i, p_r; p_{(ir)})$
- integral over convolution can be constrained to improve numerical efficiency

Soft mapping (for $s = 1$)

$$\tilde{p}_n^\mu = \Lambda^\mu_\nu[Q, (Q - p_r)/\lambda_r](p_n^\nu/\lambda_r)\,, \quad n \neq r\,, \qquad \lambda_r = \sqrt{1 - y_{rQ}}\,,$$

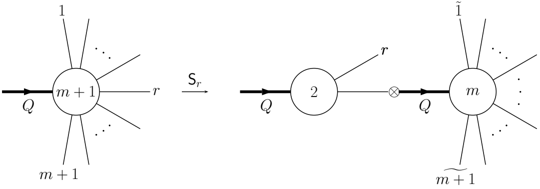
$$\Lambda^\mu_\nu[K, \widetilde{K}] = g^\mu_\nu - \frac{2(K+\widetilde{K})^\mu (K+\widetilde{K})_\nu}{(K+\widetilde{K})^2} + \frac{2K^\mu \widetilde{K}_\nu}{K^2}$$



Soft mapping (for $s = 1$)

$$\tilde{p}_n^\mu = \Lambda^\mu_\nu[Q, (Q - p_r)/\lambda_r](p_n^\nu/\lambda_r)\,, \quad n \neq r\,, \qquad \lambda_r = \sqrt{1 - y_{rQ}}\,,$$

$$\Lambda^\mu_\nu[K, \widetilde{K}] = g^\mu_\nu - \frac{2(K + \widetilde{K})^\mu (K + \widetilde{K})_\nu}{(K + \widetilde{K})^2} + \frac{2K^\mu \widetilde{K}_\nu}{K^2}$$



- momentum is conserved $\sum_n \tilde{p}_n^\mu = p_r^\mu + \sum_n p_n^\mu$
- phase-space factorization is exact
  $\mathrm{d}\phi_{m+1}(p_1, \ldots; Q) = \mathrm{d}\phi_m(\widetilde{p}_1, \ldots; Q) \otimes \mathrm{d}\phi_2(p_r, K; Q)$
- integral over convolution can be constrained to improve numerical efficiency

Compute finite integrals numerically using Monte Carlo integration

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m} =$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right) \right\}$$

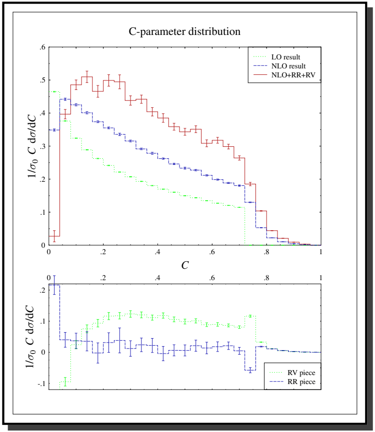
$$+ \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_m \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right) + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

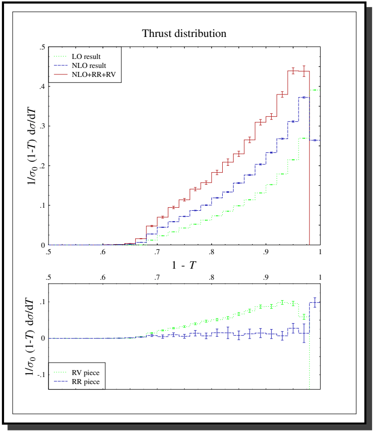
- Each integral on the r.h.s. is finite in $d = 4$ provided J is IR safe
- Subtraction terms are fully local:
  important for numerical stability and reduction of CPU time

Event shape distributions (still unphysical at NNLO, VV missing)

C-parameter

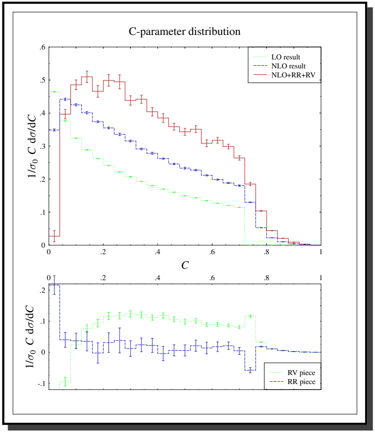


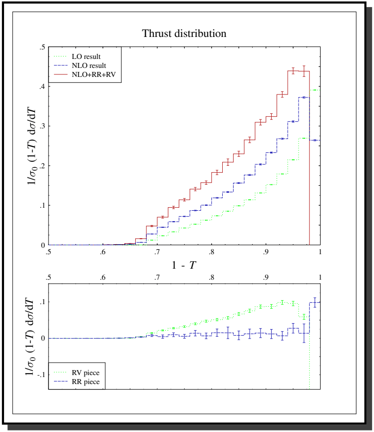
Thrust



Event shape distributions (still unphysical at NNLO, VV missing)

C-parameter | Thrust





Pricetag: one desktop pc in 50 hours

Integrate the singular factors over the factorized unresolved phase-space measures

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m} =$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_m \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \boxed{\int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right) + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right]} \right\} J_m$$

Collinear integrals

$$\int_0^{\alpha_0} \mathrm{d}\alpha\,(1-\alpha)^{2d_0-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\int \mathrm{d}\phi_2(p_i,p_r;p_{(ir)})\frac{1}{s_{ir}^{1+\kappa\epsilon}}P^{(\kappa)}_{f_if_r}(z_i,z_r;\epsilon)\,,\qquad \kappa=0,\ 1$$

$$\mathrm{d}\phi_2(p_i,p_r;p_{(ir)}) = \frac{s_{ir}^{-\epsilon}}{8\pi}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\mathrm{d}s_{ir}\,\mathrm{d}v\,\delta\big(s_{ir}-Q^2\alpha\big(\alpha+(1-\alpha)x\big)\big)$$

$$\times\,[v\,(1-v)]^{-\epsilon}\,\Theta(1-v)\Theta(v)\,,$$

Collinear integrals

$$\int_0^{\alpha_0} \mathrm{d}\alpha\, (1-\alpha)^{2d_0-1} \frac{s_{\widetilde{ir}Q}}{2\pi} \int \mathrm{d}\phi_2(p_i, p_r; p_{(ir)}) \frac{1}{s_{ir}^{1+\kappa\epsilon}} P_{f_i f_r}^{(\kappa)}(z_i, z_r; \epsilon)\,, \qquad \kappa = 0,\ 1$$

$$\frac{z_r^{k+\delta\epsilon}}{s_{ir}^{1+\kappa\epsilon}}\, g_I^{(\pm)}(z_r)\,, \qquad z_r = \frac{\alpha Q^2 + (1-\alpha) v s_{\widetilde{ir}Q}}{2\alpha Q^2 + (1-\alpha) s_{\widetilde{ir}Q}}$$

Collinear integrals

$$\int_0^{\alpha_0} d\alpha\,(1-\alpha)^{2d_0-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\int d\phi_2(p_i,p_r;p_{(ir)})\frac{1}{s_{ir}^{1+\kappa\epsilon}}P^{(\kappa)}_{f_if_r}(z_i,z_r;\epsilon)\,,\qquad \kappa=0,\ 1$$

$$\frac{z_r^{k+\delta\epsilon}}{s_{ir}^{1+\kappa\epsilon}}\,g_I^{(\pm)}(z_r)\,,\qquad z_r=\frac{\alpha Q^2+(1-\alpha)vs_{\widetilde{ir}Q}}{2\alpha Q^2+(1-\alpha)s_{\widetilde{ir}Q}}$$

δ	Function	$g_I^{(\pm)}(z)$
0	g_A	1
∓1	$g_B^{(\pm)}$	$(1-z)^{\pm\epsilon}$
0	$g_C^{(\pm)}$	$(1-z)^{\pm\epsilon}\,{}_2F_1(\pm\epsilon,\pm\epsilon,1\pm\epsilon,z)$
±1	$g_D^{(\pm)}$	${}_2F_1(\pm\epsilon,\pm\epsilon,1\pm\epsilon,1-z)$

$$\mathcal{I}\propto x\int_0^{\alpha_0}d\alpha\,\alpha^{-1-(1+\kappa)\epsilon}\,(1-\alpha)^{2d_0-1}\,[\alpha+(1-\alpha)x]^{-1-(1+\kappa)\epsilon}$$

$$\times\int_0^1 dv[v\,(1-v)]^{-\epsilon}\left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)^{k+\delta\epsilon}g\left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)$$

Soft-type integrals

- Soft integrals

$$\mathcal{J} \propto -\int_0^{y_0} \mathrm{d}y\,(1-y)^{d_0'-1}\,\frac{Q^2}{2\pi}\int \mathrm{d}\phi_2(p_r,K;Q)\left(\frac{s_{ik}}{s_{ir}s_{kr}}\right)^{1+\kappa\epsilon}$$

- Collinear-soft integrals

$$\mathcal{K} \propto \int_0^{y_0} \mathrm{d}y\,(1-y)^{d_0'-1}\,\frac{Q^2}{2\pi}\int \mathrm{d}\phi_2(p_r,K;Q)2\left(\frac{1}{s_{ir}}\frac{z_i}{z_r}\right)^{1+\kappa\epsilon}$$

$$\mathrm{d}\phi_2(p_r,K;Q) = \frac{(Q^2)^{-\epsilon}}{16\pi^2}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\mathrm{d}\varepsilon_r\,\varepsilon_r^{1-2\epsilon}\delta(y-\varepsilon_r)$$
$$\times\mathrm{d}(\cos\vartheta)\,\mathrm{d}(\cos\varphi)(\sin\vartheta)^{-2\epsilon}(\sin\varphi)^{-1-2\epsilon}$$

Soft-type integrals

- Soft integrals

$$\mathcal{J} \propto -\int_0^{y_0} \mathrm{d}y\,(1-y)^{d_0'-1}\frac{Q^2}{2\pi}\int \mathrm{d}\phi_2(p_r,K;Q)\left(\frac{s_{ik}}{s_{ir}s_{kr}}\right)^{1+\kappa\epsilon}$$

- Collinear-soft integrals

$$\mathcal{K} \propto \int_0^{y_0} \mathrm{d}y\,(1-y)^{d_0'-1}\frac{Q^2}{2\pi}\int \mathrm{d}\phi_2(p_r,K;Q)2\left(\frac{1}{s_{ir}}\frac{z_i}{z_r}\right)^{1+\kappa\epsilon}$$

$$\mathrm{d}\phi_2(p_r,K;Q) = \frac{(Q^2)^{-\epsilon}}{16\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\mathrm{d}\varepsilon_r\,\varepsilon_r^{1-2\epsilon}\delta(y-\varepsilon_r)$$
$$\times\mathrm{d}(\cos\vartheta)\,\mathrm{d}(\cos\varphi)(\sin\vartheta)^{-2\epsilon}(\sin\varphi)^{-1-2\epsilon}$$

- and iterated integrals of the above, like $\mathcal{I}{*}\mathcal{I}$, $\mathcal{J}{*}\mathcal{J}$, $\mathcal{J}{*}\mathcal{I}$, $\mathcal{K}{*}\mathcal{I}$ etc.

Three methods of computing the integrals

- Iterated sector decomposition and residuum subtraction to find Laurent expansion, compute expansion coefficients numerically
  G. Somogyi, Z. T. 2008
- Algebraic reduction, reduction to master integrals using IBP and computation of MI using differential equations
  U. Aglietti, V. Del Duca, C. Duhr, G. Somogyi, Z. T. 2008
- Expand MB representation of integrals in ϵ, and compute harmonic sum representation of the expansion coefficients
  P. Bolzoni, S. Moch, G. Somogyi, Z. T. 2008

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}^{(0)}_i(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}^{(0)}_{ik}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} = \mathrm{d}\sigma_{m+1}^{\mathrm{R}} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}_i^{(0)}(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}_{ik}^{(0)}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

$$\mathrm{C}_q^{(0)}(x;\epsilon) = \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{3}{2} - 2\ln x\right) + \mathrm{O}(\varepsilon^0)\,,$$

$$\mathrm{C}_g^{(0)}(x;\epsilon) = \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{6} - \frac{2}{3}n_\mathrm{f}\frac{T_\mathrm{R}}{C_\mathrm{A}} - 2\ln x\right) + \mathrm{O}(\varepsilon^0)\,,$$

$$\widetilde{\mathrm{S}}_{ik}^{(0)}(Y;\epsilon) = \frac{1}{\epsilon}\ln Y + \mathrm{O}(\varepsilon^0)\,.$$

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}^{(0)}_i(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}^{(0)}_{ik}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

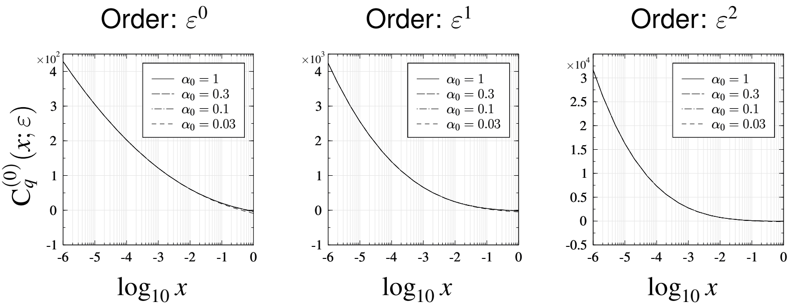


Figure: Expansion coeffecients of the functions $\mathrm{C}^{(0)}_q(x;\varepsilon)$

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}^{(0)}_i(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}^{(0)}_{ik}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

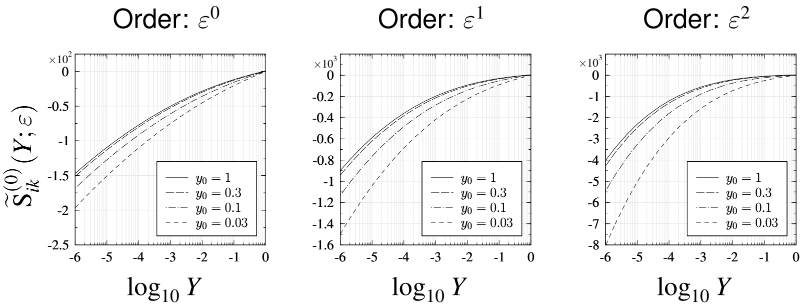


Figure: Expansion coefficients of the function $\widetilde{\mathrm{S}}^{(0)}_{ik}(Y;\varepsilon)$

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}^{(0)}_i(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}^{(0)}_{ik}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} = \mathrm{d}\sigma_{m+1}^{\mathrm{R}} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}_i^{(0)}(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}_{ik}^{(0)}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

$$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{V}} \otimes \boldsymbol{I}^{(0)}(\{p\}_m;\epsilon) + \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \boldsymbol{I}^{(1)}(\{p\}_m;\epsilon)$$

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} = \mathrm{d}\sigma_{m+1}^{\mathrm{R}} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}_i^{(0)}(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}_{ik}^{(0)}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

$$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{V}} \otimes \boldsymbol{I}^{(0)}(\{p\}_m;\epsilon) + \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \boldsymbol{I}^{(1)}(\{p\}_m;\epsilon)$$

$$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \left[\frac{1}{2}\left\{\boldsymbol{I}^{(0)}(\{p\}_m;\epsilon),\boldsymbol{I}^{(0)}(\{p\}_m;\epsilon)\right\} + \boldsymbol{I}^{R\times(0)}(\{p\}_m;\epsilon)\right]$$

Combine integrated subtraction terms with loop corrections to cancel ϵ poles

Write the final result of integrations in the forms:

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}^{(0)}_i(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \widetilde{\mathrm{S}}^{(0)}_{ik}(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

$$\int_1 \mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} = \mathrm{d}\sigma^{\mathrm{V}}_{m} \otimes \boldsymbol{I}^{(0)}(\{p\}_{m};\epsilon) + \mathrm{d}\sigma^{\mathrm{B}}_{m} \otimes \boldsymbol{I}^{(1)}(\{p\}_{m};\epsilon)$$

$$\int_1 \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2}\right)^{\mathrm{A}_1} = \mathrm{d}\sigma^{\mathrm{B}}_{m} \otimes \left[\frac{1}{2}\left\{\boldsymbol{I}^{(0)}(\{p\}_{m};\epsilon), \boldsymbol{I}^{(0)}(\{p\}_{m};\epsilon)\right\} + \boldsymbol{I}^{R\times(0)}(\{p\}_{m};\epsilon)\right]$$

$$\int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2}\right) = \mathrm{d}\sigma^{\mathrm{B}}_{m} \otimes [??]$$

Summary

- Usual steps of subtraction methods followed, but

Summary

- Usual steps of subtraction methods followed, but
- used new, pure-soft factorization to avoid overlapping singularities

Summary

- Usual steps of subtraction methods followed, but
- used new, pure-soft factorization to avoid overlapping singularities
- proposed new phase-space factorizations for defining subtraction terms

Summary

- Usual steps of subtraction methods followed, but
- used new, pure-soft factorization to avoid overlapping singularities
- proposed new phase-space factorizations for defining subtraction terms
- existing techniques (iterated sector decomposition, IBP, MB) can be extended to compute integrals over the unresolved phase spaces

Summary

- Usual steps of subtraction methods followed, but
- used new, pure-soft factorization to avoid overlapping singularities
- proposed new phase-space factorizations for defining subtraction terms
- existing techniques (iterated sector decomposition, IBP, MB) can be extended to compute integrals over the unresolved phase spaces
- Complete scheme for NNLO computations is worked out

Summary

- Usual steps of subtraction methods followed, but
- used new, pure-soft factorization to avoid overlapping singularities
- proposed new phase-space factorizations for defining subtraction terms
- existing techniques (iterated sector decomposition, IBP, MB) can be extended to compute integrals over the unresolved phase spaces
- Complete scheme for NNLO computations is worked out
- Only partially completed: integrations over the unresolved phase spaces

Summary

- Usual steps of subtraction methods followed, but
- used new, pure-soft factorization to avoid overlapping singularities
- proposed new phase-space factorizations for defining subtraction terms
- existing techniques (iterated sector decomposition, IBP, MB) can be extended to compute integrals over the unresolved phase spaces
- Complete scheme for NNLO computations is worked out
- Only partially completed: integrations over the unresolved phase spaces

Thanks for your attention!

The standard integrated approximate cross sections do not obey universal IR collinear factorization

- Due to coherent soft-gluon emission from unresolved partons only the sum $\langle\mathcal{M}_{m+1}^{(0)}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}_{m+1}^{(0)}\rangle$ factorizes in the collinear limit ($\boldsymbol{T}_{jr}=\boldsymbol{T}_j+\boldsymbol{T}_r$)

$$\mathbf{C}_{jr}\langle\mathcal{M}_{m+1}^{(0)}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}_{m+1}^{(0)}\rangle \propto \frac{1}{s_{jr}}\langle\mathcal{M}_{m}^{(0)}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}_{jr}^{(0)}|\mathcal{M}_{m}^{(0)}\rangle$$

The standard integrated approximate cross sections do not obey universal IR collinear factorization

- Due to coherent soft-gluon emission from unresolved partons only the sum $\langle\mathcal{M}^{(0)}_{m+1}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}^{(0)}_{m+1}\rangle$ factorizes in the collinear limit ($\boldsymbol{T}_{jr}=\boldsymbol{T}_j+\boldsymbol{T}_r$)

$$\mathbf{C}_{jr}\langle\mathcal{M}^{(0)}_{m+1}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}^{(0)}_{m+1}\rangle\propto\frac{1}{s_{jr}}\langle\mathcal{M}^{(0)}_{m}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}^{(0)}_{jr}|\mathcal{M}^{(0)}_{m}\rangle$$

- This factorization is violated by the factors $s_{ik}^{-\epsilon}/\epsilon^2$

$$\mathbf{C}_{jr}\frac{1}{\epsilon^2}\langle\mathcal{M}^{(0)}_{m+1}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k\,s_{jk}^{-\epsilon}+\boldsymbol{T}_r\cdot\boldsymbol{T}_k\,s_{rk}^{-\epsilon})|\mathcal{M}^{(0)}_{m+1}\rangle\propto$$
$$\times\frac{1}{s_{jr}}\Bigg[\langle\mathcal{M}^{(0)}_{m}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}^{(0)}_{jr}\left(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln s_{(jr)k}\right)|\mathcal{M}^{(0)}_{m}\rangle$$
$$-\frac{1}{\epsilon}\langle\mathcal{M}^{(0)}_{m}|\boldsymbol{T}_j\cdot\boldsymbol{T}_k\ln z_j+\boldsymbol{T}_r\cdot\boldsymbol{T}_k\ln z_r\,|\mathcal{M}^{(0)}_{m}\rangle\Bigg]$$

The standard integrated approximate cross sections do not obey universal IR collinear factorization

- Due to coherent soft-gluon emission from unresolved partons only the sum $\langle\mathcal{M}_{m+1}^{(0)}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}_{m+1}^{(0)}\rangle$ factorizes in the collinear limit ($\boldsymbol{T}_{jr}=\boldsymbol{T}_j+\boldsymbol{T}_r$)

$$\mathbf{C}_{jr}\langle\mathcal{M}_{m+1}^{(0)}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}_{m+1}^{(0)}\rangle\propto\frac{1}{s_{jr}}\langle\mathcal{M}_m^{(0)}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}_{jr}^{(0)}|\mathcal{M}_m^{(0)}\rangle$$

- This factorization is violated by the factors $s_{ik}^{-\epsilon}/\epsilon^2$

$$\mathbf{C}_{jr}\frac{1}{\epsilon^2}\langle\mathcal{M}_{m+1}^{(0)}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k\,s_{jk}^{-\epsilon}+\boldsymbol{T}_r\cdot\boldsymbol{T}_k\,s_{rk}^{-\epsilon})|\mathcal{M}_{m+1}^{(0)}\rangle\propto$$
$$\times\frac{1}{s_{jr}}\Bigg[\langle\mathcal{M}_m^{(0)}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}_{jr}^{(0)}\left(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln s_{(jr)k}\right)|\mathcal{M}_m^{(0)}\rangle$$
$$-\frac{1}{\epsilon}\langle\mathcal{M}_m^{(0)}|\boldsymbol{T}_j\cdot\boldsymbol{T}_k\ln z_j+\boldsymbol{T}_r\cdot\boldsymbol{T}_k\ln z_r\,|\mathcal{M}_m^{(0)}\rangle\Bigg]$$

- $\Rightarrow$ local subtraction requires properly defined new approximate cross sections

Collinear limit of color-connected SME

- Due to color coherence only the sum

$$|\mathcal{M}^{(0)}_{m+1;(i,l)}|^2 + |\mathcal{M}^{(0)}_{m+1;(r,l)}|^2$$

  has a universal collinear limit as $p_i||p_r$, not $|\mathcal{M}^{(0)}_{m+1;(i,l)}|^2$ or $|\mathcal{M}^{(0)}_{m+1;(r,l)}|^2$ separately
- Generally we have

$$\boldsymbol{I} = \mathcal{V}_{il}|\mathcal{M}^{(0)}_{m+1;(i,l)}(\{\tilde{p}\}^{(il)}_{m+1})|^2 + \mathcal{V}_{rl}|\mathcal{M}^{(0)}_{m+1;(r,l)}(\{\tilde{p}\}^{(rl)}_{m+1})|^2 + \dots$$

- $\mathbf{C}_{ir}\boldsymbol{I}$ exists iff
  - $\mathbf{C}_{ir}\mathcal{V}_{il} = \mathbf{C}_{ir}\mathcal{V}_{rl} \equiv \mathcal{V}_{(ir)l}$
  - $\{\tilde{p}\}^{(il)}_{m+1} \xrightarrow{i||r} \{\tilde{p}\}^{[(ir)l]}_{m}, \quad \{\tilde{p}\}^{(rl)}_{m+1} \xrightarrow{i||r} \{\tilde{p}\}^{[(ir)l]}_{m}$
- Then $\mathbf{C}_{ir}\boldsymbol{I} \propto \frac{1}{s_{ir}}\mathcal{V}_{(ir)l}\langle\mathcal{M}^{(0)}_{m}(\{\tilde{p}\}^{[(ir)l]}_{m})|\boldsymbol{T}_{ir}\boldsymbol{T}_{l}\hat{P}_{ir}|\mathcal{M}^{(0)}_{m}(\{\tilde{p}\}^{[(ir)l]}_{m})\rangle$

Monte Carlo summation over helicity in NLO computations

- proved to be useful to gain speed in multileg computations at Born level

P. Draggiotis, R. H. P. Kleiss, C. G. Papadopoulos 1998, 2002

- Gaining speed is even more important in computing real radiation

Monte Carlo summation over helicity in NLO computations

- proved to be useful to gain speed in multileg computations at Born level

P. Draggiotis, R. H. P. Kleiss, C. G. Papadopoulos 1998, 2002

- Gaining speed is even more important in computing real radiation
- The pure-soft factorization is independent of the soft-gluon helicity

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r^{\lambda},\dots)|^2 \propto \frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{2}\sum_{k\neq i}^{m}\left(\frac{s_{ik}}{s_{ir}s_{rk}}-\frac{2s_{iQ}}{s_{rQ}s_{ir}}-\frac{2s_{kQ}}{s_{rQ}s_{kr}}\right)\langle\mathcal{M}^{(0)}_{m}(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_{m}(\dots)\rangle - \boldsymbol{T}_i^2\frac{2}{s_{ir}}\frac{s_{in}}{s_{rn}}|\mathcal{M}^{(0)}_{m}(\dots)|^2\right]$$

Monte Carlo summation over helicity in NLO computations

- proved to be useful to gain speed in multileg computations at Born level

P. Draggiotis, R. H. P. Kleiss, C. G. Papadopoulos 1998, 2002

- Gaining speed is even more important in computing real radiation
- The pure-soft factorization is independent of the soft-gluon helicity
- Can define collinear and pure-soft subtractions for fixed helicities

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r^\lambda,\dots)|^2 \longrightarrow$$

$$\frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{2}\sum_{k\neq i}^{m}\left(\frac{s_{ik}}{s_{ir}s_{rk}}-\frac{2s_{iQ}}{s_{rQ}s_{ir}}-\frac{2s_{kQ}}{s_{rQ}s_{kr}}\right)\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle\right]$$

Towards event shapes

- Constructed $\mathrm{d}\sigma_5^{\mathrm{NNLO}}$ (RR) and $\mathrm{d}\sigma_4^{\mathrm{NNLO}}$ (RV) for $e^+e^- \to 3$ jets ($e^+e^- \to q\bar{q}ggg$ and $e^+e^- \to q\bar{q}gg$ subprocesses respectively)

Towards event shapes

- Constructed $\mathrm{d}\sigma_5^{\mathrm{NNLO}}$ (RR) and $\mathrm{d}\sigma_4^{\mathrm{NNLO}}$ (RV) for $e^+e^- \to 3$ jets ($e^+e^- \to q\bar{q}ggg$ and $e^+e^- \to q\bar{q}gg$ subprocesses respectively)
- Checked numerically that ($J = C$ or $1 - T$)
  - In all singly- and doubly-unresolved limits

$$\frac{\mathrm{d}\sigma_5^{\mathrm{RR,A_2}} J_3 + \mathrm{d}\sigma_5^{\mathrm{RR,A_1}} J_4 - \mathrm{d}\sigma_5^{\mathrm{RR,A_{12}}} J_3}{\mathrm{d}\sigma_5^{\mathrm{RR}}} \to 1$$

  - In all singly-unresolved limits

$$\frac{\mathrm{d}\sigma_4^{\mathrm{RV,A_1}} J_3 - \int_1 \mathrm{d}\sigma_5^{\mathrm{RR,A_1}} J_4 - \left(\int_1 \mathrm{d}\sigma_5^{\mathrm{RR,A_1}}\right)^{\mathrm{A_1}} J_3}{\mathrm{d}\sigma_4^{\mathrm{RV}}} \to 1$$

$\Rightarrow$ The counterterms are fully local,
azimuthal and color correlations fully included

Towards event shapes

- Computed the RR and RV contributions to first three moments of 3-jet event shape variables thurst (T) and C-parameter

$$\langle O^n \rangle \equiv \int O^n \frac{\mathrm{d}\sigma}{\sigma_0} = \left(\frac{\alpha_s(Q)}{2\pi}\right) A_O^{(n)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2 B_O^{(n)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^3 C_O^{(n)}$$

  where the NNLO contribution $C_O^{(n)}$ is a sum of the RR, RV and VV pieces

$$C_O^{(n)} = C_{O;5}^{(n)} + C_{O;4}^{(n)} + C_{O;3}^{(n)}$$

- The quantities $C_{O;5}^{(n)}$ and $C_{O;4}^{(n)}$ are found to be finite ($O = C$ or $O = \tau \equiv 1 - T$; $n = 1, 2, 3$)
- Up to NLO accuracy perfect agreement with known results for $e^+e^- \to 3$ jets

Towards event shapes – RR contribution

- Prediction for moments of event shapes – RR contribution

n	$C^{(n)}_{\tau;5}$	$C^{(n)}_{C;5}$
1	$-(9.27 \pm 0.34) \cdot 10^1$	$-(3.44 \pm 0.14) \cdot 10^2$
2	-3.07 ± 0.43	$-(1.42 \pm 0.03) \cdot 10^2$
3	2.01 ± 0.12	6.29 ± 1.87

- Technical details
  - No. of MC points used: $n = 40 \times 2.5 \cdot 10^5$ (VEGAS)
  - χ^2/d.o.f. as reported by VEGAS: $\chi^2/\text{d.o.f.} = 0.79$
  - No. of subtractions: 535 at 139 different PS points for each event [compare with 12 subtractions at 12 different PS points for $e^+e^- \to$ 4 jets at NLO needed in this scheme ($q\bar{q}ggg$ subprocess)]
  - Speed of code on an AMD Athlon 1.3 GHz machine with 256 MB RAM: $2.5 \cdot 10^5$ pts. $\approx$ 2.5 h

Towards event shapes – RV contribution

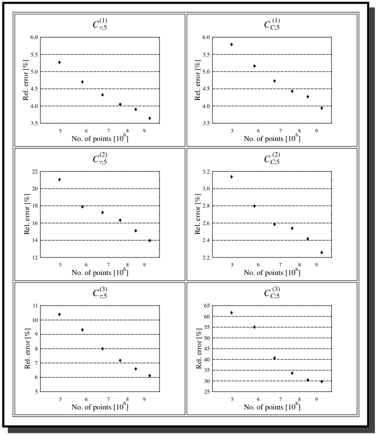
- Prediction for moments of event shapes – RV contribution

n	$C^{(n)}_{\tau;4}$	$C^{(n)}_{C;4}$
1	$(1.23 \pm 0.01) \cdot 10^3$	$(4.33 \pm 0.05) \cdot 10^3$
2	$(2.55 \pm 0.02) \cdot 10^2$	$(3.25 \pm 0.02) \cdot 10^3$
3	$(4.79 \pm 0.03) \cdot 10^1$	$(1.80 \pm 0.01) \cdot 10^3$

- Technical details
  - No. of MC points used: $n = 20 \times 2.5 \cdot 10^5$ (VEGAS)
  - χ^2/d.o.f. as reported by VEGAS: $\chi^2/\text{d.o.f.} = 1.24$
  - No. of subtractions: 15 at 7 different PS points for each event
  - Speed of code on an AMD Athlon 1.3 GHz machine with 256 MB RAM: $2.5 \cdot 10^5$ pts. $\approx$ 7 h

Rate of convergence

RR part



RV part