NLO QCD corrections to $pp \to t\bar{t}b\bar{b}$ via $q\bar{q}$ annihilation

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Outline of the talk

(1) **Introduction** - NLO corrections to multi-leg processes, $\mathrm{t\bar{t}b\bar{b}}$ production

(2) **Virtual corrections** - Reduction techniques for Feynman diagrams

(3) **Real corrections** - Dipole subtraction and phase-space slicing

(4) **First results for the LHC** - The $q\bar{q}$ channel

(1) Introduction

Importance of **multi-leg processes** at the LHC

- Most interesting signals: heavy particles decaying into jets, leptons, photons
- Irreducible backgrounds to these signals
  (often not fully accessible to measurements)

Importance of **NLO QCD corrections** at the LHC

- reduce scale uncertainties (high powers of α_S!); improve description of jets
- systematics better than Tevatron; very high statistics

Technical problems for $2 \to 3, 4, \ldots$ processes

- numerical instability of virtual corrections (Gram determinants)
- number and complexity of diagrams grow very fast

Challenges for NLO programs

- reliable predictions: **numerical stability**
- **sufficient speed**: distributions require > 1 event/sec!

Les Houches '05 prioritized wishlist

	Reaction	background for	existing calculations
1.	VVj	$\mathrm{t\bar{t}H}$, new physics	WWj: Dittmaier/Kallweit/Uwer '07; WWj: Campbell/Ellis/Zanderighi '07; WWj: Binoth/Guillet/Karg/Kauer/Sanguinetti (in progress)
2.	$\mathrm{t\bar{t}b\bar{b}}$	$\mathrm{t\bar{t}H}$	this talk
3.	$\mathrm{t\bar{t}}jj$	$\mathrm{t\bar{t}H}$	$\emptyset$
4.	$VV\mathrm{b\bar{b}}$	VBF$\to$ H $\to VV$, $\mathrm{t\bar{t}}$, NP	$\emptyset$
5.	$VVjj$	VBF$\to$ H $\to VV$	**VBF:** Jäger/Oleari/Zeppenfeld '06 + Bozzi '07
6.	$Vjjj$	new physics	$\emptyset$
7.	VVV	new physics	ZZZ: Lazopoulos/Melnikov/Petriello '07; WWZ: Hankele/Zeppenfeld '07 VVV: Binoth/Ossola/Papadopoulos/Pittau '07

State of the art

Several $2 \to 3$ (also beyond/before LH list!) but very few $2 \to 4$ calculations:

- $\mathrm{e^+e^-} \to$ 4f (EW) [Denner/Dittmaier/Roth/Wieders '05]
- $\mathrm{e^+e^-} \to \mathrm{HH}\nu\bar{\nu}$ (EW) [Boudjema/Fujimoto/Ishikawa/Kaneko/Kurihara/Shimizu/Kato/Yasui '05]
- $\gamma\gamma \to \mathrm{t\bar{t}b\bar{b}}$ (QCD) [Lei/Wen-Gan/Liang/Ren-You/Yi '07]

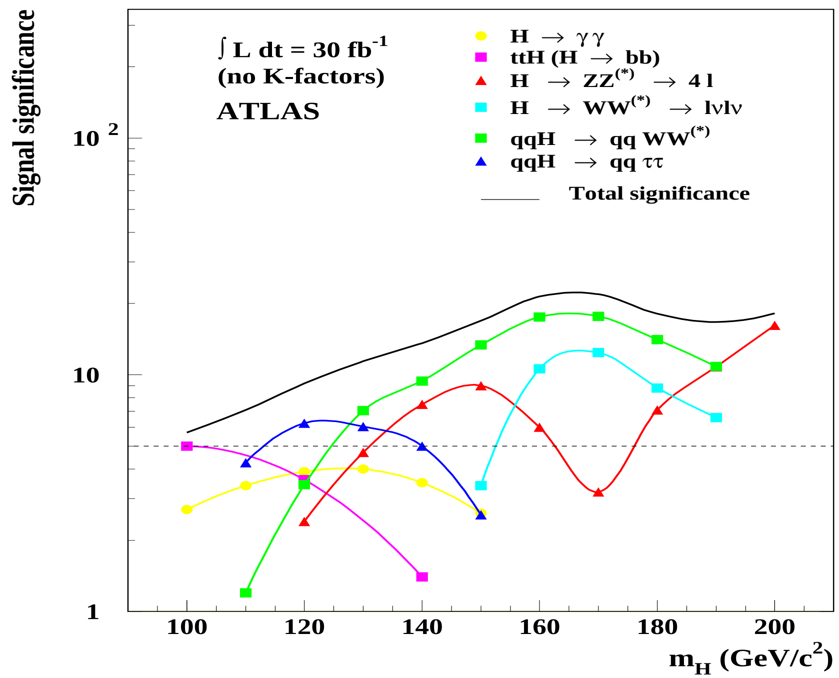
Technical motivation for $t\bar{t}b\bar{b}$

Validate NLO algorithms by solving a non-trivial problem

- $2 \to 4$ process involving hexagons
- massless and massive particles, 6 coloured legs

Phenomenological motivation for $t\bar{t}b\bar{b}$

ATLAS '03



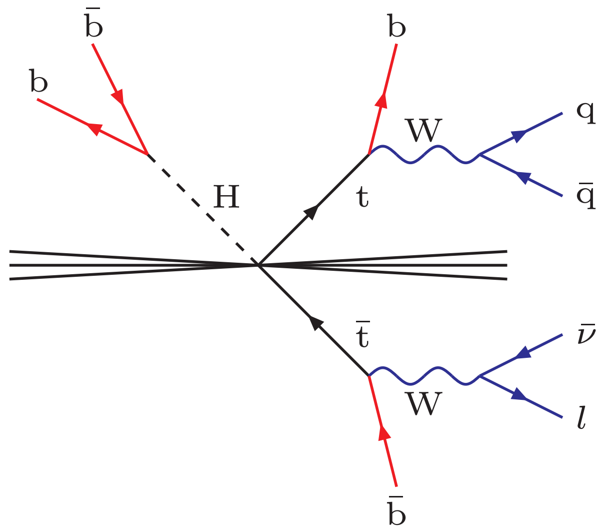
Associated $t\bar{t}H(H \to b\bar{b})$ production

- can be observed in $H \to b\bar{b}$ channel
- exploits large $BR(H \to b\bar{b})$ for light H
- measurement of top Yukawa coupling

Proposed analysis (ATLAS TDR)

- select final state $\mathrm{b\bar{b}b\bar{b}}jjl\nu$ (4 b-quarks!)
- reconstruct $\mathrm{t\bar{t}b\bar{b}}$
- select region $|m_{\mathrm{b\bar{b}}} - M_{\mathrm{H}}| < 30\,\mathrm{GeV}$

Richter-Was and Sapinski, ATL-PHYS-98-132



Backgrounds

- irreducible: $\mathrm{t\bar{t}b\bar{b}}$ (QCD+EW)
- reducible: $\mathrm{t\bar{t}}jj$

 # events and stat. significance (30 fb^{-1})

M_{H}(GeV)	110	120	130
$\mathrm{t\bar{t}H}$	60.9	41.9	25.5
$\mathrm{t\bar{t}b\bar{b}}$(QCD)	167.3	145.8	128.7
$\mathrm{t\bar{t}}jj$	66.2	54.6	41.6
$\mathrm{t\bar{t}b\bar{b}}$(EW)	21.8	18.4	15.2
$S/\sqrt{B}$	3.8	2.8	1.9
S/B	0.24	0.19	0.14

Cammin and Schumacher, ATL-PHYS-2003-024

Systematic uncertainty

Large $t\bar{t}b\bar{b}$ background

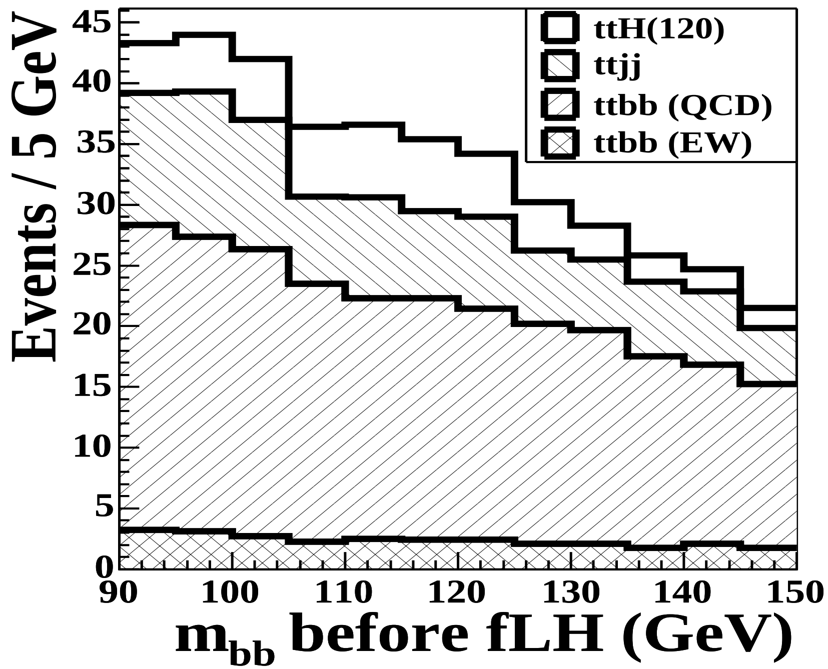
- $S/B \lesssim 0.2$
- 20% uncertainty of B kills measurement!

Systematic uncertainty of background

- only statistics in TDR: $1/\sqrt{B} < 10\%$
- but error dominated by systematics!

Normalization and shape of B

- data alone do not provide enough precision
- LO theory $\Rightarrow$ factor-4 scale dependence

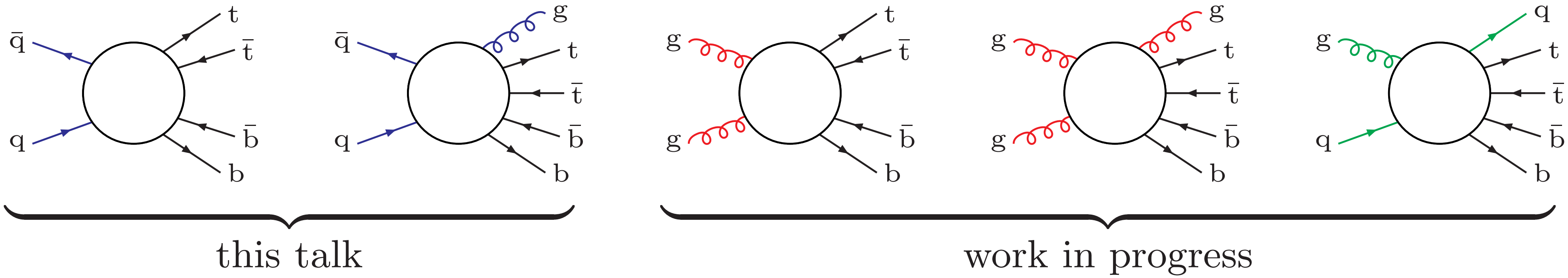


Cammin and Schumacher, ATL-PHYS-2003-024

$t\bar{t}H$ **measurement impossible without** $t\bar{t}b\bar{b}$ **at NLO!**

$2 \to 4$ and $2 \to 5$ Feynman diagrams

Quark-antiquark and gluon induced processes



Quark-antiquark channel

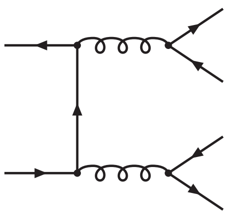
- 5 times less diagrams than gg channel
- not sufficient for LHC (small fraction of σ)
- demonstrate feasibility of calculation

diagrams and impact on σ_{LO}

	$q\bar{q}$	gg	qg
LO	7	38	
virtual	188	1003	
real	66	393	66
$\sigma/\sigma_{\text{tot}}$	5%	95%	

(2) Tree and one-loop contributions to $\bar{q}q \to t\bar{t}b\bar{b}$

Tree (7) and one-loop (188) diagrams



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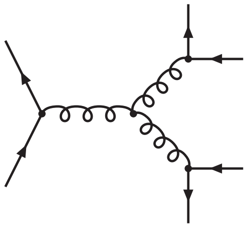

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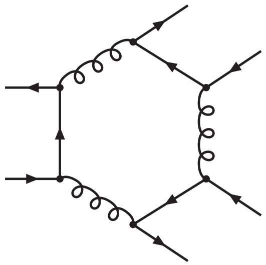
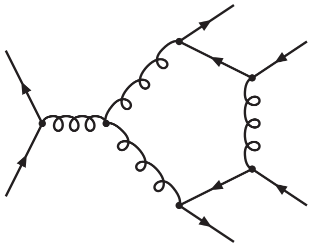


8 hexagons



24 pentagons

etc.

Two independent calculations

- diagrams generated with `FeynArts 1.0 / 3.2` [Külbeck/Böhm/Denner '90; Hahn '01]
- one calculation uses `FormCalc 5.2` [Hahn '06] for preliminary algebraic manipualations (Dirac algebra, covariant decomposition)
- bulk of reduction with two in-house `MATHEMATICA` programs
- numerics with two independent `Fortran77` codes (two libraries for tensor integrals)

Top quarks massive and bottom quarks massless

Structure of the calculation

Standard matrix elements and colour structures for **individual diagrams**

$$\mathcal{D} = \underbrace{\left[\sum_k a_k \mathcal{C}_k(\{c\})\right]}_{\substack{\text{factorized} \\ \text{colour structure}}} \sum_i \mathcal{F}_i \underbrace{\mathcal{S}_i(\{p\},\{\lambda\})}_{\substack{\text{standard matrix} \\ \text{elements}}}$$

Form factors $\mathcal{F}_i$ in terms of tensor integrals

$$\mathcal{F}_i = \sum_{j_1 \ldots j_R} \mathcal{K}_{i,j_1 \ldots j_R} \underbrace{T_{j_1 \ldots j_R}(\{p\})}_{\text{tensor loop coefficients}}$$

computed numerically diagram by diagram (no analytic reduction to scalar integrals)

Main goals

- reduction to small set of standard matrix elements $\mathcal{S}_i$
- fast and stable numerical evaluation of tensor integrals $T_{j_1 \ldots j_R}$

Colour structure

- six colour structures for $\bar{q}q \to t\bar{t}b\bar{b}$

$$1 \otimes T^a \otimes T^a, \qquad T^a \otimes 1 \otimes T^a, \qquad T^a \otimes T^a \otimes 1,$$
$$1 \otimes 1 \otimes 1, \qquad f^{abc} T^a \otimes T^b \otimes T^c, \qquad d^{abc} T^a \otimes T^b \otimes T^c$$

- exploit colour factorization for individual diagrams

Rational terms originate from $1/(D-4)$ poles of tensor loop integrals

$$\mathcal{K}_{j_1 \ldots j_R}(D) \underbrace{T_{j_1 \ldots j_R}}_{\frac{R_{j_1 \ldots j_R}}{(D-4)} + T^{\text{finite}}_{j_1 \ldots j_R}} = \mathcal{K}_{j_1 \ldots j_R}(4)\, T_{j_1 \ldots j_R} + \mathcal{K}'_{j_1 \ldots j_R}(4)\, R_{j_1 \ldots j_R} + \mathcal{O}(D-4)$$

- residues $R_{j_1 \ldots j_R}$ of tensor integrals explicitly available
- after $(D-4)$-expansion continue calculation in $D = 4$

Reduction of Standard Matrix Elements (strategy/simple examples)

After Dirac algebra + Dirac equation: $\mathcal{O}(10^3)$ Dirac structures of type

$$\underbrace{\bar{v}(p_1)\ldots\gamma^\mu\gamma^\nu\not{p}_3\ldots u(p_2)}_{\mathrm{q\bar{q}\ chain}}\ \underbrace{\bar{v}(p_3)\ldots\gamma^\mu\gamma^\nu\gamma^\rho\not{p}_6\ldots u(p_4)}_{\mathrm{t\bar{t}\ chain}}\ \underbrace{\bar{v}(p_5)\ldots\gamma^\rho\not{p}_2\not{p}_3\ldots u(p_6)}_{\mathrm{b\bar{b}\ chain}}$$

Needs additional identities in order to eliminate $\not{p}$-terms and γ-contractions

- without introducing new denominators that might spoil numerical stability

After cancellation of $1/(D-4)$ poles we work in 4 dimensions and introduce

$$\omega_\pm = \frac{1}{2}(1\pm\gamma^5) \qquad\qquad \bar{v}(p_i)\Gamma u(p_j) \Rightarrow \sum_{\lambda=\pm}\bar{v}(p_i)\Gamma\omega_\lambda u(p_j)$$

This permits to use the **Chisholm identity**

$$\mathrm{i}\varepsilon^{\alpha\beta\gamma\delta}\gamma_\delta\gamma^5 = \gamma^\alpha\gamma^\beta\gamma^\gamma - g^{\alpha\beta}\gamma^\gamma + g^{\alpha\gamma}\gamma^\beta + g^{\beta\gamma}\gamma^\alpha$$

Contracting with $\otimes\gamma_\gamma\gamma^5$ and symmetrizing $\Rightarrow$ eliminates the ε-tensor and yields

$$\mathrm{i}\varepsilon^{\alpha\beta\gamma\delta}\left[\gamma_\delta\gamma^5\otimes\gamma_\gamma\gamma^5 + (\delta\leftrightarrow\gamma)\right] = 0 \quad\Rightarrow\quad \gamma^\mu\gamma^\alpha\gamma^\beta\omega_\pm\otimes\gamma_\mu\omega_\mp = \gamma^\mu\omega_\pm\otimes\gamma^\alpha\gamma^\beta\gamma_\mu\omega_\mp$$

and similar identities that permit to exchange $\gamma^\alpha\gamma^\beta$ between γ-contracted chains

Other identies involving doubly γ-contraced chains

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}\omega_{\pm}\otimes\gamma_{\mu}\gamma^{\beta}\gamma_{\nu}\omega_{\mp} = 4\gamma^{\beta}\omega_{\pm}\otimes\gamma^{\alpha}\omega_{\mp} \qquad \text{etc.}$$

can be used (for instance)

- to reduce number of γ-contractions
- or (in the inverse direction) to shift $\not{p}$-terms and then exploit Dirac equation

$$4\bar{v}(p_1)\ldots\not{p}_4\omega_{\pm}\, u(p_2)\bar{v}(p_3)\ldots\not{p}_2\omega_{\mp}u(p_4) = \bar{v}(p_1)\ldots\gamma^{\mu}\not{p}_2\gamma^{\nu}\omega_{\pm}\, u(p_2)\bar{v}(p_3)\ldots\gamma_{\mu}\not{p}_4\gamma_{\nu}\omega_{\mp}u(p_4) =$$
$$= 4(p_2p_4)\bar{v}(p_1)\ldots\gamma^{\mu}\omega_{\pm}\, u(p_2)\bar{v}(p_3)\ldots\gamma_{\mu}\omega_{\mp}u(p_4) + \text{mass-terms}$$

In practice

- Chisolm identity + Dirac equation + momentum conservation (+ a lot of patience)
- yields several useful relations
- construct a sophisticated algorithm to reduce # of γ-contractions and $\not{p}$-terms

At the end of the day 25 types of standard matrix elements

- 10 of "massless" type: one Dirac matrix per chain

$$\bar{v}(p_1)\not{p}_i\omega_\alpha u(p_2)\ \bar{v}(p_3)\gamma^\mu\omega_\beta u(p_4)\ \bar{v}(p_5)\gamma^\mu\omega_\rho u(p_6)$$

$$\bar{v}(p_1)\not{p}_i\omega_\alpha u(p_2)\ \bar{v}(p_3)\not{p}_j\omega_\beta u(p_4)\ \bar{v}(p_5)\not{p}_k\omega_\rho u(p_6)$$

- 15 of "massive" type: 2/0 Dirac matrices inside the $t\bar{t}$ chain*

$$\bar{v}(p_1)\not{p}_i\omega_\alpha u(p_2)\ \bar{v}(p_3)\not{p}_j\gamma^\mu\omega_\beta u(p_4)\ \bar{v}(p_5)\not{p}_k\omega_\rho u(p_6)$$

$$\bar{v}(p_1)\gamma^\mu\omega_\alpha u(p_2)\ \bar{v}(p_3)\not{p}_j\not{p}_j\omega_\beta u(p_4)\ \bar{v}(p_5)\gamma^\mu\omega_\rho u(p_6)$$

$$\bar{v}(p_1)\gamma^\mu\omega_\alpha u(p_2)\ \bar{v}(p_3)\gamma^\mu\gamma^\nu\omega_\beta u(p_4)\ \bar{v}(p_5)\gamma^\nu\omega_\rho u(p_6)$$

$$\bar{v}(p_1)\gamma^\mu\omega_\alpha u(p_2)\ \bar{v}(p_3)\omega_\beta u(p_4)\ \bar{v}(p_5)\gamma^\mu\omega_\rho u(p_6)$$

$$\bar{v}(p_1)\not{p}_i\omega_\alpha u(p_2)\ \bar{v}(p_3)\omega_\beta u(p_4)\ \bar{v}(p_5)\not{p}_k\omega_\rho u(p_6)$$

*price to pay for the presence of massive top quarks

Stable reduction of tensor loop integrals (developped for $e^+e^- \to 4f$) [Denner/Dittmaier '05]

6- and 5-point tensor integrals (exploit linear dependence of $4+n$ momenta)

- reduction to lower-point integrals [Melrose '65; Denner/Dittmaier '02]
- simultaneous rank-reduction [Binoth/Guillet/Heinrich/Pilon/Schubert '05; Denner/Dittmaier '05]
- no Gram determinants $\Rightarrow$ no numerical problems in practice

4- and 3-point tensor integrals

- Passarino–Veltman reduction [Passarino/Veltman '79]
- alternative methods for small Gram determinants [Denner/Dittmaier '05]
  (analogies with techniques proposed by Ferroglia/Passera/Passarino/Uccirati '03; Binoth/Guillet/Heinrich/Pilon/Schubert '05; Ellis/Giele/Zanderighi '06)

2- and 1-point tensor integrals

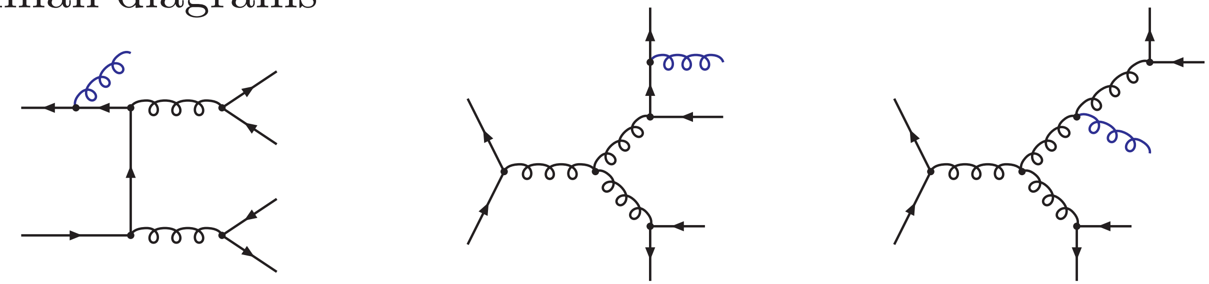
- numerically stable analytic representations [Passarino/Veltman '79; Denner/Dittmaier '05]

(3) Real corrections $\bar{q}q \to t\bar{t}b\bar{b}g$

- Also for real corrections: 2 independent calculations

Diagrams and matrix elements for $\bar{q}q \to t\bar{t}b\bar{b}g$

- 66 Feynman diagrams



etc.

- analytically with Weyl–van der Waerden spinors [Dittmaier '98] and with `Madgraph 4.1.33` [Maltoni/Stelzer]

Treatment of soft and collinear singularities

- Phase-space slicing [Giele/Glover '92; Giele et al. '93; Keller/Laenen '98; Harris/Owens '01]

$$\int d\sigma_{2\to 5} = \int_{\substack{E>\delta_s\sqrt{\hat{s}}/2 \\ \cos\theta<1-\delta_c}} d\sigma_{2\to 5} \quad + \quad F(\delta_s, \delta_c) \otimes d\sigma_{2\to 4}$$

numerical cancellations/CPU-consuming but "simple" and important check

- Dipole subtraction [Catani/Seymour '96; Dittmaier '99; Catani/Dittmaier/Seymour/Trócsányi '02]

$$\int \mathrm{d}\sigma_{2\to 5} = \int \left[\mathrm{d}\sigma_{2\to 5} - \sum_{\substack{i,j=1 \\ i\neq j}}^{6} \mathrm{d}\sigma_{2\to 5}^{\mathrm{dipole},ij} \right] + \sum_{\substack{i,j=1 \\ i\neq j}}^{6} \mathcal{F}_{ij} \otimes \mathrm{d}\sigma_{2\to 4}$$

  numerically stable/efficient but non-trivial: 30 subtraction terms

- initial-state collinear singularities cancelled by $\overline{\mathrm{MS}}$-redefinition of PDFs

Phase-space integration

- adaptive multi-channel Monte Carlo [Berends/Kleiss/Pittau '94; Kleiss/Pittau '94] as in `RACOONWW`[Denner/Dittmaier/Roth '99]/`PROFECY4f`[Bredenstein/Denner/Dittmaier/Weber '06]
- $\mathcal{O}(300)$ channels to map all peaks from propagators and dipoles

11-dimensional phase space, many channels and dipoles ⇒ CPU-time! (see later)

(4) NLO results for the LHC

The following results are **PRELIMINARY**

- not all cross checks completed
- $q\bar{q}$ channel only

LO checked against SHERPA [Gleisberg/Hoche/Krauss/Schalicke/Schumann/Winter '03]

Checks of virtual corrections

- 2 independent calculations
- UV, soft and collinear cancellations
- good agreement pointwise and after PS integration

Checks of real corrections

- only one of two calculations completed
- agreement for $2 \to 5$ matrix elements
- soft and collinear cancellations
- agreement between slicing and dipole subtraction

Setup

Parton masses

- $m_{\mathrm{t}} = 172.6\,\mathrm{GeV}$
- g, q and b massless $\Rightarrow$ recombination!

k_{T}-Jet-Algorithm [Run II Jet physics group: Blazey et al. hep-ex/0005012]

- Select partons with $|\eta| < 5$
- Reconstruct jets with $\sqrt{\Delta\phi^2 + \Delta y^2} > D = 0.8$

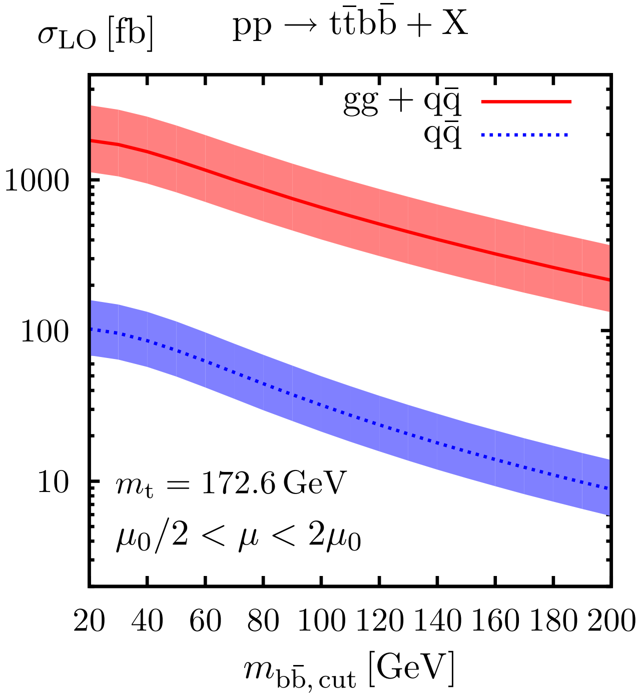
Cuts for b-jets

- require two b-jets with $p_{\mathrm{T},j} > 20\,\mathrm{GeV}$ and $y_j < 4.5$
- $\mathrm{b\bar{b}}$ invariant mass: $m_{\mathrm{b\bar{b}}} > m_{\mathrm{b\bar{b}},\mathrm{cut}}$

Strong coupling, PDFs and central scale*

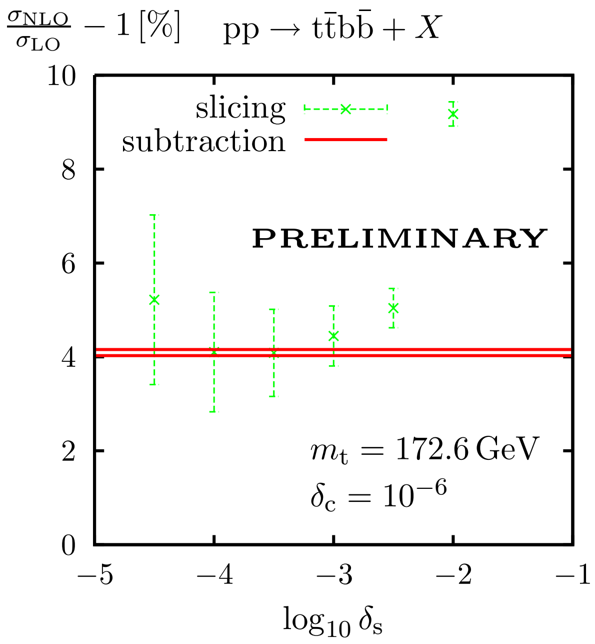
- CTEQ6M with $\alpha_{\mathrm{S}}(M_{\mathrm{Z}}) = 0.118$
- 2-loop running to $\mu = \mu_{\mathrm{R}} = \mu_{\mathrm{F}}$
- central scale $\mu_0 = m_{\mathrm{t}} + m_{\mathrm{b\bar{b}},\mathrm{cut}}/2$

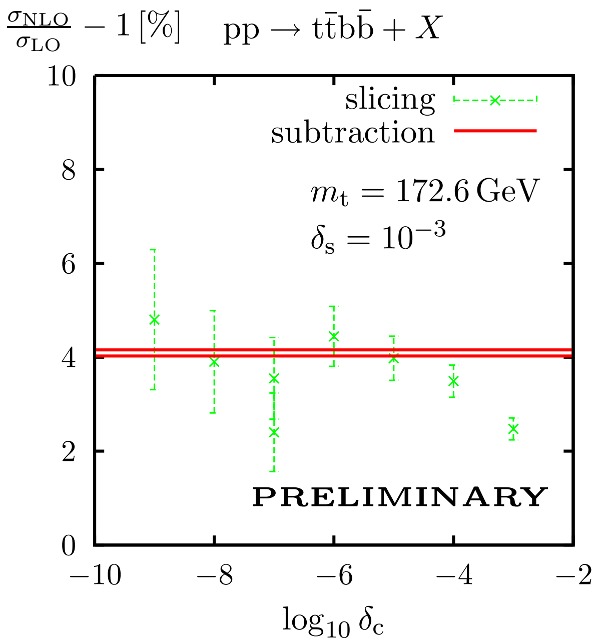
* LO obtained with LO α_{S}, LO PDFs and 1-loop running



$m_{\mathrm{b\bar{b}},\mathrm{cut}}$ dependence of $\sigma_{\rm LO}$

- relative weight $\mathrm{q\bar{q}}$:gg $\simeq 1/20$
- $\mathrm{q\bar{q}}$ and gg have similar shape
- factor-2 scale dependence





Dipole subtraction vs phase-space slicing ($q\bar{q}$ channel)

Dipole method

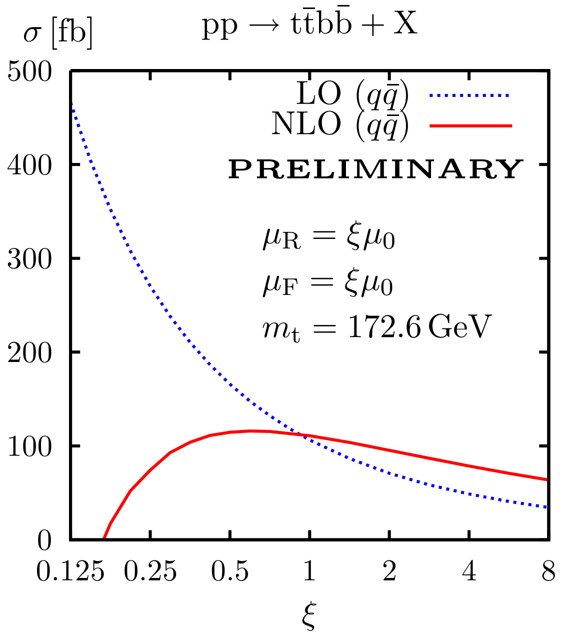
- 1.2×10^8 events $\Rightarrow$ precision $2 \times 10^{-3}\,\sigma_{LO}$
- no cut-off dependence

Phase-space slicing

- 1.2×10^8 events $\Rightarrow$ precision $2 \times 10^{-2}\,\sigma_{LO}$
- dependence on soft (δ_s) and collinear (δ_c) cut-off

Comparison

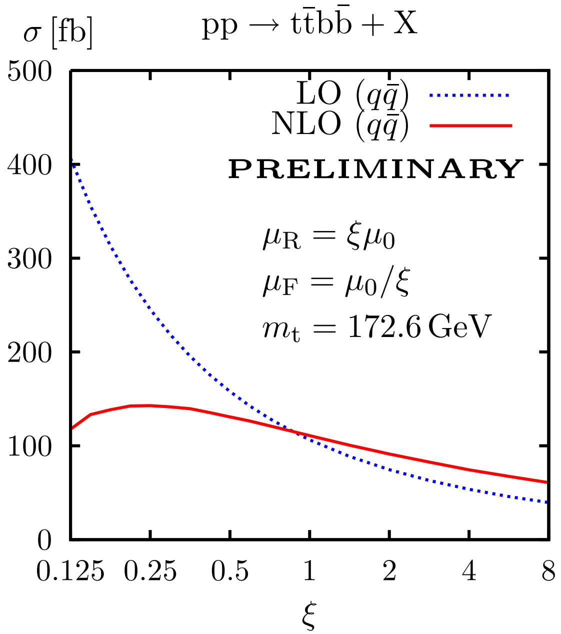
- Plateau of consistency
- Relative NLO correction of $\sim 4\%$



LO and NLO scale dependence ($q\bar{q}$ channel)

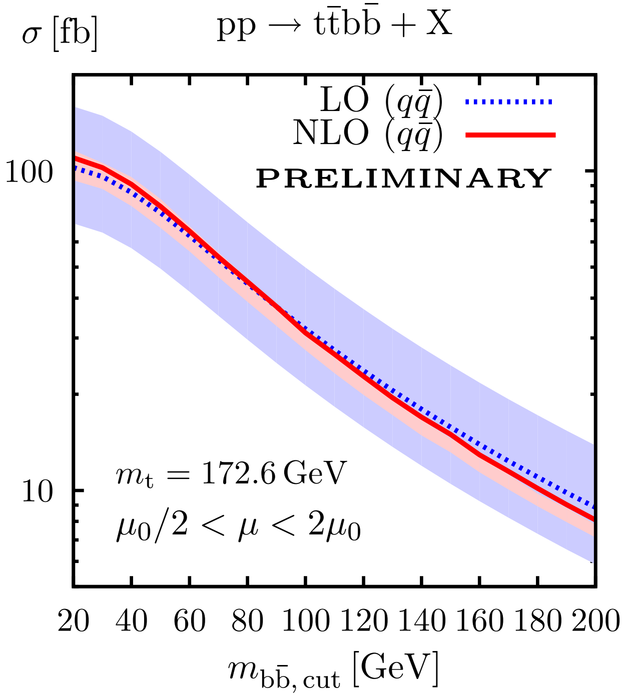
Reduction of scale dependence

- central scale $\mu_0 = m_t$
- huge LO dependence: up to factor 4
- stabilization at NLO (close to maximum)



μ_F and μ_R dependence (two plots similar)

- dominant dependence from $\alpha_S(\mu_R)^4$
- μ_F-dependence much smaller



$m_{b\bar{b},\mathrm{cut}}$ dependence of σ_{LO} and σ_{NLO}

- strong reduction of scale dependence
- NLO consistent with LO uncertainty band
- shape of $m_{b\bar{b}}$ distorted by corrections

Speed and precision of calculation

Runtime and statistical precision ($\Delta\sigma/\sigma_{\rm LO}$) with 3GHZ Intel Xeon processor

	$\sigma/\sigma_{\rm LO}$	# events (after cuts)	$\Delta\sigma/\sigma_{\rm LO}$	runtime	time/event
tree	74%	6.7×10^7	3×10^{-4}	66min	60μs
virtual	−3%	0.34×10^7	2.5×10^{-4}	12h	13ms
real + dipoles	33%	13×10^7	5×10^{-4}	34h	1ms

- speed of virtual corrections (13 ms/event) very encouraging
- for same precision ($\Delta\sigma/\sigma$) virtual corrections require less CPU-time than real corrections (scale-dependent statement!)

Conclusions

NLO calculation for $pp \rightarrow t\bar{t}b\bar{b}$

- very important for $t\bar{t}H$ measurement
- highest priority in the 2005 Les Houches wish list

Optimal testing ground for NLO methods

- $2 \rightarrow 4$ process with hexagons
- 6 coloured particles, massive top quarks

First results for $q\bar{q}$ channel available

- demonstrates feasibility/speed/stability of calculation
- not sufficient for the LHC (gg channel dominates)
- gg channel in progress

Stable reduction of 4- and 3-point integrals (for pedestrians)

Tensor integrals and covariant decomposition

$$T_{\mu_1\mu_2\ldots} = \int \frac{q_{\mu_1} q_{\mu_2}\cdots}{N_0\ldots N_{N-1}} = g^{\mu_1\mu_2} T_{00\ldots} + p_i^{\mu_1} p_j^{\mu_2} T_{ij\ldots} + \ldots$$

Contraction identities

$$2p_i^{\mu_1} T_{\mu_1\mu_2\ldots} = \int \underbrace{2p_i q}_{N_i - N_0 - f_i} \frac{q_{\mu_2}\cdots}{N_0\ldots N_{N-1}} = -f_i \int \frac{q_{\mu_2}\cdots}{N_0\ldots N_{N-1}} + \text{lower-point}$$

$$g^{\mu_1\mu_2} T_{\mu_1\mu_2\ldots} = \int \underbrace{q^2}_{N_0 + m_0^2} \frac{q_{\mu_3}\cdots}{N_0\ldots N_{N-1}} = m_0^2 \int \frac{q_{\mu_3}\cdots}{N_0\ldots N_{N-1}} + \text{lower-point}$$

Two sets of identities for rank-R coefficients

$$2(D+R-N-1) T_{00\ldots}^{(R)} = f_k T_{k\ldots}^{(R-1)} + 2m_0^2 T_{\ldots}^{(R-2)} + \text{lower-point}$$

$$Z_{ii'} T_{i'j\ldots}^{(R)} = -2\delta_{ij} T_{00\ldots}^{(R)} - f_i T_{j\ldots}^{(R-1)} + \text{lower-point}$$

Gram matrix $Z_{ij} = 2p_i p_j$

Identities in terms of adjoint Gram matrix $\tilde{Z}_{ij}$ $\quad (Z^{-1}_{ji} = \tilde{Z}_{ij}/\det(Z))$

$$2(D+R-N-1)T^{(R)}_{00\ldots} = f_k T^{(R-1)}_{k\ldots} + 2m_0^2 T^{(R-2)}_{\ldots} + \text{lower-point}$$
$$\det(Z) T^{(R)}_{ij\ldots} = -2\tilde{Z}_{ji} T^{(R)}_{00\ldots} - f_{i'} \tilde{Z}_{i'i} T^{(R-1)}_{j\ldots} + \text{lower-point}$$

Passarino-Veltman algorithm

Reduce high-rank to lower-rank

$$T^{(R)} \Leftarrow T^{(R-K)} + \text{lower-point}$$

Advantages

- Fast reduction to A_0, B_0, C_0, D_0
- works for most PS points

Disadvantages

- unstable for $\det(Z) \to 0$
- instabilities become serious for $2 \to 4$

Identities in terms of adjoint Gram matrix $\tilde{Z}_{ij}$ $\quad (Z^{-1}_{ji} = \tilde{Z}_{ij}/\det(Z))$

$$2(D+R-N-1)T^{(R)}_{00\ldots} = f_k T^{(R-1)}_{k\ldots} + 2m_0^2 T^{(R-2)}_{\ldots} + \text{lower-point}$$
$$\det(Z) T^{(R)}_{ij\ldots} = -2\tilde{Z}_{ji} T^{(R)}_{00\ldots} - f_{i'}\tilde{Z}_{i'i} T^{(R-1)}_{j\ldots} + \text{lower-point}$$

Rescue system 1

Expansion in small Gram determinant

$$T^{(R-K)} + \mathcal{O}\left(\det{}^K\right) \Leftarrow \ldots \Leftarrow T^{(R-1)} + \underbrace{\det(Z) T^{(R)}}_{\mathcal{O}\left(\det^1\right)} \Leftarrow \text{lower-point}$$

Advantages

- arbitrary $\mathcal{O}\left(\det{}^K\right)$ precision
- few terms sufficient for $\det(Z) \to 0$

Disadvantages

- high order in $\det(Z)$ requires high rank
- unstable if $f_{i'}\tilde{Z}_{i'i} \to 0$

expansions for small $f_{i'}\tilde{Z}_{i'i}$ available

Identities in terms of Gram matrix $Z_{ij} = 2p_ip_j$

$$2(D+R-N-1)T^{(R)}_{00\ldots} = f_k T^{(R-1)}_{k\ldots} + 2m_0^2 T^{(R-2)}_{\ldots} + \text{lower-point}$$

$$Z_{ii'}T^{(R)}_{i'j\ldots} = -2\delta_{ij}T^{(R)}_{00\ldots} - f_i T^{(R-1)}_{j\ldots} + \text{lower-point}$$

Rescue system 2

Change the basis: reduce to $T_{00\ldots00}$ and evaluate this integral numerically

$$\underbrace{\begin{pmatrix} 2m_0^2 & f_{i'} \\ f_i & Z_{ii'} \end{pmatrix}}_{X\,:\,\text{Cayley matrix}} \begin{pmatrix} T^{(R-1)}_{j\ldots} \\ T^{(R)}_{i'j\ldots} \end{pmatrix} = \begin{pmatrix} 2(D+R-N)T^{(R+1)}_{00j\ldots} \\ -2\delta_{ij}T^{(R)}_{00\ldots} \end{pmatrix} + \text{lower-point}$$

Advantages

- $\det(X)^{-1}$ instead of $\det(Z)^{-1}$
- $T_{00\ldots00}$ numerical integration stable

Disadvantages

- time-consuming $T_{00\ldots00}$ integration
- unstable if $\det(X)^{-1} \to 0$

expansion for small $\det(X)$ available