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SURFACE IMPEDANCE OF MULTILAYERED SUPERCONDUCTING CAVITIES

HOMSC2025 @ DESY (Hamburg)

2025-10-06

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CONTENTS

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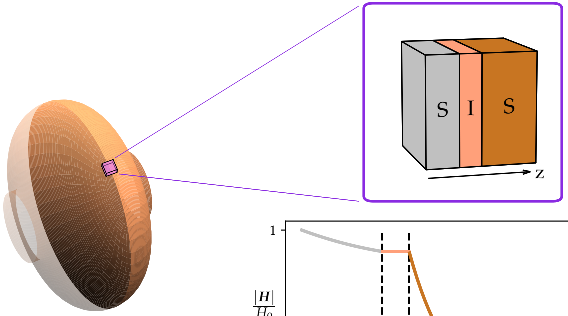
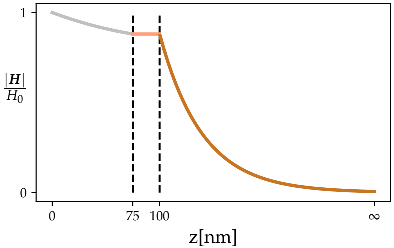
MOTIVATION

Section 1





Why ?*

1. Niobium is expensive
2. Increase in energy efficiency
3. Higher accelerating fields

How ?

1. Replace bulk Nb with Cu, Al, etc. substrates
2. Better thermal conducting substrate
3. Shield with higher H_c material

* Valente-Feliciano, A.-M. *Superconductor Science and Technology* (2016)

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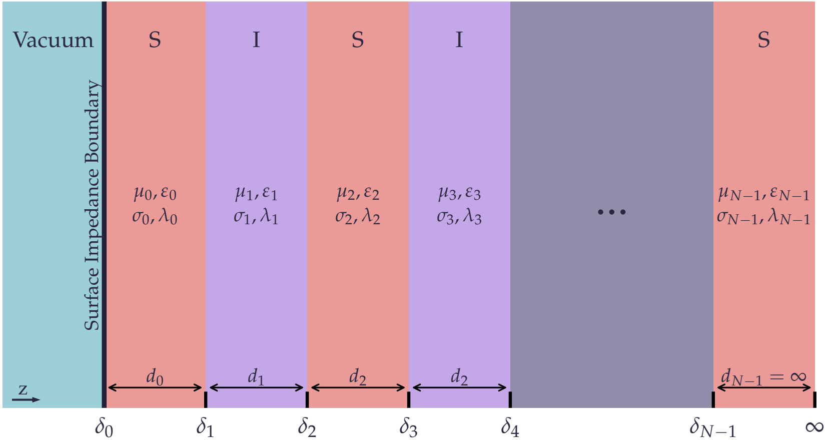
REDUCTION TO SIBC

Section 2

ANALYTICAL SOLUTION



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Solve Maxwell's equations in every layer $k \in \{0, \ldots, N-1\}$:

$$E_x^{(k)} = \begin{pmatrix} C_{2k} \\ C_{2k+1} \end{pmatrix} \cdot \begin{pmatrix} e^{j\alpha_k z} \\ e^{-j\alpha_k z} \end{pmatrix} := \boldsymbol{C}_k \cdot \boldsymbol{u}_k$$

$$H_y^{(k)} = -\frac{\alpha_k}{\omega\mu_k} \begin{pmatrix} C_{2k} \\ C_{2k+1} \end{pmatrix} \cdot \begin{pmatrix} e^{j\alpha_k z} \\ -e^{-j\alpha_k z} \end{pmatrix} := -\frac{\alpha_k}{\omega\mu_k} \boldsymbol{C}_k \cdot \boldsymbol{v}_k$$

where $\alpha \in \mathbb{C}$ depends on material type:

$$\text{Insulator}: \alpha^2 = \mu\varepsilon\omega^2$$

$$\text{Normal Conductor}: \alpha^2 = \mu\varepsilon\omega^2 - j\mu\sigma\omega$$

$$\text{Superconductor (two-fluid)}: \alpha^2 = \mu\varepsilon\omega^2 - j\mu\sigma\omega - \frac{1}{\lambda^2}$$

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Surface impedance boundary condition

$$\boldsymbol{n} \times \boldsymbol{E} = Z(\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{H}))$$

yields surface impedance

$$Z(\omega) = \frac{E_x^{(0)}(\delta_0)}{H_y^{(0)}(\delta_0)} = -\frac{\omega\mu_0}{\alpha_0}\frac{\boldsymbol{C}_0 \cdot \boldsymbol{u}(\delta_0)}{\boldsymbol{C}_0 \cdot \boldsymbol{v}(\delta_0)}$$

$\Rightarrow$ We need the coefficient vector $\boldsymbol{C}_0$

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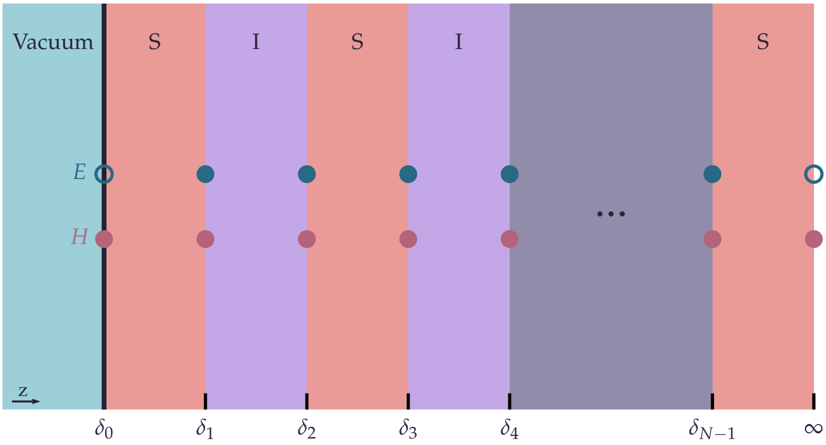

$2N$ Unknows = 1 vacuum constraint + $2(N-1)$ interface constraints + 1 infinity constraint

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Mathematically the constraints are:

- Vacuum constraint:

$$H_y^{(0)}(\delta_0) = H_0$$

- Interface constraints:

$$E_x^{(k)}(\delta_{k+1}) = E_x^{(k+1)}(\delta_{k+1})$$
$$H_y^{(k)}(\delta_{k+1}) = H_y^{(k+1)}(\delta_{k+1})$$

- Infinity constraint:

$$\lim_{z \to \infty} H_y^{(N-1)}(z) = 0$$

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Implied relations between coefficients:

- Vacuum constraint:

$$\boldsymbol{C}_0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\omega\mu_0}{\alpha_0} H_0$$

- Interface constraints:

$$\boldsymbol{C}_k \cdot \boldsymbol{u}_k(\delta_{k+1}) = \boldsymbol{C}_{k+1} \cdot \boldsymbol{u}_{k+1}(\delta_{k+1})$$

$$\boldsymbol{C}_k \cdot \boldsymbol{v}_k(\delta_{k+1}) = \gamma_k \boldsymbol{C}_{k+1} \cdot \boldsymbol{v}_{k+1}(\delta_{k+1}), \quad \gamma_k = \frac{\mu_k}{\mu_{k+1}} \frac{\alpha_{k+1}}{\alpha_k}$$

- Infinity constraint:

$$\boldsymbol{C}_{N-1} = C \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Recursion relations via linear combination of interface constraints:

$$\boldsymbol{C}_k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} e^{-j\alpha_k \delta_{k+1}} \boldsymbol{C}_{k+1} \cdot \left(\Gamma_k \boldsymbol{u}_{k+1}(\delta_{k+1})\right), \qquad \Gamma_k = \mathrm{diag}(1+\gamma_k, 1-\gamma_k)$$

$$\boldsymbol{C}_k \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} e^{j\alpha_k \delta_{k+1}} \boldsymbol{C}_{k+1} \cdot \left(\Gamma_k^\tau \boldsymbol{u}_{k+1}(\delta_{k+1})\right), \qquad \Gamma_k^\tau = \mathrm{diag}(1-\gamma_k, 1+\gamma_k)$$

yields characteristic matrix equation for each layer:

$$\boldsymbol{C}_k = \tau_{k,k+1} \boldsymbol{C}_{k+1}, \quad \tau_{k,k+1} = \frac{1}{2} \begin{pmatrix} e^{-j\alpha_k \delta_{k+1}} \left[\Gamma_k \boldsymbol{u}_{k+1}(\delta_{k+1})\right]^T \\ e^{j\alpha_k \delta_{k+1}} \left[\Gamma_k^\tau \boldsymbol{u}_{k+1}(\delta_{k+1})\right]^T \end{pmatrix}$$

and consequently a single transfer matrix:

$$\boldsymbol{C}_0 = T \boldsymbol{C}_{N-1} := \left[\prod_{k=0}^{N-2} \tau_{k,k+1}\right] \boldsymbol{C}_{N-1}$$

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Now notice:

$$\boldsymbol{C}_0 = T\boldsymbol{C}_{N-1} = CT\begin{pmatrix}0\\1\end{pmatrix} = C\begin{pmatrix}T_{01}\\T_{11}\end{pmatrix}$$

So we find the surface impedance:

$$Z(\omega) = -\frac{\omega\mu_0}{\alpha_0}\frac{T_{01}+T_{11}}{T_{01}-T_{11}}$$

- Transfer matrix depends on all material parameters
- $Z(\omega)$ is determined by transfer matrix, which we can compute
- $Z(\omega)$ is a non-linear complex-valued function of ω

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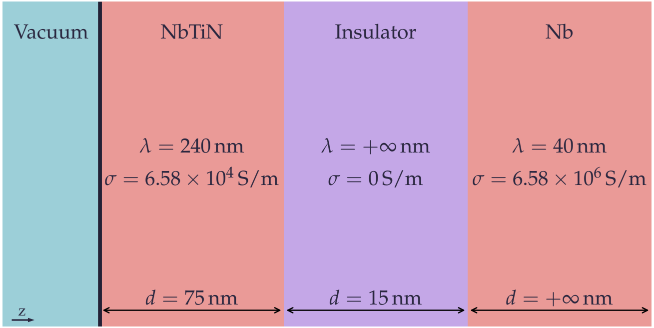
EXAMPLE

Bulk Nb with generic insulator layer and NbTiN sputtering at $f = 1.3\,\text{GHz}^*$



$$Z(\omega) = 21.462 \times 10^{-9}\,\Omega + 1.225 \times 10^{-3} j\,\Omega$$

* Keckert S., *PhD Dissertation, Universität Siegen* (2020)

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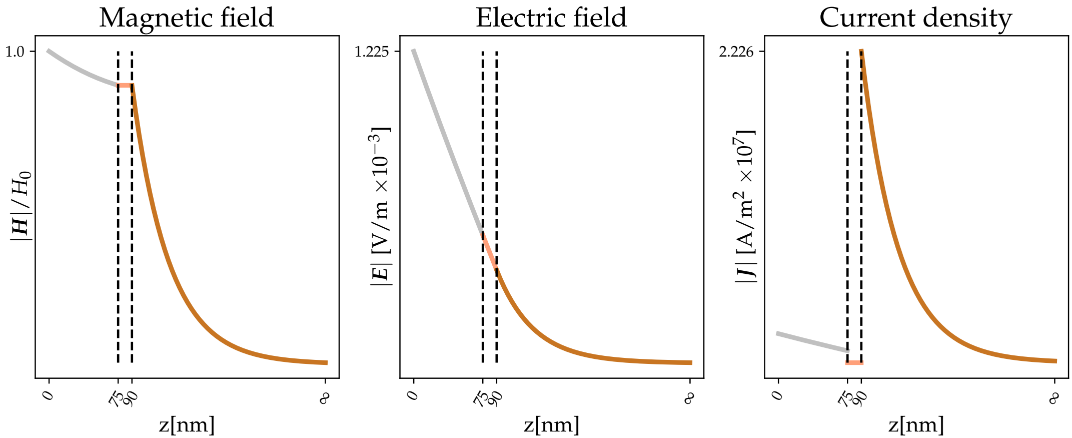

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EIGENVALUE PROBLEM

Section 3

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WEAK FORMULATION

Weak formulation from Maxwell's equation with surface impedance boundary leads to the following weak formulation:

$$\frac{1}{\kappa^2}\int_\Omega \mathrm{d}V\ [(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{v})] + \frac{\tau}{\kappa}\frac{Z_0}{Z(\omega)}\int_{\partial\Omega}\mathrm{d}S\ [(\boldsymbol{n}\times\boldsymbol{u})\cdot(\boldsymbol{n}\times\boldsymbol{v})] + \tau^2\int_\Omega \mathrm{d}V\ [\boldsymbol{u}\cdot\boldsymbol{v}] = 0$$

where $\boldsymbol{u}$ is the trial function and $\boldsymbol{v}$ is the test function, and

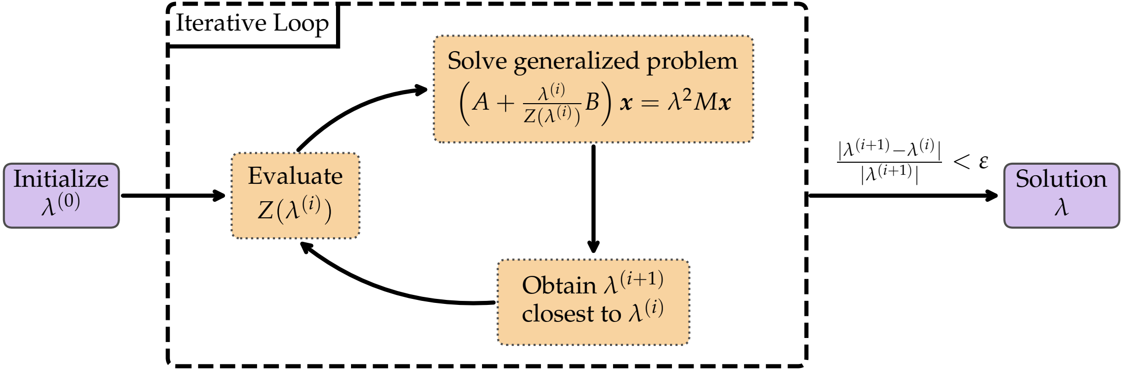
$$\kappa = \frac{2\pi f}{c}, \quad \tau = -\frac{1}{2Q} + j$$

The resulting non-linear complex eigenvalue problem is then of the form:

$$A\boldsymbol{x} + \frac{\lambda}{Z(\lambda)}B\boldsymbol{x} = \lambda^2 M\boldsymbol{x}$$

NONLINEAR PROBLEM

We solve the non-linear problem using a fixed point iteration



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OUTLOOK & CONCLUSION

Section 4

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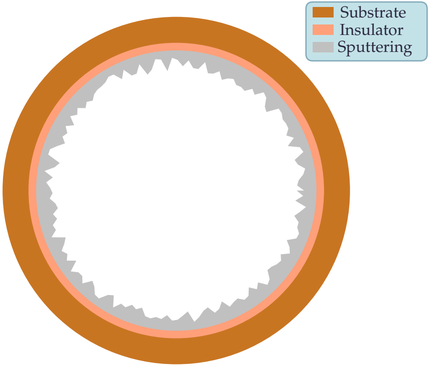
OUTLOOK



- Replace thickness of sputtering layer with stochastic process, i.e. $Z(\omega) \to Z(\mathbf{r}, \omega)$
- Model stochastic process with Karhunen-Loeve expansion informed by experimental data.
- Use polynomial chaos expansion to efficiently compute statistics of the quality factor, e.g. mean and standard deviation.

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CONCLUSION

- Multilayered cavities can be modelled via SIBC
- Analytical solution can support many multilayer variations
- Resulting eigenvalue problem is non-linear and complex-valued, but can be handled with fixed-point iteration

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Questions?