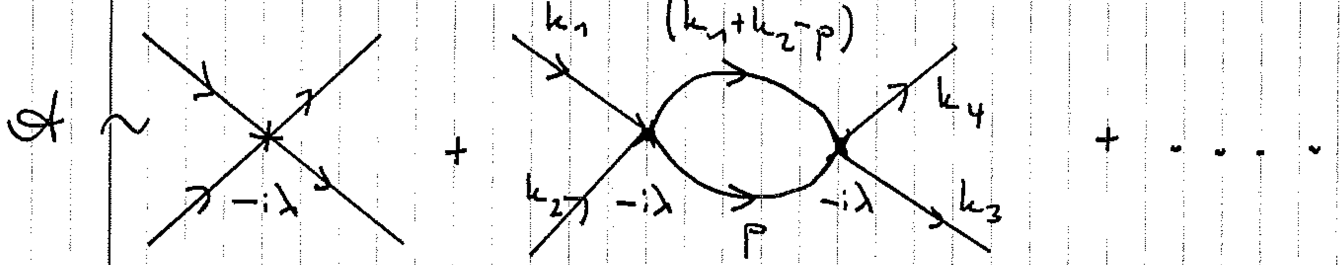
STRING THEORY

I) Motivation

* In Quantum Field Theory (QFT) fundamental particles are described as pointlike objects.

* While well-confirmed experimentally at low energies ($\lesssim 7$ TeV), this leads to conceptual problems at high energies (in the ultra-violet (UV)).

Consider e.g. $2 \to 2$ scattering in ϕ^4 theory



tree-level 1-loop higher loop

The integral $\displaystyle\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} \frac{i}{(k_1+k_2-p)^2-m^2}$

leads to a UV divergence as $|p| \longrightarrow \infty$

Since $|p| \to \infty$ probes the physics at $\Delta x \to 0$, this UV divergence indicates a breakdown of the theory at small distances.

* For most practical purposes this problem can be overcome by the theory of renormalisation, but at the cost of introducing a number of input parameters which cannot be predicted by the theory.

Most drastic examples are the <u>Higgs mass</u> or the <u>cosmological constant</u>. Explaining their unnaturally small values is among the most pressing problems of modern physics.

* According to the modern <u>Wilsonian point of view</u>, a renormalised <u>QFT</u> is an <u>effective description</u> of physics <u>at low energies</u>. Our ignorance about the true degrees of freedom in the UV is absorbed in our choice of input parameters (= renormalisation constants.)

* Even more drastically, Einstein's theory of gravity is not even renormalisable as a perturbative QFT (since its coupling constant $\sqrt{G_N}$ has mass dimension -1)

  Technically this means we cannot remove the cutoff as can be done in QCD or other renorm. QFTs.

<u>Note:</u> There exist interesting attempts to define GR as a non-perturbatively renormalisable QFT. Even if this worked, this would not solve the problem that a renormalisable theory is just an effective theory (à la Wilson) as can be seen by the unavoidable presence of renormalisation constants, see above.

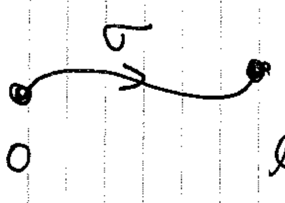
* Wanted is thus a theory which

  i) at low energies precisely <u>reproduces</u> the experimentally tested <u>effective description</u> of particles via QFT & general relativity (GR)

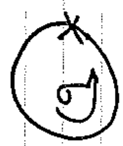
iii) at high energies is <u>free of UV divergences</u> without the need to renormalise.

String Theory is such a theory!

<u>Idea</u>: Replace the concept of a point particle by a string-like object of the 2 possible topologies:



open string



closed string

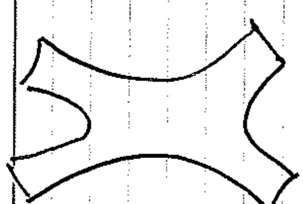
Energy scale $M_s \sim \ell_s^{-1}$ $\ell_s \sim$ "string length"

* At energies $\ll M_s$, a string looks like a point particle.

$\rightarrow$ $M_s \overset{!}{\gg} 7\,\text{TeV}$ (to match with experiment)

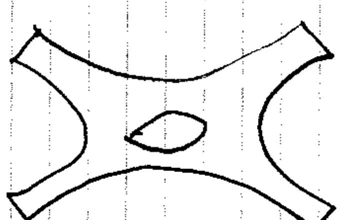
* At energies $\sim M_s$ the theory behaves very differently than a QFT.

The analogue of the $2 \rightarrow 2$ scattering is:



tree-level

+



1-loop

+ ...

$\updownarrow$

$\boxed{\text{UV } \underline{\underline{\text{finite}}}}$

Intuitive reason: Absence of pointlike interaction vertex

(We will calculate this & see where UV finiteness technically comes from!)

Furthermore we will see that the

* open string sector <u>predicts</u> existence of gauge interactions of particle physics at low energies
* closed string sector <u>predicts</u> GR at low energies.

Thus the theory is a candidate for a fundamental, unified theory of Quantum Gravity that is

* UV finite — at least perturbatively
* free of <u>dimensionless</u> input parameters. Only free parameter is M_s, which sets the scale of the theory:

$$\underbrace{7\ \text{TeV} \lesssim M_s}_{\text{experimentally}} \lesssim \underbrace{M_{pl} \cong 10^{19}\ \text{GeV}}_{\text{theoretical considerations}}$$

II. The classical bosonic string

II.1 The bosonic string action

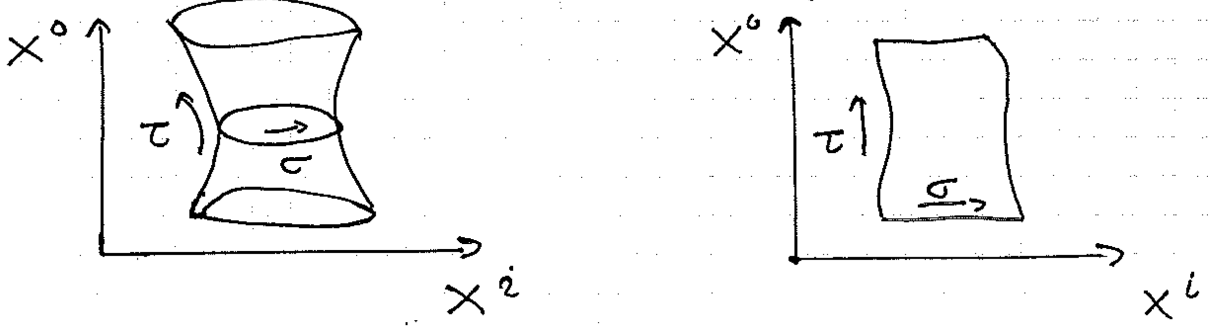
II.1.a) Nambu-Goto action

A string is a 1-dimensional object. There are 2 possible topologies:

closed | open

$0 \leq \sigma \leq \ell$

- The propagation of a string in $\mathbb{R}^{1,d-1}$ now defines a 2-dimensional <u>worldsheet (WS) Σ</u>



Σ parametrised by $\tau \in \mathbb{R}$ and $0 \leq \sigma \leq \ell$

$$\Sigma : \quad (\tau,\sigma) \longmapsto X^\mu(\tau,\sigma) \subset \mathbb{R}^{1,d-1}$$

This map is called the <u>string map</u>. It provides an embedding of the WS into the ambient space.

Notation: oftentimes $(\tau,\sigma) \equiv \xi^a \qquad a = 0,1$

- In complete analogy to point particle one defines the <u>Nambu-Goto action</u>

$$\boxed{S_{NG} = -T \int_\Sigma dA}$$

where dA is the area element of Σ:

$$dA = \left(-\det\left\{\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}\right\}\right)^{1/2} d^2\xi$$

(Note: For point-particle: $ds = \sqrt{-\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\eta_{\mu\nu}}\, d\tau$

Convention: $\eta_{\mu\nu} = (-1, +1, \dots, +1)$!

X^μ and ξ^a have mass dimension (-1)

$$\Rightarrow [T] = [\text{mass}^2] = [\text{length}^{-2}]$$

T := string tension

Def.: $\boxed{T = \frac{1}{2\pi\alpha'}}$ $\quad [\sqrt{\alpha'}] = [\text{length}]$

α' : Regge slope

Def.: $\boxed{l_s = 2\pi\sqrt{\alpha'}}$: string length

Def : $M_s = (\alpha')^{-1/2}$: string (mass) scale

II. 2. b) The Polyakov action

It is convenient to eliminate the square root in S_{NG}.

We introduce an auxiliary field, the WS metric $h_{ab}(\xi^a)$ and define the Brink – Di Vecchia – Howe (BDH) or Polyakov action

$$\boxed{\begin{aligned} S_P &= -\frac{T}{2} \int_\Sigma d^2\xi \sqrt{-\det h}\; h^{ab}(\xi)\, \underbrace{\partial_a X^\mu(\xi)\, \partial_b X^\nu(\xi)\, \eta_{\mu\nu}}_{\shortparallel} \\ &= -\frac{T}{2} \int_\Sigma d^2\xi \sqrt{-\det h}\; h^{ab} \cdot G_{ab} \end{aligned}}$$

Comments:

- $h_{ab}(\xi^a)$ is a 2-tensor on the WS : $h_{ab} = h_{ba}$
- $X^\mu(\xi^a)$ is a scalar on WS (no "a,b" index) but a spacetime vector ("μ" index)

- The metric h has signature $(-,+)$ $\Rightarrow$ overall minus in S_P

$\Rightarrow$ S_P describes d two-dim. scalar fields $X^\mu(\xi^a)$ coupled to the dynamical WS metric $h_{ab}(\xi^a)$

$\equiv$ 2-dim. gravity coupled to scalars

Energy momentum tensor

The energy momentum tensor T_{ab} measures the response of S_p w.r.t. a metric variation δh_{ab}:

$$\delta S_p = \frac{1}{4\pi} \int d^2\xi \sqrt{-h}\; T_{ab}\; \delta h^{ab}$$

i.e. $$\boxed{T_{ab} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S_p}{\delta h^{ab}}}$$ $(h \equiv \det h_{ab})$

Using $\delta h = -h_{ab} (\delta h^{ab}) \cdot h$

this yields:

$$\boxed{T_{ab} = -\frac{1}{\alpha'} \left(G_{ab} - \frac{1}{2} h_{ab}\, h^{cd} G_{cd} \right)}$$

where $G_{ab} = \partial_a X \circ \partial_b X$

The e.o.m. for h_{ab} are $\boxed{T_{ab} = 0}$

$$\Rightarrow \boxed{G_{ab} = \frac{1}{2} (h \circ G)\, h_{ab}}$$ by e.o.m. for h_{ab}

One finds that on-shell for h_{ab}, S_p and S_{NG} agree:

$$S_p\left[X, h_{ab}\big|_{T_{ab}=0}\right] = S_{NG}$$

II.2 c) Symmetries of the Polyakov action

Distinguish: · spacetime symmetries acting on $\mathbb{R}^{1,d-1}$

· WS symmetries acting on Σ

i) spacetime symmetries

manifest d-dimensional Poincaré-Invariance

$$X^\mu(\xi) \longmapsto \Lambda^\mu{}_\nu X^\nu(\xi) + V^\mu$$

$$\text{for} \qquad \Lambda \in SO(1,d-1)$$

Poincaré invariance becomes a global internal symmetry of the 2-dim. field theory

ii) WS symmetries

1. Local diffeomorphism invariance under

$$\xi^a \longmapsto \tilde{\xi}^a(\xi) = \xi^a - \varepsilon^a(\xi) \quad + \mathcal{O}(\varepsilon^2)$$

The various fields transform according to their tensorial nature

* $X^\mu(\xi)$: scalar field from WS perspective

$$\begin{aligned} X^\mu(\xi) \longmapsto \tilde{X}^\mu(\tilde{\xi}) &:= X^\mu(\xi(\tilde{\xi})) \\ &= X^\mu(\xi) + \varepsilon^c \partial_c X^\mu(\xi) \\ &\quad + \mathcal{O}(\varepsilon^2) \\ &=: X^\mu(\xi) + \delta X^\mu(\xi) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

i.e. $\boxed{\delta X^\mu = \varepsilon^c \partial_c X^\mu}$ to linear order in ε

2 <u>Local conformal</u> = Weyl invariance

$$\boxed{\delta X^\mu = 0 \quad , \quad \delta h_{ab} = 2\,\Lambda(\xi)\, h_{ab}}$$

<u>Crucial</u>: This is special to the fact that WS is in $D=2$

For general D dimensional theory: $\quad h_{ab} \mapsto \Omega \cdot h_{ab}$

$$\Rightarrow \sqrt{-\det h} \longmapsto (\Omega)^{D/2} \sqrt{-\det h}$$

$$h^{ab} \longmapsto (\Omega)^{-1}\, h^{ab}$$

$$\Rightarrow \int d^D\xi \sqrt{-\det h}\; h^{ab} G_{ab} \text{ invariant iff } D=2$$

> This Weyl invariance will become pivotal for a consistent quantisation.

$\Rightarrow$ String Theory is a special (unique ??) generalisation of point particle theory. Higher dim. analogues (membranes) hard (impossible ??) to quantise.

II.2.d) Flat worldsheet coordinates

The local diffeomorphism and Weyl invariance can be used to "fix the metric" h_{ab} by locally gauging away all its parameters. This can be seen by counting degrees of freedom:

For a D-dimensional WS:

$$\left.\begin{array}{ll} h_{ab} : & \frac{1}{2} D(D+1) \text{ d.o.f.} \\ \text{diffeomorph.} + \text{Weyl}: & D+1 \text{ d.o.f.} \end{array}\right\} \begin{array}{l} \frac{D}{2}(D+1) - (D+1) \\ \text{d.o.f. cannot be} \\ \text{gauged away locally} \end{array}$$

$D = 2 \implies$ All parameters can locally be gauged away.

$\implies$ Locally metric can be made flat $\quad R = 0$

The diffeom. invariance can be used to achieve

$$h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For now we work in flat gauge: $h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\implies \boxed{S_p = \frac{T}{2} \int d\tau\, d\sigma \left((\partial_\tau X)^2 - (\partial_\sigma X)^2 \right)}$$

Light cone coordinates: $\boxed{\xi^+ = \tau + \sigma \quad , \quad \xi^- = \tau - \sigma}$

$$ds^2 = -d\tau^2 + d\sigma^2 = -d\xi^+ d\xi^-$$

$$\Longrightarrow h_{++} = 0 = h_{--} \; ; \quad h_{+-} = h_{-+} = -\frac{1}{2}$$

$$h^{+-} = h^{-+} = -2$$

$$\Longrightarrow V_+ = h_{+-} V^- = -\frac{1}{2} V^- \quad \text{etc.}$$

$$\partial_\pm = \frac{\partial}{\partial \xi^\pm} = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right)$$

$$d\tau\, d\sigma = d\xi^+ d\xi^- \det \frac{\partial(\sigma,\tau)}{\partial(\xi^+,\xi^-)} = \frac{1}{2} d\xi^+ d\xi^-$$

S_P in Lightcone gauge (LCG)

$$\boxed{S_P = +T \int d^2\xi^\pm \; \partial_+ X^\mu \partial_- X^\nu \eta_{\mu\nu}}$$

Energy momentum in LCG:

$$T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X \cdot \partial_b X - \frac{1}{2} h_{ab} h^{cd} \partial_c X \cdot \partial_d X\right)$$

$$\Longrightarrow \bullet \; \boxed{T_{+-} \equiv 0 \equiv T_{-+}}$$

$$\bullet \; T_{++} = -\frac{1}{\alpha'} \partial_+ X \cdot \partial_+ X$$

$$T_{--} = -\frac{1}{\alpha'} \partial_- X \cdot \partial_- X$$

Crucial: Before going to flat gauge, the e.o.m. for h_{ab} gave $T_{ab} \overset{!}{=} 0$.

This condition has to be implemented <u>as a constraint</u> in flat coordinates

$$\boxed{T_{++} \overset{!}{=} 0 \overset{!}{=} T_{--}}$$

(will be important upon quantisation)

Even after gauge fixing there is left a <u>large residual symmetry</u>.

The generators are the conformal Killing vectors:

Conformal Killing transformation (CKT)

$$\boxed{\xi^+ \mapsto \tilde{\xi}^+(\xi^+) = \xi^+ - \varepsilon^+(\xi^+) \qquad \xi^- \mapsto \tilde{\xi}^-(\xi^-) = \xi^- - \varepsilon^-(\xi^-)}$$

Since its effect on the metric can be undone by a Weyl trafo, it is not fixed by $h_{ab} = \eta_{ab}$ $\Longrightarrow$ <u>residual symmetry</u>!

II.3 Oscillator expansions

II.3.a) Equations of motion & boundary conditions

$$S_p = 2T \int d\tau d\sigma \; \partial_+ X \cdot \partial_- X$$
$$= \frac{T}{2} \int d\tau d\sigma \left((\partial_\tau X)^2 - (\partial_\sigma X)^2 \right)$$

To obtain the e.o.m. for X^μ we vary S_p s.t. $\delta X^\mu(\tau = -\infty)$ and $\delta X^\mu(\tau = +\infty)$ vanish

$$\delta S_p = \frac{T}{2} \int d\tau d\sigma \left[+ 2\, \partial_\tau X \; \partial_\tau \delta X - 2\, \partial_\sigma X \; \partial_\sigma \delta X \right]$$
$$= -T \int d\tau d\sigma \left[(\partial_\tau^2 - \partial_\sigma^2) X \cdot \delta X \right]$$
$$+ T \int d\sigma \; \partial_\tau X \cdot \delta X \Big|_{\tau = -\infty}^{+\infty}$$
$$- T \int d\tau \; \partial_\sigma X \cdot \delta X \Big|_{\sigma = 0}^{\ell}$$

- The τ-boundary term vanishes by assumption $\delta X \Big|_{\tau = -\infty}^{\tau + \infty} = 0$
- If also σ-boundary term vanishes, then the e.o.m. are:

$$\left.\boxed{\begin{aligned} &(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0 \\ \Leftrightarrow\; &\partial_+ \partial_- X^\mu = 0 \end{aligned}}\right\} \text{Free wave equation}$$

We now impose boundary conditions such that this is the case:

i) Closed string:

Boundary terms at $\sigma = 0$ & $\sigma = \ell$ cancel each other.
Most generally, this is the case for

$$X^\mu(\tau, \sigma = 0) = M^\mu{}_\nu X^\nu(\tau, \sigma = \ell)$$

for M an $O(1, d-1)$ matrix.

However, if we want to interpret $X^\mu(\tau,\sigma)$ as string embedding coordinates & preserve Poincaré invariance in $\mathbb{R}^{1,d-1}$, then we need:

$$\boxed{X^\mu(\tau, \sigma = 0) = X^\mu(\tau, \sigma = \ell) \qquad \text{periodic boundary cond.}}$$

ii) Open string:

Boundary terms vanish at $\sigma = 0$ & $\sigma = \ell$ separately.
For each X^μ and for each $\sigma = 0, \ell$ we can impose either

a) Neumann b.c. $\qquad \partial_\sigma X^\mu \Big|_{\substack{\sigma = 0 \text{ and/or} \\ \sigma = \ell}} = 0$

b) Dirichlet b.c. $\qquad \delta X^\mu \Big|_{\substack{\sigma = 0 \text{ and/or} \\ \sigma = \ell}} = 0 \quad \longleftrightarrow$ string endpoint fixed in μ-direction

$\Rightarrow$ For each μ can have (NN), (DD), (ND), (D,N)

II. 3 b) Closed string expansion

$$\partial_+ \partial_- X^\mu = 0 \quad \longrightarrow \quad X^\mu = X^\mu_L(\xi^+) + X^\mu_R(\xi^-)$$

subject to $\quad X^\mu(\tau,\sigma) = X^\mu(\tau, \sigma + \ell) \qquad (*)$

Solution:

* The most general solution for $\partial_+ X^\mu_L(\xi^+)$ & $\partial_- X^\mu_L(\xi^-)$ subject to $(*)$ is a Fourier expansion

$$\boxed{\begin{aligned} \partial_- X^\mu_R(\xi^-) &= \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha^\mu_n \, e^{-\frac{2\pi}{\ell} i n (\tau-\sigma)} \\ \partial_+ X^\mu_L(\xi^+) &= \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}^\mu_n \, e^{-\frac{2\pi}{\ell} i n (\tau+\sigma)} \end{aligned}}$$

* Integration gives

$$\boxed{\begin{aligned} X^\mu(\tau,\sigma) = x^\mu &+ \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}}\, \alpha^\mu_0 (\tau-\sigma) + \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}}\, \tilde{\alpha}^\mu_0(\tau+\sigma) \\ &+ i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\frac{\alpha^\mu_n}{n} e^{-\frac{2\pi i}{\ell} n(\tau-\sigma)} + \frac{\tilde{\alpha}^\mu_n}{n} e^{-\frac{2\pi i}{\ell} n(\tau+\sigma)} \right) \end{aligned}}$$

* Periodicity requires: $\boxed{\alpha^\mu_0 \overset{!}{=} \tilde{\alpha}^\mu_0 \equiv \sqrt{\frac{\alpha'}{2}}\, p^\mu}$
  but $\alpha^\mu_n, \tilde{\alpha}^\mu_n$ for $n \neq 0$ are independent left/right Fourier modes

* Reality $X^\mu(\tau,\sigma) = \left(X^\mu(\tau,\sigma)\right)^*$ implies

$$x^\mu = x^{\mu *}, \quad p^\mu = p^{\mu *} \quad (\alpha^\mu_n)^* = \alpha^\mu_{-n}$$

* Conjugate momentum density:

$$\boxed{\Pi^\mu(\tau,\sigma) := \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu(\tau,\sigma)} = T\,\dot{X}^\mu(\tau,\sigma)}$$

Interpretation:

- p^μ is the total <u>spacetime momentum</u> of string.
  Namely:
$$P^\mu = \int_0^\ell d\sigma\, \Pi^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'} \int_0^\ell d\sigma\, \dot{X}^\mu = p^\mu$$

- x^μ is the <u>center-of-mass position</u> of string at $\tau = 0$
$$q^\mu = \frac{1}{\ell} \int_0^\ell d\sigma\, X^\mu = x^\mu + \frac{2\pi\alpha'}{\ell}\, p^\mu \cdot \tau$$

- The <u>total angular momentum</u> is
$$J^{\mu\nu} = \int_0^\ell d\sigma\, \left(X^\mu \Pi^\nu - X^\nu \Pi^\mu \right) = \ell^{\mu\nu} + E^{\mu\nu} + \bar{E}^{\mu\nu}$$

$$\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \qquad \text{c.o.m.} - \text{piece}$$

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right)$$

II 3. c) Open string expansion

i) **Neumann b.c. at both ends $\sigma = 0$ & $\sigma = \ell$:**

$$X'^{\mu}(\tau,\sigma) = 0 \quad \text{for} \quad \sigma = 0 \text{ and } \sigma = \ell \qquad (*)$$

* It is convenient to extend the domain of σ to $\sigma = [-\ell, \ell]$ by defining:

$$\hat{X}^{\mu}(\tau,\sigma) := \begin{cases} X^{\mu}(\tau,\sigma) & \sigma \in [0,\ell] \\ X^{\mu}(\tau,-\sigma) & \sigma \in [-\ell,0] \end{cases}$$

Requiring that $\hat{X}^{\mu}(\tau,\sigma)$ be smooth and periodic w/ periodicity 2ℓ is equivalent to $(*)$ for X^{μ}

* Most general solution for $\partial_{\pm}\hat{X}^{\mu}(\tau,\sigma)$ by Fourier expansion with α_n^{μ} & $\tilde{\alpha}_n^{\mu}$.

* $0 \overset{!}{=} \partial_\sigma X^{\mu}(\tau,\sigma)\Big|_{\sigma=0,\ell} = (\partial_+ - \partial_-)X^{\mu}(\tau,\sigma)\Big|_{\sigma=0,\ell} \Longrightarrow \boxed{\alpha_n^{\mu} \overset{!}{=} \tilde{\alpha}_n^{\mu}}$

* Final solution:

(NN) • $$X^{\mu}(\tau,\sigma) = x^{\mu} + \frac{2\pi\alpha'}{\ell} p^{\mu}\tau + i\sqrt{2\alpha'} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \frac{1}{n}\alpha_n^{\mu} e^{-i\frac{\pi}{\ell}n\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right)$$

• $$\boxed{\partial_{\pm}X^{\mu}(\tau,\sigma) = \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^{\mu} e^{-i\pi/\ell \, n(\tau\pm\sigma)}}$$

$$\boxed{\alpha_0^{\mu} = \sqrt{2\alpha'}\, p^{\mu}} \qquad (\alpha_n^{\mu})^* = \alpha_{-n}^{\mu}$$

Interpretation:

- String can move freely (i.e. endpoints $\sigma = 0, \ell$ free) but NN b.c. implement that there is no momentum flow off the string:

Recall that $P^\alpha_\mu = -T\sqrt{-h}\,\partial^\alpha X_\mu$, i.e.

$$P^\sigma_\mu \Big|_{\sigma=0,\ell} = -T(X')^\mu\Big|_{\sigma=0,\ell} = 0 \quad \text{iff} \quad X'^\mu\Big|_{\sigma=0,\ell} = 0$$

- x^μ, p^μ are the c.o.m. position & momentum
- $$J^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^\ell d\sigma\,\left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right)$$

$$= \ell^{\mu\nu} + E^{\mu\nu} \quad \text{as before}$$

- $$\{x^\mu, p^\nu\} = \eta^{\mu\nu}$$

$$\{\alpha^\mu_m, \alpha^\nu_n\} = -i m\, \delta_{m+n,0}\, \eta^{\mu\nu}$$

- Hamiltonian: $$\boxed{H = \frac{\pi}{2\ell}\sum_{n=-\infty}^{\infty}\left(\alpha_{-n}\cdot\alpha_n\right)}$$

iii) Dirichlet b.c. at both ends:

$$\delta X^\mu = 0 \qquad \sigma = 0, \ell$$, i.e. strings endpoints are fixed

$$\Rightarrow \dot{X}^\mu = 0 \qquad \sigma = 0, \ell$$

General solution:

$$X^\mu(\tau, \sigma = 0) = x_0^\mu, \qquad X^\mu(\tau, \sigma = \ell) = x_1^\mu$$

vanishing momentum $p^\mu = 0$

Solution

$$(DD) \quad X^\mu(\tau,\sigma) = x_0^\mu + \frac{1}{\ell}(x_1^\mu - x_0^\mu)\sigma + \sqrt{2\alpha'} \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{\ell} n\tau} \sin\left(\frac{\pi n \sigma}{\ell}\right)$$

Def.: $$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}} \frac{1}{\pi} (x_1^\mu - x_0^\mu)$$

$$\boxed{\partial_\pm X^\mu = \pm \frac{\pi}{\ell} \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-i\frac{\pi n}{\ell}(\tau \pm \sigma)}}$$

$\Rightarrow$ c.o.m. position:

$$q^\mu = \frac{1}{\ell} \int_0^\ell d\sigma \, X^\mu(\tau,\sigma) = \frac{1}{2}(x_0^\mu + x_1^\mu)$$

The Hamiltonian acquires an extra contribution from the potential energy of the stretched string:

$$\boxed{H = \frac{T}{2\ell}(x_1^\mu - x_0^\mu)^2 + \frac{\pi}{2\ell} \sum_{n \neq 0} \alpha_{-n} \cdot \alpha_n}$$

iii) Mixed (DN) : $X'^\mu = 0 \quad \sigma = 0$
$\dot{X}^\mu = 0 \quad \sigma = \ell$

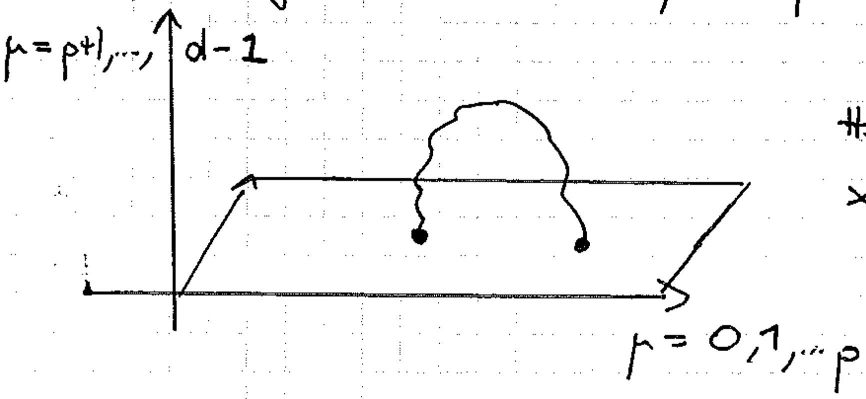
$$\boxed{(ND) \quad X^\mu(\tau,\sigma) = x^\mu + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{\ell}n\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right)}$$

Note: · Fourier modes are half-integer $n \in \mathbb{Z}+\frac{1}{2}$
· c.o.m. momentum vanishes
· x^μ : position at $\sigma = \ell$

II.3.d) The concept of D-branes

Consider open string in $\mathbb{R}^{1,d-1}$: In each dimension we can choose (NN), (DD) or (DN) b.c. independently.

Example: (NN) for X^μ $\mu = 0,\dots,p$
(DD) for X^μ $\mu = p+1,\dots,d-1$



Here: $x_0^\mu = x_1^\mu$ for (DD)

Open string moves freely along a $(p+1)$-dim. hyper-surface & is fixed in the directions normal to it.

This hypersurface is called a Dp-brane.

A Dp-brane is by itself a dynamical object :
There is non-zero momentum flow off the string in the (DD) directions normal to it. Total momentum conservation implies momentum exchange between the string & the Dp-brane.

(Note: Dp-brane only breaks translational invariance spontaneously.)

The dynamics of the D-brane is described by an analogue of the Nambu-Goto action and will be studied later. We will find that the tension scales like $\frac{1}{g_s}$ with g_s: string coupling.

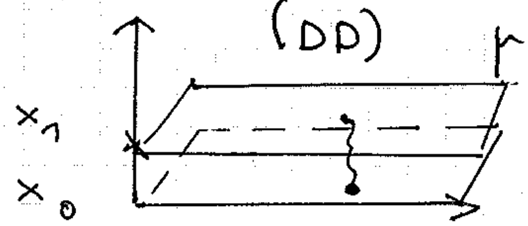
$\Longleftrightarrow$ D-branes are <u>non-perturbative</u> objects.

<u>More general brane configurations</u> include:

* parallel branes:

e.g. (NN) $\mu = 0, \dots, p-1, p$

(DD) $\mu = p+1, \dots, d-1$



$x_0^\mu \neq x_1^\mu$

* branes of different dimension:

Dp-brane & Dq-brane $\Longleftrightarrow$ (DN) strings

* branes at angles (later)

Special case: (NN) in all d dimensions

D(d-1)-brane filling all of $\mathbb{R}^{1,d-1}$

III) Quantisation of the bosonic string

There are ≥ 3 ways to quantise the bosonic string:

- Old covariant quantisation (OCQ):
  Virasoro constraints implemented at quantum level
  → manifestly Lorentz invariant
  → unitarity only in "critical" dimension
- Light-cone quantisation (LCQ):
  constraints implemented before quantisation
  → manifestly unitary
  → Lorentz invariance only in "critical" dimension
- Path-integral quantisation
  uses Faddeev-Popov gauge fixing
  BRST algebra satisfied in critical dimension

III.1) Commutation relations

In canonical quantisation, the classical fields $X^\mu(\tau,\sigma)$ & their canonical momenta $\Pi^\mu(\tau,\sigma)$ are promoted to operators $\hat{X}^\mu(\tau,\sigma)$, $\hat{\Pi}^\mu(\tau,\sigma)$ and

$$\{\ ,\ \}_{P.B.} \longrightarrow \frac{1}{i}[\ ,\ ]$$

$$\boxed{\begin{aligned}[\hat{X}^\mu(\tau,\sigma),\hat{\Pi}^\nu(\tau,\sigma')] &= i\eta^{\mu\nu}\,\delta(\sigma-\sigma') \\ [\hat{X}^\mu(\tau,\sigma),\hat{X}^\nu(\tau,\sigma')] = 0 &= [\hat{\Pi}^\mu(\tau,\sigma),\hat{\Pi}^\nu(\tau,\sigma')]\end{aligned}}$$

Also the oscillator modes are promoted to operators w/ commutation relations

$$\boxed{\begin{aligned} &[x^\mu, p^\nu] = i\eta^{\mu\nu} \\ &[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] \\ &[\tilde{\alpha}_m^\mu, \alpha_n^\nu] = 0 \end{aligned}}$$

Hermiticity $\hat{x}^\mu = (\hat{x}^\mu)^\dagger$, $\hat{\pi}^\mu = (\hat{\pi}^\mu)^\dagger$ implies

$$\alpha_m^\mu = (\alpha_{-m}^\mu)^\dagger$$

As always products of non-commuting operators are "ambiguous" in going from classical to quantum.

Def.: Normal ordering

$$:\alpha_m \alpha_k: = \begin{cases} \alpha_m \alpha_k & \text{if } m \leq k \\ \alpha_k \alpha_m & \text{if } k < m \end{cases}$$

Compare: Harmonic oscillator:

$[a, a^\dagger] = 1 \rightarrow$ identify $\forall m$: $\alpha_{+|m|}$

$\rightarrow$ Identify $\forall m$:

$$\boxed{\begin{aligned} &\alpha_{|m|}^i \longleftrightarrow \text{annihilation operator} \\ &\alpha_{-|m|}^i \longrightarrow \text{creation operator} \end{aligned}}$$

$\rightarrow$ Fock space: $|0;p\rangle$ ground state (no oscillators) w/ 4-momentum p^μ

defined by: $\alpha_m^i |0;p\rangle = 0 \;\; \forall m > 0$

- $\alpha_{-1}^i |0,p\rangle$: first excited level
- $\alpha_{-2}^i |0,p\rangle$, $\alpha_{-1}^i \alpha_{-1}^i |0,p\rangle$ etc. 2nd excited level

Qu: What about α_m^0 (wrong sign)?

What is the mass of these states?

III.2) Light-cone quantisation (LCQ)

III.2.a) Light-cone gauge

This scheme works with the flat WS metric $h_{ab} = \eta_{ab}$, but uses the residual reparametrisation invariance (conformal symmetry)

$$\xi^+ \mapsto \tilde{\xi}^+(\xi^+), \quad \xi^- \mapsto \tilde{\xi}^-(\xi^-) \qquad (*)$$

to implement the Virasoro constraints $T_{ab} = 0$ before quantisation.

First step: Light-cone coordinates for spacetime

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{d-1}\right), \quad X^i \qquad i = 1, \dots, d-2$$

$\Rightarrow$ metric : $\eta_{+-} = -1 = \eta_{-+}, \quad \eta_{ij} = 1$

$$X \cdot X = -2X^+X^- + \underbrace{X^iX^i}_{\equiv (X_\perp)^2} \quad (\text{sum over } i)$$

Second step: Use $(*)$ to transform (τ, σ) as

$$\tilde{\tau} = \frac{1}{2}\left(\tilde{\xi}^+(\xi^+) + \tilde{\xi}^-(\xi^-)\right)$$

$$\tilde{\sigma} = \frac{1}{2}\left(\tilde{\xi}^+(\xi^+) - \tilde{\xi}^-(\xi^-)\right)$$

$\tilde{\tau}$ satisfies the free wave equation

$$\partial_+\partial_-\tilde{\tau} = 0$$

$\Leftrightarrow$ same equation as $X^\mu(\tau,\sigma)$: $\partial_+\partial_- X^\mu(\tau,\sigma) = 0$

$\Rightarrow$ One can use $(*)$ to identify $\tilde{\tau}$ with one of the X^μ.

We choose $X^+(\tau,\sigma)$:

$$\tilde{\tau} = \frac{\ell}{2\pi\alpha'}\frac{1}{p^+}X^+ - \underbrace{x^+}_{\text{integration constant}}$$

Now relabel $(\tilde{\tau},\tilde{\sigma}) \to (\tau,\sigma)$,

i.e. $\boxed{X^+(\tau,\sigma) = \frac{2\pi\alpha'}{\ell} p^+ \tau + x^+}$ $(*)$

We have effectively used the infinite dimensional conformal symmetry $(*)$ to set $\alpha_m^+ = 0 \ \forall m \neq 0$!

<u>Note</u>: This procedure is not manifestly Lorentz invariant as it singles out X^+.

Advantage of light-cone gauge (LCG)

We can now solve explicitly for Virasoro constraints :

$\boxed{T_{ab} = 0 \iff (\dot{X} \pm X')^2 = 0}$ (Recall: $T_{\pm\pm} = -\frac{1}{\alpha'}\partial_\pm X \cdot \partial_\pm X$)

In LQG:

$$0 \stackrel{!}{=} -2(\dot{X} \pm X')^+(\dot{X} \pm X')^- + (\dot{X} \pm X')^{i}(\dot{X} \pm X')^{i}$$

using $(*)$ from above $\Rightarrow \boxed{(\dot{X} \pm X')^- = \frac{1}{2}\cdot\frac{\ell}{2\pi\alpha'}\frac{1}{p^+}(\dot{X} \pm X')^2_\perp}$ $(**)$

$\Rightarrow$ Only the transverse d.o.f. are independent, X^+ and X^- are determined by them.

Consider e.g. open string w/ (NN) b.c. $\forall \mu$:

Evaluate $(**)$ to find :

$$\boxed{\alpha^-_n = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\left(\frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha^i_{n-m}\,\alpha^i_m\right)} \quad \text{(classically)}$$

$\forall\, n \in \mathbb{Z}$ including $\alpha^-_0 = \sqrt{2\alpha'}\,p^-$

from difference of 2 constraints in $(**)$

Similarly for closed strings :

$$\boxed{\overset{(\sim)}{\alpha}{}^-_n = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\left(\sum_{m=-\infty}^{\infty}\overset{(\sim)}{\alpha}{}^i_{n-m}\,\overset{(\sim)}{\alpha}{}^i_m\right)} \quad \text{and} \quad \boxed{N_\perp = \tilde{N}_\perp}$$

i.e. $\alpha^-_0 = \tilde{\alpha}^-_0$

III.2.b) Quantisation in LCG

$$\boxed{\begin{aligned} [\alpha^i_m, \alpha^j_n] &= m\,\delta_{m+n}\,\delta^{ij} \qquad \text{etc.} \\ [p^+, q^-] &= i \end{aligned}}$$

The classical expression for α^-_n has to be modified by means of a normal ordering constant :

For $n=0$: $\frac{1}{2}\sum_{m=-\infty}^{+\infty}\alpha^i_{-m}\alpha^i_m = \frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\alpha^i_m: - a = \frac{1}{2}(\alpha^i_0)^2 + \sum_{m=1}^{\infty}\alpha^i_{-m}\alpha^i_m - a$

Normal ordering constant: a : will be determined by consistency of the quantum theory

Open (NN) :

$$\boxed{\alpha_n^- = \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \left(\frac{1}{2} \sum_{m=-\infty}^{\infty} :\alpha^i_{n-m} \alpha^i_m : - a \, \delta_{n,0} \right)}$$

The <u>mass shell condition</u> is :

$$M^2 = -p^2 = 2 p^+ p^- - p^i p^i$$

where:
$$p^\mu = \frac{1}{\sqrt{2\alpha'}} \alpha_0^\mu \qquad \text{(open, NN)}$$
$$p^\mu = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_0^\mu \qquad \text{(closed)}$$

a) <u>open, NN $\forall \mu$:</u>

$$M^2_{op} = \frac{1}{2\alpha'} \left(2 \cdot \sum_{m=1}^{\infty} :\alpha^i_{-m} \alpha^i_m : + \alpha_0^i \alpha_0^i - 2a - \right.$$
$$\left. - \alpha_0^i \alpha_0^i \right)$$

$$\boxed{M^2_{op} = \frac{1}{\alpha'} (N_\perp - a) \qquad N_\perp = \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n}$$

b) <u>closed</u>

$$\boxed{M^2_{op} = \frac{2}{\alpha'} (N_\perp + \tilde{N}_\perp - 2a) \quad \text{w/ } N_\perp = \tilde{N}_\perp \ (\text{"level-matching"})}$$

<u>Note:</u> One can generalise the above expressions for open strings with b. c.

(NN) : X^+, X^-, X^k

(DD) : X^ℓ

DN/ND: X^a

$$\boxed{\begin{aligned} M^2_{cl.} &= \frac{1}{\alpha'}\left(N_\perp + \alpha' (T \Delta x)^2 - a \right) \\ N_\perp &= \sum_{n=1}^{\infty} \left(\alpha^k_{-n} \alpha^k_n + \alpha^l_{-n} \alpha^l_n \right) + \sum_{r \in \mathbb{N}_0 + 1/2} \alpha^a_{-r} \alpha^a_r \end{aligned}}$$

In LCQ the critical values for a and the number of spacetime dimensions follows by the requirement of Lorentz invariance:

Recall: Generators of Lorentz transformations (open NN)

$$J^{\mu\nu} = T \int_0^l (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)\, d\sigma$$

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} \qquad \mu \neq \nu$$

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right)$$

Note: No normal ordering ambiguity since $\mu \neq \nu$

These should satisfy the Lorentz algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho} J^{\nu\sigma} + i\eta^{\nu\sigma} J^{\mu\rho} - i\eta^{\mu\sigma} J^{\nu\rho} - i\eta^{\nu\rho} J^{\mu\sigma}$$

The transformation J^{i-} mixes $X^{\pm}$ with X^{i}.
Since X^{+}/X^{-} are singled out in LCG, a breakdown of Lorentz invariance might show up in an anomaly of the algebra satisfied by J^{-i}.

In absence of an anomaly we should get

$$[J^{-i}, J^{-j}] = i\underbrace{\eta^{--}}_{=0} J^{ij} + i\eta^{ij} \underbrace{J^{--}}_{=0} - i\underbrace{\eta^{-j}}_{=0} J^{i-} - i\underbrace{\eta^{i-}}_{=0} J^{-j} = 0$$

However explicit, long and painful computation reveals:
(for open NN $\forall\mu$):

$$[J^{-i}, J^{-j}] = -\frac{1}{(p^{+})^{2}} \sum_{m=1}^{\infty} \Delta_m \left(\alpha^{i}_{-m}\alpha^{j}_{m} - \alpha^{j}_{-m}\alpha^{i}_{m} \right)$$

with $\Delta_m = m \cdot \frac{26-d}{12} + \frac{1}{m}\left(\frac{d-26}{12} + 2(1-a) \right)$

$$\boxed{\Rightarrow [J^{-i}, J^{-j}] = 0 \iff d = 26,\ a = 1}$$

A similar computation fixes $d=26$, $a=1$ for closed string; $d=26$ also holds in presence of lower-dimensional branes (i.e. with sectors NN, DD or (DN) with the correct value of "a" determined later)

III.2.c) Open string spectrum along D-branes

Consider first (NN) b.c. $\forall\ X^+, X^-, X^i$: $d=26$

General fact:

A state in d dimensions forms an irreducible representation under the subgroup of $SO(1, d-1)$ that leaves its momentum invariant = "little group"

* $p^2 = 0$ $p = (\omega, \omega, 0)$ little group: $SO(d-2)$
* $p^2 < 0$ $p = (p, 0)$ little group: $SO(d-1)$

Indeed:

- tachyonic ground state $|0,p\rangle$: $\alpha^i_n |0,p\rangle = 0 \ \forall\, n>0$

$$M^2|0,p\rangle = \tfrac{1}{\alpha'}(N_\perp - a)|0,p\rangle = -\tfrac{1}{\alpha'}|0,p\rangle \quad \text{since } N_\perp|0,p\rangle = 0$$

$\Longrightarrow$ will be absent in final superstring theory

- massless transverse vector $\quad \zeta_i \alpha^i_{-1}|0,p\rangle \quad M^2 = 0$

$\Longrightarrow$ vector / fundamental repr. of $SO(24)$

$$\zeta_i \ \hat{=}\ \square \qquad \text{(vector repr.)}$$

- all higher states are massive & form complete irreducible repr. of $SO(25)$

e.g. 2nd level

$$\zeta_i \alpha^i_{-2}|0,p\rangle \ \&\ \zeta_{ij}\alpha^i_{-1}\alpha^j_{-1}|0,p\rangle, \quad M^2 = \tfrac{1}{\alpha'}$$

$$SO(24): \quad \underbrace{\square}_{24} \qquad \underbrace{\square\!\square}_{\text{symmetric traceless}} \ + \ 1$$

$$\#: \frac{24\cdot 25}{2} - 1$$

$\Longrightarrow$ combine into $\square\!\square$ of $SO(25)$: 324 d.o.f.

Regge trajectory

One finds an infinite tower of string excitations

$$\boxed{\alpha' M^2 = N - 1} \qquad N = 0, 1, 2, \dots,$$

$$\frac{1}{\sqrt{\alpha'}} = \frac{2\pi}{\ell_s} \qquad \text{natural "unit" of mass}$$

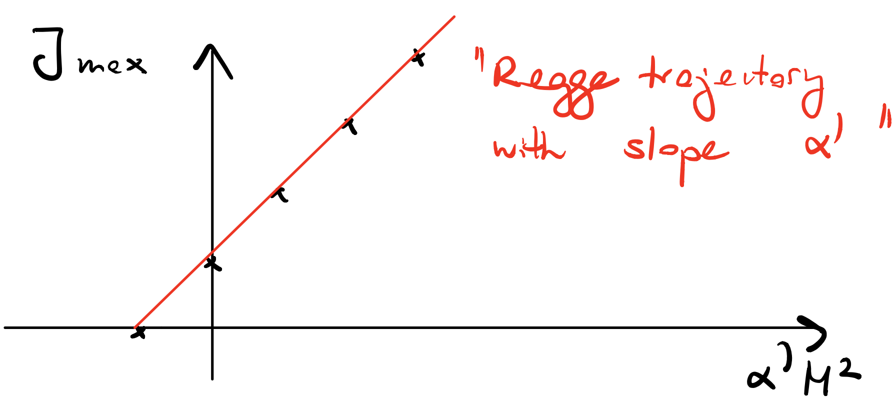
At level N, the spectrum contains a symmetric N-tensor

$$\zeta_{i_1 \dots i_N} \, \alpha_{-1}^{i_1} \dots \alpha_{-1}^{i_N} |0; p\rangle$$

of spin $J = N$.

This is in fact the maximal spin per mass level:

$$\boxed{J_{max} = \alpha' M^2 + 1} \quad : \text{ max. spin at mass level } M$$



Partition Function

The partition function is

$$Z(q) := \mathrm{tr}_{\mathcal{H}}\, q^{\hat{N}_\perp} \qquad \hat{N}_\perp \text{ number operator}$$

$$=: \sum_{N=0}^{\infty} d_N\, q^N$$

Result: $\boxed{Z(q) = q^2\, \eta(\tau)^{-24}}$ Dedekind η-function

where $q = e^{2\pi i \tau}$

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right)$$

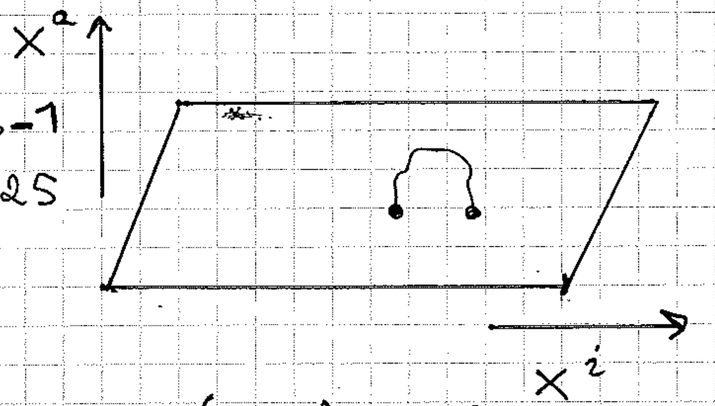
For large N: $\boxed{d_N = N^{-27/4}\, e^{4\pi\sqrt{N}} + \ldots \quad \text{(Hardy - Ramanujan)}}$

For more info & derivations: See Ex. # 6

Consider now the open string spectrum in presence of a Dp-brane within $\mathbb{R}^{1,25}$:

(NN) $i = +, -, 1, \dots, p-1$

(DD) $a = p+1, \dots, 25$



The light-cone coordinates $(+,-)$ must be along (NN) directions, i.e. $X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^p\right)$

$\Longrightarrow$ excitations along brane : $X^i_{\perp}$ $\quad i = 1, \dots, p-1$

normal excitations orthogonal : X^a $\quad a = p+1, \dots, 25$

- Ground state : $|0, p\rangle$ momentum in (NN) along the Dp-brane
- First level :

i) $\alpha^i_{-1} |0, p\rangle \qquad i = 1, \dots, p-1 \qquad M^2 = 0$

$\Rightarrow$ massless vector from perspective of Dp-brane propagating along the brane ($\square$ of $SO(p-1)$)
By general arguments of QFT this must be a gauge potential, as can be verified by computation of its interactions (later)

$\Rightarrow$ A single Dp-brane hosts a $U(1)$ gauge theory!

ii) $\alpha^a_{-1} |0, p\rangle \qquad a = p+1, \dots, 25 \qquad M^2 = 0$

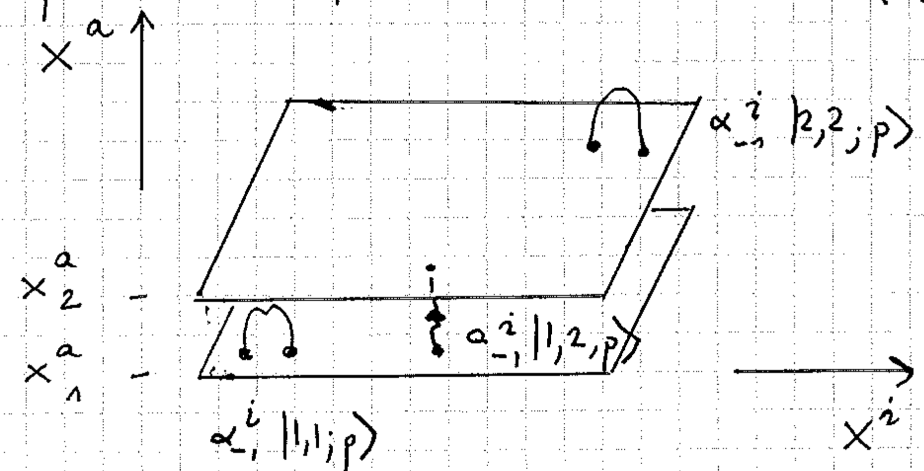
$\Rightarrow$ collection of $(24-p)$ massless scalar fields as seen from perspective of Dp-brane

i.e. $\forall$ a we have 1 excitation $\alpha_{-1}^{a}|0,p\rangle$ that does not carry an index "i" along Dp-brane.

Interpretation: These are the Goldstone bosons associated with spontaneous breaking of 26 dim. Poincaré invariance by Dp-brane.

Note: NN in all directions is special case: D25 brane filling $\mathbb{R}^{1,d-1}$

Consider two parallel Dp-branes at distance $(x_2^a - x_1^a)$



2 sets of massless vectors:

$\alpha_{-1}^{i}|11,p\rangle$: vector with both ends on brane 1 : $M_{11}^2=0$

$\alpha_{-1}^{i}|22,p\rangle$ both ends on brane 2 : $M_{22}^2=0$

(and same for scalars: $\alpha_{-1}^{a}|11,p\rangle$, $\alpha_{-1}^{a}|22,p\rangle$)

In addition: strings stretched between brane 1 & brane 2

$\alpha_{-1}^{i}|12,p\rangle$ (both orientations)

$\alpha_{-1}^{i}|21,p\rangle$

$$\alpha' M_{12}^2 = \alpha' M_{21}^2 = N - a + \alpha' (T \Delta x)^2$$

$$M_{12}^2 = M_{21}^2 = \frac{1}{2\pi\alpha'} \frac{(x_2^a - x_1^a)^2}{2\pi\alpha'} \Longrightarrow \text{massive vectors along brane 1 \& brane 2}$$

(same for scalars $\alpha_{-1}^a |12, p\rangle$, $\alpha_{-1}^a |21, p\rangle$)

Now consider limit $x_2^a \longrightarrow x_1^a$ by moving the 2 Dp-branes together:

We end up with 4 massless vectors

$$\alpha_{-1}^i \, | ij, p\rangle \qquad i, j = 1, 2$$

and 4 sets of massless scalars $\alpha_{-1}^a |ij, p\rangle$.

More generally for N coincident Dp-branes

N^2 massless vectors $\alpha_{-1}^i |ij, p\rangle \quad ij = 1, \ldots, N$

N^2 sets of massless scalars $\alpha_{-1}^a |ij, p\rangle$

The labels i, j are called Chan-Paton factors.

Take a basis $\lambda_{kl}^a \quad a = 1, \ldots, N^2$ of $(N \times N)$ matrices

$$|kl, p\rangle \equiv \lambda_{kl}^a \, |a, p\rangle$$

The appearance of N^2 massless vectors signals an enhancement of gauge symmetry to non-abelian gauge symmetry.

Observe: N^2 is the dimension of the adjoint representation of $U(N)$. $U(N) \ni M: \quad M^{+} = M^{-1}$

Indeed: Consistent gauge interactions if

$$\lambda^a = (\lambda^a)^+ \qquad a = 1, \dots, N^2 \qquad (\text{later})$$

i.e. λ^a span the Lie algebra of $U(N)$. $\left(M = e^{i \lambda^a m_a}\right)$

$$\boxed{\text{A stack of } N \text{ coincident Dp-branes "carries" a } U(N) \text{ gauge theory.}}$$

Note: Moving N Dp-branes apart breaks
$U(N) \longrightarrow U(1)^N$ because only N massless vector remain.

Indeed: Physics of Dp-branes "knows" a lot about gauge theories!

$\rightarrow$ Modern way to think about gauge theories is via "geometric engineering": Think of brane dynamics!

III. 2. d) Closed string spectrum

- Level matching: $N = \tilde{N}$, $a = \tilde{a}$
- $M_{cl}^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(N-1)$

- Ground state $\quad |0;p\rangle \qquad M^2 = -\frac{4}{\alpha'}$

- First level $N = \tilde{N} = 1$

$$\boxed{|\psi\rangle = \zeta_{ij}\, \alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j}\, |0;p\rangle \quad i,j = 1,\dots,24} \qquad M^2 = 0$$

Decompose square matrix / 2-tensor ζ_{ij} into irreducible representations of $SO(24)$

$$\zeta_{ij} = \underbrace{\zeta_{(ij)}}_{\substack{\text{symmetric}\\ \text{traceless}}} + \underbrace{\zeta_{[ij]}}_{\substack{\text{anti-}\\ \text{symmetric}}} + \underbrace{\zeta^{(0)}}_{\text{trace}}$$

$$\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0;p\rangle = \left[\frac{1}{2}\left(\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j} + \alpha_{-1}^{j}\tilde{\alpha}_{-1}^{i}\right) - \frac{1}{d-2}\delta^{ij}\alpha_{-1}^{k}\tilde{\alpha}_{-1}^{k}\right]|0;p\rangle$$
$$+ \alpha_{-1}^{[i}\tilde{\alpha}_{-1}^{j]}\,|0;p\rangle$$
$$+ \frac{1}{d-2}\delta^{ij}\alpha_{-1}^{k}\alpha_{-1}^{k}\,|0;p\rangle$$

Interpretation

- $\zeta_{(ij)}$: massless spin-2 particle transversely polarised

has the correct d.o.f. to furnish the graviton of GR

Confirmed by explicit analysis of interactions (see later)

- $\mathfrak{Z}_{[ij]}$: extra so-called Kalb-Ramond field

$\mathfrak{Z}_{[ij]} \equiv B_{ij}$ antisymmetric tensor field

Fact: B_{ij} furnishes a generalisation of the concept of a gauge potential

Gauge field: $A \equiv \underbrace{A_\mu}_{\text{gauge potential}} dx^\mu$ in form notation

$F = dA = \partial_\mu A_\nu \, dx^\mu \wedge dx^\nu$
invariant under
$A \to A + d\chi$

Higher rank tensor: $B \equiv \frac{1}{2} B_{\mu\nu} \, dx^\mu \wedge dx^\nu$

It likewise enjoys a gauge symmetry w/ field strength

$$H = dB \equiv \frac{1}{2} \partial_\mu B_{\nu\rho} \, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

inv. under $B \to B + d\Lambda$

- $\mathfrak{Z}^{(0)}$: scalar field Φ called dilaton

Effective action:

At $E \ll M_s$, an effective action governs physics of light (massless) particles:

$$S_{eff} = \frac{2\pi}{\ell_s^{24}} \int d^{26}x \sqrt{-G} \; R - \frac{1}{12} e^{-\frac{\phi}{3}} H_{\mu\nu\rho} H^{\mu\nu\rho}$$

$$- \frac{1}{8} \partial_\mu \Phi \partial^\mu \phi + \underbrace{\mathcal{O}(\alpha')}_{\text{higher mass towers}}$$

$\frac{2\pi}{\ell_s^{24}} = M^{24}_{pl,26}$: Planck scale in 26-dim. spacetime

IV Superstring

Include fermionic sector on WS

$$X^\mu(\tau,\sigma) \xleftrightarrow[\text{symmetry}]{\text{Super-}} \psi^\mu_A(\tau,\sigma)$$

spinor in 2d

Various consistent types of such theories in critical dimension $d=10$, related by dualities, w/out tachyonic ground state.

Note: Includes also fermionic excitations

Closed:

Type IIA

massless level:
- $G_{\mu\nu}$, Φ, $B_{\mu\nu}$ as in bosonic
- C_μ } RR gauge potentials
- $C_{\mu\nu\rho}$ } higher forms

+ supersymmetric fermionic partners

χ_α $\bar{\chi}_{\dot\alpha}$ 'dilatinos' spin 1/2

ψ^μ_α, $\bar{\psi}^\mu_{\dot\alpha}$ " gravitinos spin 3/2

Type IIB
- $G_{\mu\nu}$, Φ, $B_{\mu\nu}$

- $C, \quad C_{\mu\nu}, \quad C_{\mu\nu\rho\sigma}$
- $\chi_\alpha, \tilde{\chi}_\alpha$
- $\psi^\mu{}_\alpha, \tilde{\psi}^\mu{}_\alpha$

Heterotic $SO(32)$, $E_8 \times E_8$ } possible gauge symmetries in $d=10$

Type I : Open + closed w/ $SO(32)$

+ a few non-SUSY theories