particle motion in FRW space-time

geodesic equation [sect. 1.2.1 of Baumann]:

$$\frac{\dot p}{p} = -\frac{\dot a}{a} \Rightarrow p \sim \frac{1}{a}$$

physical 3-momentum $\to 0$ in expanding space-time "momentum redshift"

for massless particles:

$$E = p \sim \frac{1}{a} \quad \text{decays, too.}$$

photons redshift:

$$\lambda = \frac{\hbar}{p} \Rightarrow \lambda \sim a$$

$$t_0 > t_1 \Rightarrow a(t_0) > a(t_1)$$

now

$$\Rightarrow \lambda_1 = \frac{a(t_0)}{a(t_1)}\lambda_0 > \lambda_1$$

redshift parameter:

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1}$$

$$\Rightarrow 1 + z = \frac{a(t_0)}{a(t_1)} = \frac{1}{a(t_1)}$$

convention: $a(t_0) = 1$

For nearby sources ($z < 1$) we can expand:

$$a(t_1) = a(t_0)\cdot\left[1 + (t_1 - t_0)\cdot H_0 + \dots\right]$$

$$H \equiv \frac{\dot a}{a}, \quad H_0 \equiv \left.\frac{\dot a}{a}\right|_{t=t_0}$$

"Hubble parameter (now)"

So:

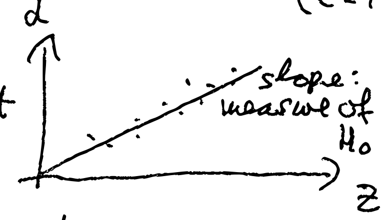
$$1 = (1+z)\cdot\left[1 + (t_1 - t_0)\cdot H_0\right] = 1 + z + (1+z)(t_1 - t_0)\cdot H_0$$

$$\Leftrightarrow 0 = z + (t_1 - t_0)H_0 + \cancel{z\cdot(t_1 - t_0)}\cdot H_0$$

$$\Leftrightarrow z \simeq \underbrace{(t_0 - t_1)}_{\text{physical distance traveled by light } (c=1)}\cdot H_0$$

So: $z \simeq H_0 \cdot d$

distance-redshift diagram



$$H_0 = 100 \cdot h \cdot km \cdot s^{-1} \cdot Mpc^{-1}$$

0.67 ± 0.01

d: deduced from luminosity assuming "standard candles" (objects of known absolute luminosity)

Dynamics of Expansion – Friedmann equation

derivation from GR ...

(i) definitions:

$$\Gamma^{\mu}_{\nu\rho} \equiv \frac{1}{2} g^{\mu\sigma} \cdot \left(-\partial_\sigma g_{\nu\rho} + \partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu}\right) \quad \text{"Christoffel symbols"}$$

$$R_{\mu\nu} \equiv \partial_\sigma \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\sigma\mu} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\rho\mu} \quad \text{"Ricci tensor"}$$

$$R \equiv g^{\mu\nu} R_{\mu\nu} \quad \text{"Ricci scalar"}$$

(ii) non-vanishing $\Gamma^{\mu}_{\nu\rho}$ for FRW-metric:

$$\Gamma^{0}_{ij} = -\frac{\dot a}{a} g_{ij}\,, \quad \Gamma^{i}_{0j} = \frac{\dot a}{a} \cdot \delta^{i}_{j}$$

$$\Gamma^{i}_{jk} = \frac{1}{2} g^{il} \cdot \left(-\partial_l g_{jk} + \partial_j g_{lk} + \partial_k g_{lj}\right)$$

(iii) Ricci tensor components:

$$R_{00} = -3\,\frac{\ddot a}{a}$$

$$R_{ij} = -\left[\frac{\ddot a}{a} + 2\left(\frac{\dot a}{a}\right)^2 + 2 \cdot \frac{k}{a^2}\right] g_{ij}$$

$$\Rightarrow R = -6 \cdot \left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)$$

GR follows from an action principle
→ Einstein-Hilbert action:
unique action satisfying
(i) diffeomorphism invariance
(ii) locality
(iii) action & e.o.m. are 2-derivative in ∂_t & Levi-Civita connection: $\Gamma^{\mu}_{\nu\rho}$ are the Christoffel symbols

$$S_{EH} = \frac{M_P^2}{2} \cdot \int d^4x \sqrt{-g}\,\left(R - 2 \cdot \Lambda\right)$$

adding matter via a local 2-derivative matter action: $S_{EH} \to S = S_{EH} + S_m$ generates the energy-momentum tensor $T_{\mu\nu}$, the source of the Einstein eq.s, of a perfect fluid compatible with spatial homog. & isotropy, by varying S_m w.r.t. $g_{\mu\nu}$:

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \cdot \frac{\delta S_m}{\delta g_{\mu\nu}} = (p+\rho) u^\mu u^\nu - p \cdot g_{\mu\nu}$$

fluid at rest: $u^\mu = (1, \vec{0})$

Thus with: $\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{2}\cdot\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda\cdot g_{\mu\nu}\right)$

$\delta S / \delta g^{\mu\nu} = 0$ implies:

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\equiv G_{\mu\nu} \text{ "Einstein tensor"}} - \Lambda\cdot g_{\mu\nu} = \frac{1}{M_p^2}\cdot T_{\mu\nu}$$

$\mu = \nu = 0$ component:

$$\cancel{-3\frac{\ddot a}{a}} - \frac{1}{2}\cancel{(-6)}\cdot\left(\cancel{\frac{\ddot a}{a}} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right) - \Lambda = \frac{\rho}{M_p^2}$$

$$\Leftrightarrow \boxed{\frac{\dot a^2}{a^2} = \frac{\rho}{3M_p^2} + \frac{\Lambda}{3} - \frac{k}{a^2}} \quad \text{"1}^{\text{st}}\text{ Friedmann equation" (E1)}$$

ij - component:

$$-\left[\frac{\ddot a}{a} + 2\cdot\frac{\dot a^2}{a^2} + \frac{2k}{a^2}\right] g_{ij} - \frac{1}{2}(-6)\left[\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right] g_{ij}$$

$$-\Lambda\cdot g_{ij} = \frac{1}{M_p^2}\left(-p\cdot g_{ij}\right)$$

$$\Leftrightarrow \boxed{2\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2} - \Lambda = -\frac{p}{M_p^2}} \quad \text{"2}^{\text{nd}}\text{ Friedmann equation" (E2)}$$

all other components vanish.

energy-momentum is conserved:

$$0 \overset{!}{=} \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\sigma\mu}$$

0-component:

$$0 = \partial_0 T^{00} + \Gamma^\mu_{\mu 0} T^{00} + \Gamma^0_{ij} T^{ij}$$

$$= \dot\rho + 3\frac{\dot a}{a}\rho - \frac{\dot a}{a}\underbrace{g_{ij}\cdot(-p\cdot g^{ij})}_{=3}$$

$$= \dot\rho + 3\frac{\dot a}{a}(\rho + p) \quad \text{"continuity eq."}$$

can also be obtained from (E1) & (E2):

$$\partial_t(E1) + 3\frac{\dot a}{a}(E1) - \frac{\dot a}{a}(E2) = \nabla_\mu T^{\mu\nu} = 0$$

notation: $\rho_\Lambda \equiv M_p^2 \Lambda = -p_\Lambda$

$\rho_{tot} = \rho + \rho_\Lambda$, $p_{tot} = p + p_\Lambda$

(E1) & (E2) become:

$$H^2 = \frac{\rho_{tot}}{3M_p^2} - \frac{k}{a^2}, \quad 2\frac{\ddot a}{a} + H^2 + \frac{k}{a^2} = -\frac{p_{tot}}{M_p^2}$$

so (E1) - (E2): $\frac{\ddot a}{a} = -\frac{1}{6M_p^2}\cdot\left(\rho_{tot} + 3p_{tot}\right)$

critical density: $\rho_c \equiv 3 M_p^2 H^2 \left(\simeq 8.1 \cdot 10^{-47} h^2 \cdot \mathrm{GeV}^4 \text{ now}\right)$

density parameter: $\Omega \equiv \frac{\rho}{\rho_c}$

$\rightsquigarrow$ (E1) becomes: $1 = \Omega - \frac{k}{(aH)^2}$

$k=0$: $\rho = \rho_c$, $\Omega = 1$ "flat"

$k=+1$: $\rho > \rho_c$, $\Omega > 1$

$k=-1$: $\rho < \rho_c$: $\Omega < 1$

define curvature parameter:

$$\Omega_k \equiv -\frac{k}{a^2 H^2}$$

$\Rightarrow$ at all times we have:

$$1 = \Omega + \Omega_k$$

To get $a(t)$, need at last $p = p(\rho)$ "equation of state": $\boxed{p = w \cdot \rho}$ often a good approx.

For a flat single-component universe ($w = w_i$, $\Omega = \Omega_i$):

$$\frac{\dot a}{a} = H(\rho_c) \cdot \sqrt{\Omega_i}\, a^{-\frac{3}{2}(1+w_i)}$$

- non-relativistic component: $p \ll \rho$

$$\dot\rho + 3H(\rho + p) = 0 \Rightarrow \frac{\dot\rho}{\rho} = -3\frac{\dot a}{a}$$

$$\Rightarrow \rho = \rho_0 \cdot \left(\frac{a_0}{a}\right)^3$$

if $k=0$: $\left(\frac{\dot a}{a}\right)^2 = \frac{\rho_0}{3M_p^2}\left(\frac{a_0}{a}\right)^3 = H_0^2 \cdot \frac{1}{a^3}$

$$\Rightarrow a(t) \sim t^{2/3}$$

- relativistic component - radiation: $p = \frac{1}{3}\rho$

$$\dot\rho + 3H(\rho + p) = 0 \Leftrightarrow \dot\rho = -4H\rho \Rightarrow \rho \sim \frac{1}{a^4}$$

$k=0$: (a^{-3} from number density dilution due to 3-volume expansion, extra a^{-1} from energy/frequency redshift)

$a \sim t^{1/2}$

- general component with: $p = w \cdot \rho$

$$\rho \sim a^{-3\cdot(1+w)}, \quad k=0: \; a \sim t^{\frac{2}{3(1+w)}}$$

$\rightsquigarrow$ can write (E1) as:

$$H^2 = \frac{1}{3M_p^2} \cdot \left(\rho_{m_0} \cdot a^{-3} + \rho_{\mathrm{rad.}0} \cdot a^{-4} + \rho_{k_0} \cdot a^{-2}\right)$$

$$H^2 = H_0^2 \cdot \left(\Omega_{m_0} \cdot a^{-3} + \Omega_{rad,0} \cdot a^{-4} + \Omega_{k_0} \cdot a^{-2} + \Omega_\Lambda\right)$$

observations: $\Omega_{m_0} \simeq 0.32$, $\Omega_{\Lambda_0} \simeq 0.68$

$\Omega_{r_0} \simeq 9.4 \cdot 10^{-5} \Rightarrow |\Omega_{\Lambda_0}| < 10^{-2}$

$$\frac{da}{dt} = H_0 a_0 \cdot \sqrt{\Omega_m \frac{a_0}{a} + \Omega_r \frac{a_0^2}{a^2} + \Omega_k + \Omega_\Lambda \frac{a^2}{a_0^2}}$$

now define: $x \equiv \frac{a}{a_0}$ $(a_0 = 1) \rightarrow \int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dx}{dx/dt}$

present age of the universe:

$$\int_0^{t_0} dt = H_0^{-1} \cdot \int_0^1 \frac{dx}{\sqrt{1 - \Omega_0 + \Omega_r x^{-2} + \Omega_m x^{-1} + \Omega_\Lambda x^2}}$$

$(\Omega_k = 1 - \Omega_0)$

simplified 1-component universes:

- radiation-dominated $\Omega_r = \Omega_0 = 1 \rightarrow t_0 = \frac{1}{2} H_0^{-1}$
- matter-dominated $\Omega_m = \Omega_0 = 1 \rightarrow t_0 = \frac{2}{3} H_0^{-1}$
- curvature " " $\Omega_0 = 0 \rightarrow t_0 = H_0^{-1}$
- vacuum energy $\Omega_\Lambda = \Omega_0 = 1 \rightarrow t_0 = \infty$

we can apply this to different epochs $t \sim H^{-1}$, e.g. age of universe at electroweak epoch:

$$H \sim \frac{\sqrt{\varsigma}}{M_P} \sim \frac{(100\,\mathrm{GeV})^2}{10^{19}\,\mathrm{GeV}} \sim 10^{-15}\,\mathrm{GeV} \Rightarrow t \sim 10^{15}\,\mathrm{GeV}^{-1} \simeq 10^{-9}\,\mathrm{s}$$