Thermal history of the universe

Particle decoupling

- To understand the history of the universe, compare particle interaction rates Γ with rate of cosmological expansion H

$\Gamma \gg H$: expansion irrelevant
$\rightarrow$ thermal equilibrium

$\Gamma \lesssim H$: particles decouple

$$\Gamma = n \cdot \sigma \cdot v$$

particle number density (n), interaction cross section (σ), relative particle velocity (v)

$[\Gamma] = 1 \qquad [n] = 3 \qquad [\sigma] = -2\,, \quad [v] = 0$

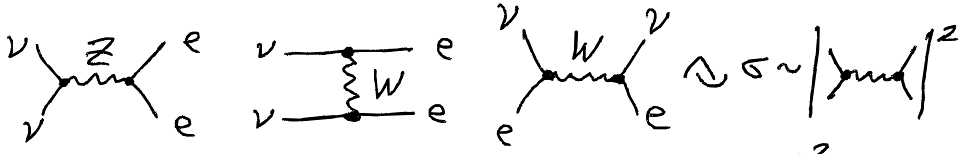
at high-$T \gtrsim 100$ GeV, all known particles ultra-relativistic: $v \simeq 1$, $n \sim T^3$

$$H = \sqrt{\frac{\rho}{3 M_P^2}}$$

in radiation-dominated universe $\rho \sim T^4$

$$\Rightarrow \quad H \sim \frac{T^2}{M_P} \qquad [H] = 1$$

Decoupling of EW interactions



at $T > 100$ GeV: $\sigma \sim \dfrac{\alpha^2}{T^2}$, $\alpha = \dfrac{g^2}{4\pi} \simeq 0.03$

(gauge bosons massless!)

$$\Rightarrow \Gamma = n\sigma v \sim T^3 \frac{\alpha^2}{T^2} \sim \alpha^2 T$$

$$\frac{\Gamma}{H} \sim \alpha^2 \frac{M_P}{T} \sim \frac{10^{16}\,\text{GeV}}{T}$$

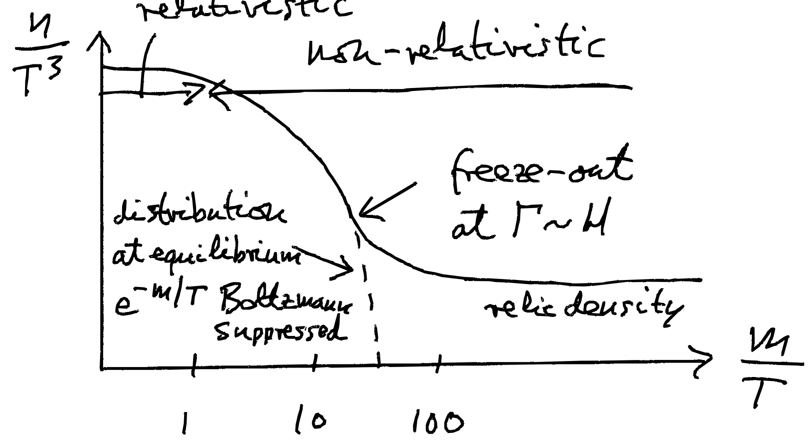
- for $100\,\text{GeV} < T < 10^{16}\,\text{GeV}$: thermal equilibrium for $T \lesssim 100\,\text{GeV}$: $\sigma \sim \frac{\alpha^2}{M_W^4} T^2 \sim G_F^2 T^2$

$$M_W \simeq 80\,\text{GeV},\ G_F \simeq 1.2 \cdot 10^{-5}\,\text{GeV}^{-2}$$

$$\Rightarrow \frac{\Gamma}{H} \sim \frac{G_F^2 T^5}{T^2} M_p \sim \left(\frac{T}{1\,\text{MeV}}\right)^3$$

EW interactions decouple at $T \sim 1\,\text{MeV}$.

typically, for stable massive particles:



if all particles had remained in equilibrium until today $\to$ universe would be mainly photons. Any massive particle species exponentially suppressed.

Thermodynamics

gas of particles in kinetic equilibrium

$$E(\vec{p}) = \sqrt{p^2 + m^2}$$

distrib. funct. $f(\vec{p}) = \frac{1}{e^{\frac{E-\mu}{T}} \pm 1}$, $+$: fermions, $-$: bosons

note: no depend. on $\vec{x}$ (homogeneity), $\vec{p}$ (isotropy)

$$\mu = -T \left.\frac{\partial S}{\partial N}\right|_{U,V}$$

chemical potential: characterizes response of system to change in particle #

- particle # density: $n = \underbrace{\frac{g}{(2\pi)^3}}_{\text{density of states in phase } \{\vec{x},\vec{p}\}} \int d^3p\, f(p)$, g = # of internal deg. of freedom
- energy density: $\rho = \frac{g}{(2\pi)^3} \cdot \int d^3p\, f(p) \cdot E(p)$
- pressure density: $p = \frac{g}{(2\pi)^3} \cdot \int d^3p\, f(p) \cdot \frac{\vec{p}^{\,2}}{3E}$

$\to$ for massless particles $p = w\rho = \frac{1}{3}\rho$

thermal eq. : $T_i = T$

chemical eq.: $\mu_i + \mu_j = \mu_k + \mu_l$ for $i+j \leftrightarrow k+l$

if no conserved particle # (eg. photons gas)

$\Rightarrow \mu = 0$ in equilibrium

$\rightsquigarrow$ $X + \bar{X} \leftrightarrow \gamma + \gamma \Rightarrow \mu_X = -\mu_{\bar{X}}$

For fermions present today (e^- & neutrinos)

$\rightsquigarrow$ at early times : $|\mu| \ll T$ @ $T \gg m$

μ_ν not known, but usually assumed small

$\rightsquigarrow$ neglect μ's for now.

$$n = \frac{g}{(2\pi)^3} \cdot \int \frac{dp \cdot 4\pi p^2}{e^{\frac{\sqrt{p^2+m^2}}{T}} \pm 1}, \quad \rho = \int \frac{dp \cdot 4\pi p^2 \sqrt{p^2+m^2}}{e^{\frac{\sqrt{p^2+m^2}}{T}} \pm 1}$$

define: $x \equiv \frac{m}{T}$, $\xi \equiv \frac{p}{T}$

$$\Rightarrow n = \frac{g}{2\pi^2} \cdot T^3 \cdot I_\pm(x)$$

$$\rho = \frac{g}{2\pi^2} \cdot T^4 \cdot J_\pm(x)$$

$$I_\pm(x) = \int_0^\infty d\xi \frac{\xi^2}{e^{\sqrt{\xi^2+x^2}} \pm 1}, \quad J_\pm(x) = \int_0^\infty \frac{d\xi \cdot \xi^2 \sqrt{\xi^2+x^2}}{e^{\sqrt{\xi^2+x^2}} \pm 1}$$

generically evaluated numerically

analytical results in relativistic & non-relativistic limits - useful integrals:

$$\int_0^\infty d\xi \frac{\xi^n}{e^\xi - 1} = \zeta(n+1) \cdot \Gamma(n+1), \quad \int_0^\infty d\xi \cdot \xi^n \cdot e^{-\xi^2} = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right)$$

$\zeta(z)$: Riemann zeta function

$\Gamma(n+1) = \int_0^\infty dt \cdot t^n \cdot e^{-t}$ Euler Gamma function

in relativistic limit ($x \to 0$) :

$$I_-(0) = \zeta(3)\Gamma(3) = 2 \cdot \zeta(3) \qquad \left(\frac{1}{e^\xi + 1} = \frac{1}{e^\xi - 1} - \frac{1}{e^{2\xi} - 1} \right)$$

$$I_+(0) = \int_0^\infty d\xi \frac{\xi^2}{e^\xi + 1} = \frac{3}{4} I_-(0)$$

$$\Rightarrow \begin{cases} n = K \cdot \frac{\zeta(3) g}{\pi^2} T^3, & K = \begin{cases} 1, & \text{bosons} \\ 3/4, & \text{fermions} \end{cases} \\ \rho = K \cdot \frac{\pi^2 g}{30} T^4, & K = \begin{cases} 1, & \text{bosons} \\ 7/8, & \text{fermions} \end{cases} \end{cases}$$

<u>relic photons - CMB</u>: temperature now $T_0 \simeq 2.73\,K$

$$n_{\gamma,0} = \frac{2\zeta(3)}{\pi^2} T_0^3 \simeq 0.243 \cdot 2.73^3 \cdot 83\,cm^{-3}$$

$$\simeq 410 \text{ photons}/cm^3$$

$1\,GeV \simeq 1.2\cdot 10^{13} K$ <u>and</u> $1\,GeV \simeq (2\cdot 10^{-14} cm)^{-1}$

$1\,GeV^3 \simeq 1.3\cdot 10^{41} cm^{-3}$

$$\Rightarrow 1\,K^3 \simeq (1.2\cdot 10^{13})^{-3} \cdot (2\cdot 10^{-14} cm)^{-3} \simeq 83\,cm^{-3}$$

$$\rho_{\gamma,0} = \frac{\pi^2}{15}\cdot T_0^4 \Rightarrow \Omega_{\gamma,0} h^2 \simeq 2.5\cdot 10^{-5}$$

$$p = \frac{\rho}{3} \quad , \quad \langle E \rangle = \frac{\rho}{n} = \frac{\pi^4}{30\cdot\zeta(3)}\cdot T$$

non-relativistic limit: $m \gg T \Leftrightarrow x \gg 1$

$$I_\pm(x) = \sqrt{\frac{\pi}{2}}\cdot x^{3/2}\cdot e^{-x}$$

$$\Rightarrow n = g\left(\frac{mT}{2\pi}\right)^{3/2}\cdot e^{-m/T}, \quad \rho = mn$$

restoring μ: $n = g\left(\frac{mT}{2\pi}\right)^{3/2}\cdot e^{-\frac{m-\mu}{T}}$

- ρ_R: total ρ of all species in equil. expressed in terms of photon temp. T
- some species i may be in thermal equil. but have $T_i \neq T$
- since $\rho_{non-relat.} \ll \rho_{relat.} \Rightarrow \rho_R \simeq \rho_{relat.}$

$$\rho_R = \frac{\pi^2}{30} g_* \cdot T^4$$

g_* : counts the total # of effectively massless d.o.f. (those species i with $m_i \ll T$)

$$g_* = \underbrace{\sum g_i \left(\frac{T_i}{T}\right)^4}_{\text{bosons}} + \frac{7}{8} \underbrace{\sum g_i \left(\frac{T_i}{T}\right)^4}_{\text{fermions}}$$

At $T \gtrsim 100$ GeV, add up all SM particles.

$g_b = 28$ (γ: 2; $W^\pm, Z$: (3x3); gluons: (2x8); Higgs: 1)

$g_f = 90$ (quarks: 6x12, $\ell^\pm$: 3x4, ν: 3x2)

$$\Rightarrow g_* = g_b + \frac{7}{8} g_f = 106.75$$

temperature evolution of g_*

$$n_i = g_i \left(\frac{mT}{2\pi}\right)^{3/2} \cdot e^{-m_i/T}$$ in non-relat. limit

$\Rightarrow$ at $T < m_i$ particles no longer contribute to g_*

- $T > 100\ \text{GeV}$ $\qquad g_* = 106.75$

- $t\bar{t}$-annihilation $\quad g_t = 12$
  at $T \sim \frac{m_t}{6} \sim 30\ \text{GeV}$
  $m_t \simeq 170\ \text{GeV}$
  $$g_* = 106.75 \;\Big|\; -\tfrac{7}{8}\cdot 12 \;=\; 96.25$$

- $T \sim 10\ \text{GeV}$ $\quad g_H = 1,\ g_Z = 3$, $\ g_W = 6$
  $m_H \sim 125\ \text{GeV}$
  $m_Z, m_W \sim 80..90\ \text{GeV}$
  $$g_* = 96.25 \;\Big|\; -10 \;=\; 86.25$$

- $T \sim 0.6\ \text{GeV}$ $\quad g_b = 12$
  $m_b \sim 4\ \text{GeV}$
  $$g_* = 86.25 \;\Big|\; -\tfrac{7}{8}12 \;=\; 75.75$$

- $T \sim 0.2\ \text{GeV}$ $\quad g_\tau = 4$, $\ g_c = 12$
  $m_\tau \sim 1.7\ \text{GeV}$
  $m_c \sim 1\ \text{GeV}$
  $$g_* = 75.75 \;\Big|\; -\tfrac{7}{8}\cdot 16 \;=\; 61.75$$

- $T \sim 0.1\ \text{GeV}$
  $m_s \sim 0.1\ \text{GeV}$
  $m_d \sim 0.005\ \text{GeV}$
  $m_u \sim 0.002\ \text{GeV}$

  QCD phase transition! naively: $g_u = g_d = g_s = 12$ $\rightarrow$ but: only strongly interacting d.o.f. are now pions $\pi^{\pm 0} \Rightarrow g_\pi = 3$

  $\Downarrow$ hadronization

$m_\mu \simeq 0.106\ \text{GeV}$
$m_e \simeq 0.511\ \text{GeV}$
$m_\nu < 0.6\ \text{eV}$

γ,	μ,	e,	ν,	π
2	4	4	6	3

$$g_* = 5 + \tfrac{7}{8}\cdot 14 \;\Big|\; = 17.25$$

- $T \sim 15\ \text{MeV}$ $\quad \pi$ and μ disappear
  $$g_* = 17.25 - 3 \;\Big|\; -\tfrac{7}{8}\cdot 4 \;=\; 10.75$$

- e^+e^--annihilation
  $T \sim 1\ \text{MeV}$ $\quad$ only species left: γ, e, ν
  but $g_* \neq 10.75 - \frac{7}{8}\cdot 4$

entropy density

$$s = \frac{\rho + P - \mu \cdot n}{T}, \quad s = \frac{S}{V}$$

we neglect μ : $s = \sum_i \frac{\rho_i + P_i}{T_i}$

$$s = \frac{2\pi^2}{15} g_{*s}(T) \cdot T^3$$

$$g_{*s}(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3$$

$$g_{*}(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4$$

most of the time, all species have the same T:

$$\Rightarrow g_{*s} = g_* \quad \text{if } T_i = T \ \forall i$$

$$n_\gamma = \zeta(3) \frac{2}{\pi^2} T^3 \Rightarrow s = \frac{2\pi^4}{45} \cdot \frac{g_{*s}}{\zeta(3)} n_\gamma \simeq 1.8 g_{*s} n_\gamma$$

for bosons of species i:

$$\frac{n_i}{s} = \frac{\zeta(3)}{\pi^2} \cdot g_i \frac{45}{2\pi^2 g_{*s}} = \frac{45 \cdot \zeta(3)}{2\pi^4} \cdot \frac{g_i}{g_{*s}}$$

$\frac{n_i}{s}$ = const. if g_{*s} is const.

i.e. if no species is being created or destroyed

today: $g_{*s} = 3.91 \Rightarrow s \simeq 7.04 \cdot n_\gamma$

entropy conservation

1st law: $T\,dS = dE + p\,dV - \mu\,dN$

with: $E = \rho V \Rightarrow dE = V d\rho + \rho\, dV$

if quantum numbers conserved in comoving volume $V = a^3$, $dN = 0$

$$\Rightarrow T\,dS = (\rho + P)\,dV + V\,d\rho$$

$$\Rightarrow T \cdot \frac{dS}{dt} = (\rho + P)\, 3 a^2 \dot{a} + a^3 \dot{\rho} = a^3 \cdot \left[(\rho + P)\, 3H + \dot{\rho}\right]$$

using the continuity eq.:

$$\dot{\rho} + 3H(\rho + P) = 0$$

$\Rightarrow \frac{dS}{dt} = 0$ total entropy per comoving volume is conserved.

$\Rightarrow \boxed{s\, a^3 = \text{const.}}$ for closed system in equilibrium

$\Rightarrow T \sim \frac{1}{a} \cdot \frac{1}{g_{*s}(T)^{1/3}}$

for particles in equilibrium

neutrino decoupling & temperature

$\nu\bar{\nu} \leftrightarrow e^- e^+$

$\nu e \leftrightarrow \nu e$

$\sigma \sim G_F^2 T^2$

$\frac{\Gamma}{H} \sim \left(\frac{T}{\text{MeV}}\right)^3$

Neutrinos decouple just before e^+e^--annihilation and do not inherit any of this energy.

When T drops below m_e, the entropy in e^+e^- is transferred to the photons, but not to the decoupled neutrinos...

$\Rightarrow$ ν & γ acquire $\neq T$!

After ν's freeze out at t_F, neutrinos are still described by ultra-relat. distrib. with present-day temp.: $T_{\nu,0} = T_{\nu,F} \cdot \frac{a(t_F)}{a_0}$

next, use entropy conservation of the $e^+e^-\gamma$-system in equilibrium after ν-freeze-out:

$$g_{*s}(T) \cdot T^3 a^3 = \text{const.}$$

before e^+e^--annihilation at $T_{\nu,F} = T_{\gamma,F}$

$$g_{*s} = 2 + \frac{7}{8}(2+2) = \frac{11}{2}$$

$$\left(\frac{11}{2}\right)^{1/3} \underbrace{T_{\nu,F}\cdot a(t_F)}_{T_{\nu,0}} = 2^{1/3}\cdot T_{\gamma,0}$$

$$\Leftrightarrow \boxed{\frac{T_{\gamma,0}}{T_{\nu,0}} = \left(\frac{11}{4}\right)^{1/3} \simeq 1.4}$$ photons are heated

$$\rho_\nu = 3\cdot\frac{7}{8}\cdot\left(\frac{4}{11}\right)^{4/3}\cdot\rho_\gamma$$ if ν's were massless

$$\Rightarrow \Omega_{\nu,0}h^2 \simeq 1.7\cdot 10^{-5}$$