Quantum description → perturbations

generalized Hawking radiation:

→ QF in grav. system with event horizon of size R_h

$\Rightarrow$ light fields radiate quanta w/ thermal distro and: $T \sim \frac{1}{R_h}$

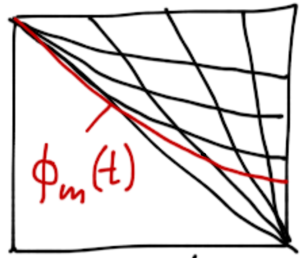
- BHs: $R_h = 2GM \Rightarrow T_{BH} \sim \frac{1}{M}$
- dS: $R_h \sim H^{-1}$

$$\Rightarrow T_{dS} \sim H$$

$$\Rightarrow \langle \delta\phi_k^2 \rangle, \langle \delta g^2_{\mu\nu_k} \rangle \sim H^2$$

define time-slicing by $\phi_m(t)$:

- slow-roll inflation: time translation inv. of dS slightly broken



$\rightsquigarrow$ Goldstone boson $\pi(\vec{x},t)$ due to spontaneous time transl. symm. breaking:

$$t \to u(\vec{x},t) = t + \pi(\vec{x},t)$$

- fluctuations of background fields:

$$\delta\phi_m(t) = \phi_m(t + \pi(\vec{x},t)) - \phi_m(t)$$

Einstein eq.s couple $\delta g_{\mu\nu}$ to π.
Use gauge freedoms of the metric:
i) spat. flat gauge: $g_{ij} = a^2 \cdot \delta_{ij}$

$\delta g_{00}, \delta g_{0i}$ linked to π by Einstein eq.s

$\Rightarrow$ only fluctuation mode is π

ii) comoving gauge : $\delta\phi_m = 0$

π <u>eaten</u> by $\delta g_{\mu\nu}$, $g_{ij} = a^2 \cdot e^{2\zeta(\vec{x},t)}\delta_{ij}$

$\Rightarrow$ relation : $\zeta = -H\pi + \dots$

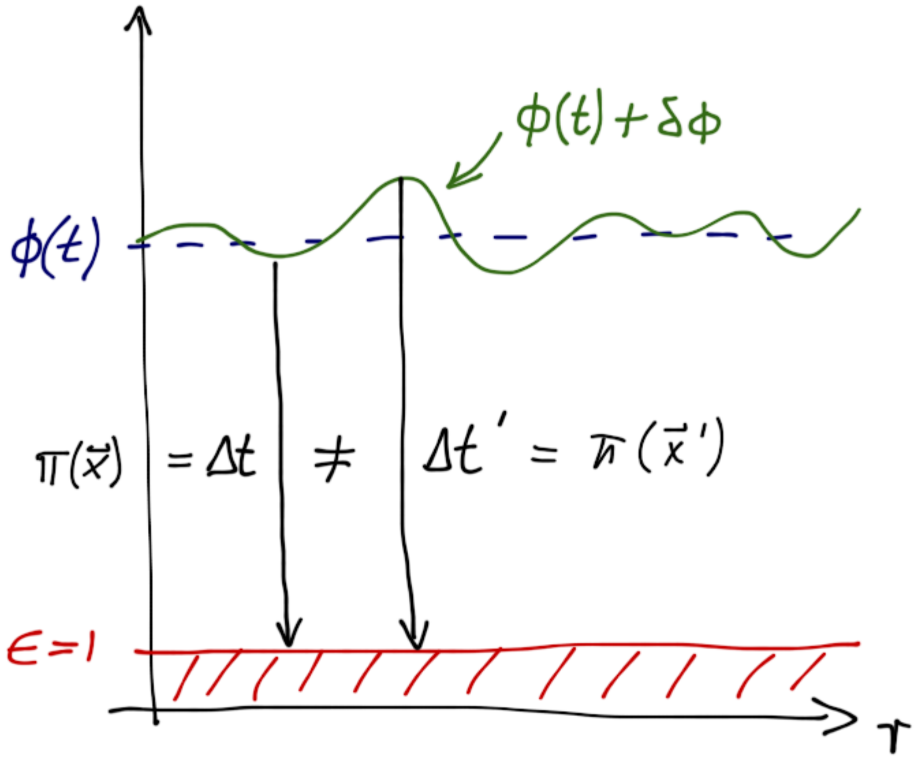
<u>eff. action</u> : 'chiral lagrangian' of dS inv. breaking

$$S = \int d^4x \sqrt{-g}\left(u, (\partial_\mu u)^2, \Box u, \dots\right)$$

$$= \int d^4x \sqrt{-g}\, M_p^2 \dot{H} \cdot \left[\dot{\pi}^2 - \frac{1}{a^2}(\partial_i \pi)^2 + 3\epsilon_H H^2 \pi^2 + \dots\right]$$

$$= \int d^4x \cdot a^3 \cdot 2M_p^2 \epsilon_H \left[\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i \zeta)^2 + \dots\right]$$

we can display this graphically :



$$ds^2 = dt^2 - a^2 \cdot (1 + 2\zeta + \dots) \cdot \delta_{ij}\, dx^i dx^j$$

things simplify in conformal time:

$$dt = a \cdot d\tau$$

$$\Rightarrow S = M_p^2 \cdot \int d\tau\, d^3x \cdot \underbrace{2a^2 \epsilon_H}_{\equiv z^2} \cdot \left[(\zeta')^2 - (\partial_i \zeta)^2 \right]$$

canonically normalize ζ and go to Fourier modes of field:

$$\zeta \to v = M_p \cdot z \cdot \zeta$$

$$v = \int \frac{d^3k}{(2\pi)^3} \cdot v_k\, e^{i\vec{k}\vec{r}}$$

$$S = \int \frac{d^3k}{(2\pi)^3} S_k$$

$$S_k = \int d\tau \cdot \left[(v_k')^2 - \left(k^2 - \frac{z''}{z}\right) v_k^2 \right]$$

e.o.m. from $\delta S / \delta v_k$:

$$v_k'' - \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

'Mukhanov-Sasaki eq.'

for quasi-dS:

$$\frac{z''}{z} = \frac{2}{\tau^2} \left(1 + \mathcal{O}(\epsilon, \xi)\right)$$

mode solutions:

$$v_k = \alpha \cdot \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + \beta \cdot \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right)$$

for $\tau = -\frac{1}{aH} \to \infty$ (early times = small distances) curvature negligible & modes should be like Minkowski vacuum modes:

$$\lim_{\tau\to-\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}$$ 'Bunch-Davies vacuum'

$$\Rightarrow \alpha = 1 , \beta = 0$$

quantization now straight forward:

$$v_k \to \hat{v}_k = \hat{a}_{\vec{k}} v_k + \hat{a}^+_{-\vec{k}} v^*_{-k}$$

compute 2-point function:

$$\langle \hat{v}_k \hat{v}_{k'} \rangle = \langle 0 | \hat{v}_k \hat{v}_{k'} | 0 \rangle$$

$$= \langle 0 | (a_{\vec{k}} v_k + a^+_{-\vec{k}} v^*_k)(a_{\vec{k}'} v_{k'} + a^+_{-\vec{k}'} v^*_{k'}) | 0 \rangle$$

$$= v_k v^*_{k'} \langle 0 | [a_{\vec{k}}, a^+_{-\vec{k}'}] | 0 \rangle$$

$$= |v_k|^2 \cdot \delta^{(3)}(\vec{k} + \vec{k}')$$

$$\Rightarrow |v_k|^2 = \frac{1}{2k^3} \cdot \frac{1}{\tau^2} = \frac{a^2 H^2}{2k^3}$$

$$v = z\zeta , \quad \Delta^2_v \equiv \frac{k^3}{2\pi^2} |v_k|^2 \Big| (M_p = 1)$$

dim.-less power spectru

$$\Rightarrow \Delta^2_\zeta = \frac{1}{z^2} \Delta^2_v = \frac{1}{8\pi^2} \cdot \frac{H^2}{\epsilon} \quad (\epsilon_H \approx \epsilon)$$

$$\leftarrow \dot{\phi} = -\frac{V'}{3H} = -H\sqrt{2\epsilon}$$

$$= \frac{1}{4\pi^2} \cdot \frac{H^4}{\dot{\phi}^2}$$

similarly, tensor fluctuations → primordial gravitational waves:

$$ds^2 = dt^2 - a^2 \cdot (e^{2\zeta} \cdot \delta_{ij} + h_{ij}) \cdot dx^i dx^j$$

$$\Rightarrow S = \int d\tau\, d^3x \cdot \frac{M_p^2}{4} a^2 \cdot \left[(h_{ij}')^2 - (\vec\nabla h_{ij})^2 \right]$$

$$u_{ij} = M_p \frac{a}{2} h_{ij}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int d\tau \cdot \sum_\gamma \left[(u_{k,\gamma}')^2 - \left(k^2 - \frac{a''}{a}\right) u_{k,\gamma}^2 \right]$$

↖ 2 graviton polarizations

$$\Rightarrow \langle \hat u_{k,\gamma} \hat u_{k',\gamma} \rangle = |u_{k,\gamma}|^2 \cdot \delta^{(3)}(\vec k + \vec k')$$

$$\Rightarrow \Delta_T^2 = \underset{\substack{\uparrow \\ \sum_\gamma}}{2} \cdot \frac{4}{M_p^2 a^2} \Delta_u^2 = \frac{4k^3}{\pi^2} \cdot \frac{1}{M_p^2 a^2} \cdot \underbrace{|u_{k,\gamma}|^2}_{=|v_k|^2} = \frac{2}{\pi^2} \cdot \frac{H^2}{M_p^2}$$

$$\Rightarrow \boxed{r \equiv \frac{\Delta_T^2}{\Delta_\zeta^2} = 16\epsilon}$$

'tensor-to-scalar ratio'

in slow-roll: $\dot\phi = -\frac{V'}{3H}$

$$\Rightarrow \Delta_\zeta^2 = \frac{1}{12\pi^2} \cdot \frac{V^3}{V'^2} = \frac{1}{24\pi^2} \cdot \frac{V}{\epsilon}$$

spectral tilt n_s of Δ_ζ^2:

$$n_s = 1 + \frac{d \ln \Delta_\zeta^2}{d \ln k}, \quad \Delta_\zeta^2 = A_s \cdot \left(\frac{k}{k_*}\right)^{n_s - 1}$$

use:
- $k = a\, k_{ph}\big|_{\text{hor. cross.}} = e^{-N_e} H$

  N_e e-folds before end of infl.

  $\Rightarrow d \ln k = -dN_e$
- $\frac{d\phi}{dN_e} = \frac{\dot\phi}{H} = -\frac{V'}{V}$

$$\Rightarrow n_s = 1 - 6\epsilon + 2\eta$$

2nd significance of r:

compute $N_e = \int H\,dt = \int \frac{d\varphi}{\sqrt{2\epsilon}}$

$$\Rightarrow \quad N_e \simeq \frac{\Delta\phi}{M_p} \cdot \frac{1}{\sqrt{2\epsilon}}$$

$$\Leftrightarrow \quad r = 16\epsilon \simeq \frac{8}{N_e^2} \cdot \left(\frac{\Delta\phi}{M_p}\right)^2$$

$$\Rightarrow \quad \boxed{r \simeq 0.003 \cdot \left(\frac{50}{N_e}\right)^2 \cdot \left(\frac{\Delta\phi}{M_p}\right)^2}$$

'Lyth bound'

$\rightsquigarrow$ $r \sim 0.01$ corresponds to boundary between large-field and small-field inflation.