Classical Theory of Optimal Detection in Context of Radio Astronomy

Astroparticle Physics Seminar (DESY)

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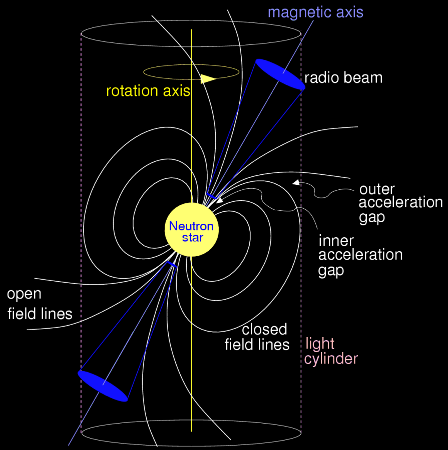
What are the pulsars?



Lorimer and Kramer (2005) *Handbook of pulsar astronomy.*

Supernova core collapse ("death" of a star)

- Radius: 10^6 km $\to$ 10 km (10^{-10} for cross section)
- Magnetic field: 100 G $\to 10^{12}$ G (magnetic flux conservation)
- Rotation period: $P_0 \to 10^{10} \times P_0$ (angular momentum conservation) (0.5 s)

Fast-spinning neutron star

Lighthouse effect — radio beam periodically shines to the observer

Propagation of the pulsar signal

Interstellar medium is not empty, but it is a cold, ionized plasma.

refractive index $\longrightarrow \mu = \sqrt{1 - \left(\frac{f_p}{f}\right)^2}$ plasma freq. $\longrightarrow f_p = \sqrt{\frac{e^2}{\pi m_e} n_e}$ group velocity $\longrightarrow v_g = \mu c$

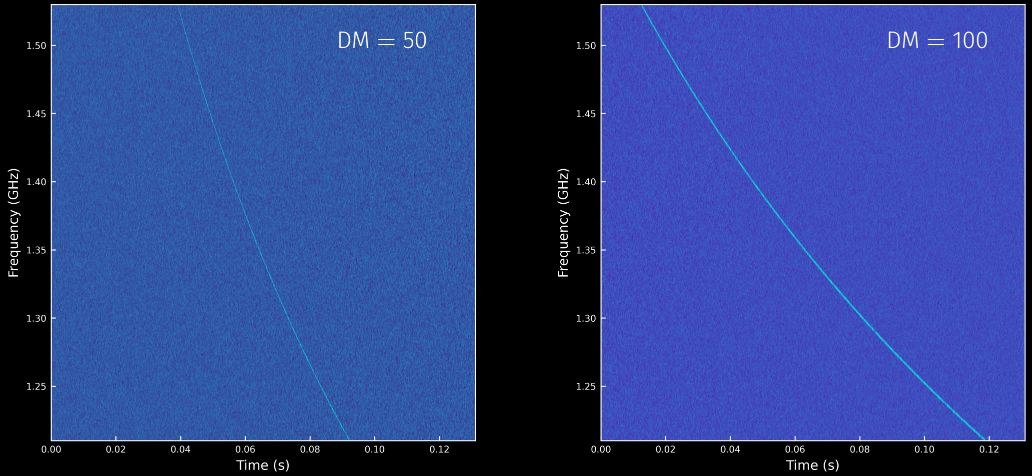
electron number density

dispersion measure

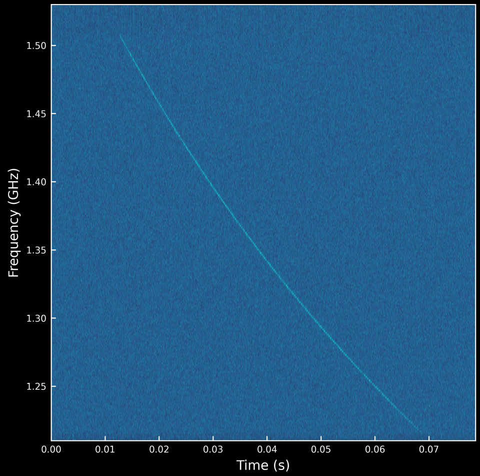
time delay $\longrightarrow \tau = \int_0^d \frac{\mathrm{d}l}{v_g} - \frac{d}{c} \approx \frac{1}{c} \int_0^d \left(1 + \frac{f_p^2}{2f^2}\right) \mathrm{d}l - \frac{d}{c} = \frac{e^2}{2\pi m_e c} \frac{\int_0^d n_e \,\mathrm{d}l}{f^2} = \mathcal{D} \times \frac{\mathrm{DM}}{f^2}$

dispersion constant

Pulsar signals — toy model



Real pulsar signal



Andrey Kazantzev / MPIfR — Crab Pulsar / Effelsberg

- Signals are weak
- Noise level is high
- Radio interferences of wide- and narrow-band type
- Detection is important for many science cases (pulsar timing, etc.)

Bayesian signal detection — I

Data stream: A_1 — signal is **in** the data, A_0 — signal is **not in** the data.

Decision: A_1^* — signal is **in** the data, A_0^* — signal is **not in** the data.

Possible situations:

- $A_0^* A_0$ — correct non-detection, $\hat{F} = P(A_0^*|A_0)$,
- $A_1^* A_0$ — "false alarm" or "false positive" detection, $F = P(A_1^*|A_0)$,
- $A_0^* A_1$ — missing the signal, $\hat{D} = P(A_0^*|A_1)$,
- $A_1^* A_1$ — correct detection of the signal, $D = P(A_1^*|A_1)$.

Bayesian signal detection — II

Stream: A_1 — signal, A_0 — no signal
Decision: A_1^* — signal, A_0^* — no signal

$$\hat{F} = P(A_0^*|A_0) \quad F = P(A_1^*|A_0)$$
$$\hat{D} = P(A_0^*|A_1) \quad D = P(A_1^*|A_1)$$
$$\hat{F} + F = 1, \quad \hat{D} + D = 1$$

$$P(A_0^*, A_0) + P(A_1^*, A_0) + P(A_0^*, A_1) + P(A_1^*, A_1) = 1$$

Mean risk: $\bar{r} = \sum_i r_i P_i =$

$$= r_1 P(A_0^*, A_0) + r_2 P(A_1^*, A_0) + r_3 P(A_0^*, A_1) + r_4 P(A_1^*, A_1) = r_F P(A_1^*, A_0) + r_{\hat{D}} P(A_0^*, A_1)$$

$$P(A_1^*, A_0) = P(A_0) P(A_1^*|A_0) = P(A_0) F, \quad P(A_0^*, A_1) = P(A_1) P(A_0^*|A_1) = P(A_1) \hat{D}$$

$$\bar{r} = r_F F P(A_0) + r_{\hat{D}} \hat{D} P(A_1)$$

Detection criteria

$$\text{Mean risk:}\quad \bar{r} = r_F F P(A_0) + r_{\hat{D}} \hat{D} P(A_1)$$

Bayes' criterion: $\operatorname{argmin} \bar{r}$

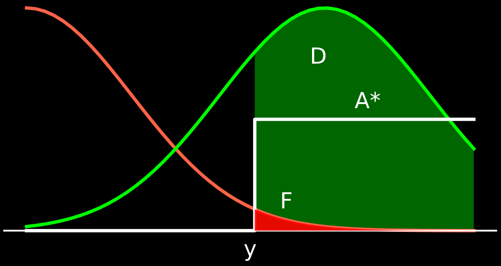
Criterion of "ideal observer": $\operatorname{argmin} \bar{r} = \operatorname{argmin}(FP(A_0) + \hat{D}P(A_1)) \qquad (r_F = 1, r_{\hat{D}} = 1)$

Neyman–Pearson criterion: $\operatorname{argmax}(D)$ while F is fixed (fixed "false alarm" rate)

$$\bar{r} = r_{\hat{D}} P(A_1) - [D - l_0 F] r_{\hat{D}} P(A_1), \qquad l_0 = r_F P(A_0) / r_{\hat{D}} P(A_1)$$

Weighting criterion: $\operatorname{argmax}\ (D - l_0 F)$

Statistical Signal Detection



Example 1. $y = Ax + n, (A = 0, 1),$ $\quad p(y|A_0) = p_n(y),$ $\quad p(y|A_1) = p_{sn}(y).$

$$D = \int_{-\infty}^{\infty} A^*(y) p_{sn}(y)\, \mathrm{d}y \qquad F = \int_{-\infty}^{\infty} A^*(y) p_n(y)\, \mathrm{d}y$$

Statistical Signal Detection

$$y = Ax + n,\ (A = 0, 1), \qquad p(y|A_0) = p_{\mathrm{n}}(y), \qquad p(y|A_1) = p_{\mathrm{sn}}(y).$$

$$D = \int_{-\infty}^{\infty} A^*(y) p_{\mathrm{sn}}(y)\,\mathrm{d}y \qquad F = \int_{-\infty}^{\infty} A^*(y) p_{\mathrm{n}}(y)\,\mathrm{d}y$$

$$D - l_0 F = \int_{-\infty}^{\infty} A^*(y) p_{\mathrm{n}}(y)[l(y) - l_0]\,\mathrm{d}y \quad \to A^*(y) = 1 \text{ when } \ l(y) - l_0 > 0 \quad l(y) = \frac{p_{\mathrm{sn}}(y)}{p_{\mathrm{n}}(y)} \text{ — likelihood ratio}$$

If the “noise” is Gaussian:

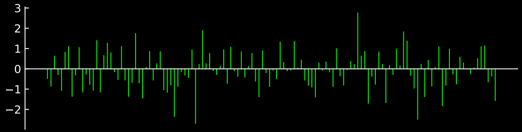
$$l(y) = \mathbf{exp}\left(-\frac{x^2}{2n_0^2}\right)\mathbf{exp}\left(\frac{xy}{n_0^2}\right) \quad \text{or} \quad l(y) = \mathbf{exp}\left(-\frac{1}{2n_0^2}\sum_i x_i^2 \Delta t\right)\mathbf{exp}\left(\frac{1}{n_0^2}\sum_i x_i y_i \Delta t\right)$$

$\sum_i xy\Delta t$ — correlation sum (or correlation integral in continuous case)

Correlation sum/integral is the optimal filter Bayesian sense (maximizes sensitivity)

General formulation of the detection problem in radio astronomy

The Effelsberg telescope



Bayes' approach to the signal detection:

$$\left.\begin{aligned} H_0 &: r(t) = w(t) \\ H_1 &: r(t) = w(t) + s(t) \end{aligned}\right\} t \in (T),\ w(t) \in \mathbb{C},\ s(t) \in \mathbb{C}$$

$w(t)$ — white noise, $s(t)$ — Gaussian signal:

$$w(t) : \mathbb{E}[w(t)] = 0 \qquad \mathbb{E}[w(t)w^*(t')] = N_0(1+i)\delta(t-t')$$

$$s(t) : \mathbb{E}[s(t)] = m(t) \qquad \mathbb{E}[(s(t)-m(t))(s(t')-m(t'))^*] = K_s(t,t')$$

Classical solution

By using the Karhunen–Loève expansion:

$$s(t) = \sum_n s_n \phi_n(t) \qquad \phi_n(t) - \text{orthogonal functions}$$

$$\mathbb{E}[(s_n - m_n)(s_k - m_k)^*] = \lambda_n \delta_{nk} \qquad m(t) = \sum_n m_n \phi_n(t)$$

$$l_R + l_D \underset{H_0}{\overset{H_1}{\gtrless}} \gamma : \qquad \begin{cases} l_R = \dfrac{1}{N_0} \displaystyle\sum_n \dfrac{\lambda_n}{\lambda_n + N_0} \left| \int_{(T)} r(t) \phi_n^*(t) \, \mathrm{d}t \right|^2 \\ l_D = 2 \displaystyle\sum_n \dfrac{1}{\lambda_n + N_0} \mathrm{Re} \left\{ \left[\int_{(T)} r(t) \phi_n^*(t) \, \mathrm{d}t \right] \left[\int_{(T)} m(t) \phi_n^*(t) \, \mathrm{d}t \right]^* \right\} \end{cases}$$

- Threshold γ is determined with Bayes or Neyman-Pearson criterion (fixed false-detection rate)
- Corresponds to the coherent de-dispersion for a known deterministic signal

Classical solution for a deterministic signal (matched filter)

$$s(t) = m(t) = \phi_0(t) = f(t) \text{ — fully known signal}$$

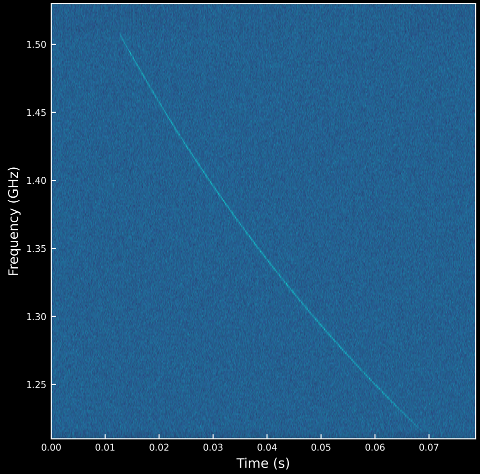
$$\mathbb{E}[(s_n - m_n)(s_k - m_k)^*] = 0 \qquad \lambda_n = 0$$

$$l_R = \frac{1}{N_0} \sum_n \frac{\lambda_n}{\lambda_n + N_0} \left| \int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right|^2 = 0$$

$$l_D = 2 \sum_n \frac{1}{\lambda_n + N_0} \mathrm{Re}\left\{ \left[\int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right] \left[\int_{(T)} m(t)\phi_n^*(t)\,\mathrm{d}t \right]^* \right\} = \underbrace{\frac{2E}{N_0} \mathrm{Re}\left\{ \int_{(T)} r(t) f^*(t)\,\mathrm{d}t \right\}}_{\text{matched filter}}$$

$$E = \int_{(T)} f(t) f^*(t)\,\mathrm{d}t \text{ — energy of the signal}$$

Time-frequency analysis and spectrogram



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- Natural feeling how signals look like or sound
- Certain signals have particular signature on the time-frequency plane
- Spectrogram is the most common way to analyze the signals

Spectrogram and its properties

Short-time Fourier transform:

$$\text{spectrogram} - |S_t(\omega)|^2 = \left|\mathcal{F}\{s(\tau)\,\underbrace{h(\tau - t)}_{\text{window}}\}\right|^2 = \left|\int \mathrm{e}^{-i\omega\tau} s(\tau)\,\underbrace{h(\tau - t)}_{\text{window}}\,\mathrm{d}\tau\right|^2$$

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Properties:

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Properties:

- total energy conservation: $E_{\mathrm{SP}} = \int\int |S_t(\omega)|^2 \, dt \, d\omega = \int \underbrace{|s(t)|^2}_{\text{signal energy}} dt \times \int |h(t)|^2 \, dt$

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- time marginal: $P(t) = \int |S_t(\omega)|^2 \, d\omega = \int |s(\tau)|^2 |h(\tau - t)|^2 \, dt$

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- frequency marginal: $P(\omega) = \int |S_t(\omega)|^2\,dt = \int |S(\omega')|^2 |H(\omega - \omega')|^2\,d\omega'$

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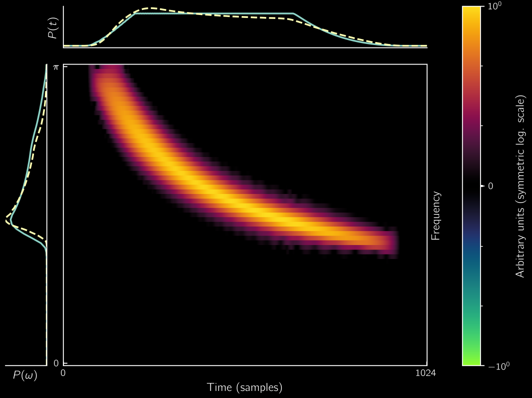
Marginals are not conserved $\rightarrow$ energy is scrambled in the t-f plane

Spectrogram — illustrative example (fully known signal)

- Easy to understand and interpret
- Both marginals are not conserved
- Suitable for optimal detector?



Cohen's class of time-frequency distributions

$$C_{xy}(t,\omega;\Pi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(t-t',\omega-\omega')\, W_{xy}(t',\omega')\,\mathrm{d}t'\,\frac{\mathrm{d}\omega'}{2\pi}$$

$$W_{xy} = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right) y^*\left(t-\frac{\tau}{2}\right) \mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau$$ — cross-Wigner-Ville distribution

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$\Pi(t,\omega)$ — arbitrary normalized function:

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$\Pi(t,\omega)$ — arbitrary normalized function:

- $\Pi(t,\omega) = \Pi_W(t,\omega) = 2\pi\delta(t)\delta(\omega)$ gives auto Wigner-Ville distribution

$$C_{xx}(t,\omega;\Pi_W) = W_{xx}(t,\omega)$$

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- $\Pi(t,\omega) = \Pi_S(t,\omega) = W_{hh}(t,\omega)$ gives classical spectrogram

$$C_{xx}(t,\omega;\Pi_S) = S_{xx} = \left|\int_{-\infty}^{\infty} x(u)h^*(t-u)\,\mathrm{e}^{-i\omega u}\,\mathrm{d}u\right|^2 = \left|\mathcal{F}\left\{x(u)h^*(t-u)\right\}\right|^2$$

Cohen's class of time-frequency distributions

$$C_{xy}(t, \omega; \Pi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi(t - t', \omega - \omega') \, W_{xy}(t', \omega') \, \mathrm{d}t' \, \frac{\mathrm{d}\omega'}{2\pi}$$

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- many other variants...

Cohen's class optimal detector

$$l_R^{(MF)} = \frac{1}{N_0} \sum_n \frac{\lambda_n}{\lambda_n + N_0} \left| \int_{(T)} r(t) \phi_n^*(t) \, \mathrm{d}t \right|^2 , \quad l_D^{(MF)} = 2 \sum_n \frac{1}{\lambda_n + N_0} \mathrm{Re} \left\{ \left[\int_{(T)} r(t) \phi_n^*(t) \, \mathrm{d}t \right] \left[\int_{(T)} m(t) \phi_n^*(t) \, \mathrm{d}t \right]^* \right\}$$

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$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{x_1x_2}(t,\omega;\Pi)\, C_{x_3x_4}^*(t,\omega;\Pi)\,\mathrm{d}t\,\frac{\mathrm{d}\omega}{2\pi} = \left[\int_{-\infty}^{\infty} x_1(t)x_3^*(t)\,\mathrm{d}t \right] \left[\int_{-\infty}^{\infty} x_2(t)x_4^*(t)\,\mathrm{d}t \right]^* \qquad \text{— Moyal relation}$$

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$$|\pi(\eta,\tau)| = 1 \qquad \pi(\eta,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Pi(t,\omega)\,\mathrm{e}^{-i(\eta t+\omega\tau)}\,\mathrm{d}t\,\frac{\mathrm{d}\omega}{2\pi} \qquad \rightarrow \qquad \text{spectrogram is not suitable (!)}$$

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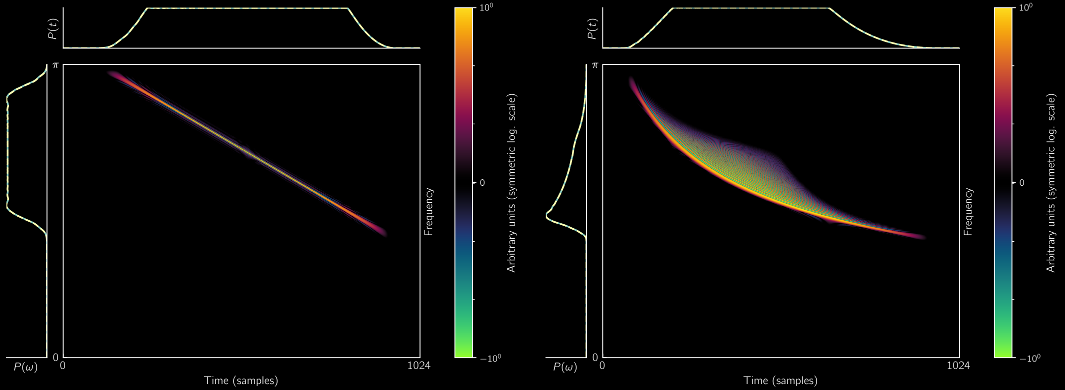
$$\left.\begin{aligned} l_R &= \frac{1}{N_0}\int_{-\infty}^{\infty}\int_{(T)} W_{rr}(t,\omega)\left[\sum_n \frac{\lambda_n}{\lambda_n+N_0}W^*_{\phi_n\phi_n}(t,\omega)\right]\mathrm{d}t\,\frac{\mathrm{d}\omega}{2\pi} \\ l_D &= 2\int_{-\infty}^{\infty}\int_{(T)} \mathrm{Re}\left\{W_{rm}(t,\omega)\right\}\left[\sum_n \frac{1}{\lambda_n+N_0}W^*_{\phi_n\phi_n}(t,\omega)\right]\mathrm{d}t\,\frac{\mathrm{d}\omega}{2\pi}\end{aligned}\right\}$$

opt. det. for Wigner-Ville distributions ($\Pi(t,\omega) = 2\pi\delta(t)\delta(\omega)$)

Property of perfect localization (Wigner-Ville distribution)



Bertrads' class of time-frequency distributions

For analytical signals $X(\nu)$ and $Y(\nu)$:

$$B_{xy}(t,f) = f^{2r-q+2} \int_{-\infty}^{\infty} X(f\lambda(u))\, Y^*(f\lambda(-u))\, \mu(u)\, \mathrm{e}^{i2\pi tf[\lambda(u)-\lambda(-u)]}\, \mathrm{d}u$$

r, q — fixed parameters

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r, q — fixed parameters

$$\lambda(u) = \left(k\frac{\mathrm{e}^{-u}-1}{\mathrm{e}^{-ku}-1}\right)^{1/(k-1)}$$

$k \neq 0, 1$, k — arbitrary number characterizing distribution

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$k \neq 0, 1$, k — arbitrary number characterizing distribution

$$\mu^{\mathrm{L}}(u) = \left|\frac{\mathrm{d}}{\mathrm{d}u}\left[\lambda(u)-\lambda(-u)\right]\right|\left[\lambda(u)\lambda(-u)\right]^{r+1}$$

— localized form (in the t-f plane)

$$\mu^{\mathrm{A}}(u) = \left[\lambda(u)\lambda(-u)\right]^{r+1}$$

— auxiliary form

Bertrads’ class of time-frequency distributions

For analytical signals $X(\nu)$ and $Y(\nu)$:

$$B_{xy}(t,f) = f^{2r-q+2}\int_{-\infty}^{\infty} X(f\lambda(u))\, Y^*(f\lambda(-u))\,\mu(u)\,\mathrm{e}^{i2\pi tf[\lambda(u)-\lambda(-u)]}\,\mathrm{d}u$$ r, q — fixed parameters

$$\lambda(u) = \left(k\frac{\mathrm{e}^{-u}-1}{\mathrm{e}^{-ku}-1}\right)^{1/(k-1)}$$ $k \neq 0, 1,\ k$ — arbitrary number characterizing distribution

$$\mu^{\mathrm{L}}(u) = \left|\frac{\mathrm{d}}{\mathrm{d}u}[\lambda(u)-\lambda(-u)]\right|[\lambda(u)\lambda(-u)]^{r+1}$$ — localized form (in the t-f plane)

$$\mu^{\mathrm{A}}(u) = [\lambda(u)\lambda(-u)]^{r+1}$$ — auxiliary form

General internal product property:

$$\int_0^{\infty}\int_{-\infty}^{\infty} B^{\mathrm{L}}_{x_1x_2}(t,f)\,B^{\mathrm{A}*}_{x_3x_4}(t,f)\,f^{2q}\,\mathrm{d}t\,\mathrm{d}f = \left[\int_0^{\infty} x_1(f)\,x_3^*(f)\,f^{2r+1}\,\mathrm{d}f\right]\left[\int_0^{\infty} x_2(f)\,x_4^*(f)\,f^{2r+1}\,\mathrm{d}f\right]^*$$

Bertrads' class optimal detector

$$l_R^{(MF)} = \frac{1}{N_0} \sum_n \frac{\lambda_n}{\lambda_n + N_0} \left| \int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right|^2, \quad l_D^{(MF)} = 2 \sum_n \frac{1}{\lambda_n + N_0} \mathrm{Re}\left\{ \left[\int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right] \left[\int_{(T)} m(t)\phi_n^*(t)\,\mathrm{d}t \right]^* \right\}$$

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$$\int_0^\infty \int_{-\infty}^\infty B^{\mathrm{L}}_{x_1 x_2}(t,f)\, B^{\mathrm{A}*}_{x_3 x_4}(t,f)\,\mathrm{d}t\,\mathrm{d}f = \left[\int_0^\infty x_1(f)\, x_3^*(f)\mathrm{d}f \right] \left[\int_0^\infty x_2(f)\, x_4^*(f)\mathrm{d}f \right]^* \qquad q = 0, \quad r = {}^1/_2$$

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$$\left.\begin{aligned} l_R &= \frac{1}{N_0}\int_{-\infty}^\infty\int_{(T)} B^{\mathrm{L}}_{rr}(t,\omega)\left[\sum_n \frac{\lambda_n}{\lambda_n+N_0}B^{\mathrm{A}*}_{\phi_n\phi_n}(t,\omega)\right]\,\mathrm{d}t\,\frac{\mathrm{d}\omega}{2\pi} \\ l_D &= 2\int_{-\infty}^\infty\int_{(T)} \mathrm{Re}\left\{B^{\mathrm{L}}_{rm}(t,\omega)\right\}\left[\sum_n \frac{1}{\lambda_n+N_0}B^{\mathrm{A}*}_{\phi_n\phi_n}(t,\omega)\right]\,\mathrm{d}t\,\frac{\mathrm{d}\omega}{2\pi}\end{aligned}\right\}$$

opt. det. for Bertrands' class
(k is arbitrary)
Submitted to *IEEE Trans. on Signal Processing*

Bertrads' class optimal detector

$$l_R^{(MF)} = \frac{1}{N_0} \sum_n \frac{\lambda_n}{\lambda_n + N_0} \left| \int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right|^2, \quad l_D^{(MF)} = 2 \sum_n \frac{1}{\lambda_n + N_0} \mathrm{Re} \left\{ \left[\int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right] \left[\int_{(T)} m(t)\phi_n^*(t)\,\mathrm{d}t \right]^* \right\}$$

$$\int_0^\infty \int_{-\infty}^\infty B^{\mathrm{L}}_{x_1 x_2}(t,f)\, B^{\mathrm{A}*}_{x_3 x_4}(t,f)\,\mathrm{d}t\,\mathrm{d}f = \left[\int_0^\infty x_1(f)\, x_3^*(f)\mathrm{d}f \right] \left[\int_0^\infty x_2(f)\, x_4^*(f)\mathrm{d}f \right]^* \qquad q = 0, \quad r = {}^1\!/_2$$

$$l_R = \frac{1}{N_0} \int_{-\infty}^\infty \int_{(T)} B^{\mathrm{L}}_{rr}(t,\omega) \left[\sum_n \frac{\lambda_n}{\lambda_n + N_0} B^{\mathrm{A}*}_{\phi_n \phi_n}(t,\omega) \right] \mathrm{d}t\, \frac{\mathrm{d}\omega}{2\pi}$$

$$l_D = 2 \int_{-\infty}^\infty \int_{(T)} \mathrm{Re}\left\{ B^{\mathrm{L}}_{rm}(t,\omega) \right\} \left[\sum_n \frac{1}{\lambda_n + N_0} B^{\mathrm{A}*}_{\phi_n \phi_n}(t,\omega) \right] \mathrm{d}t\, \frac{\mathrm{d}\omega}{2\pi}$$

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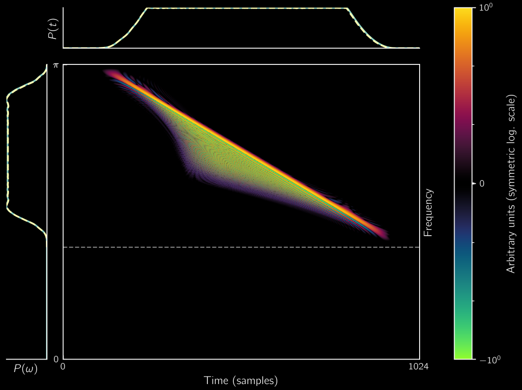
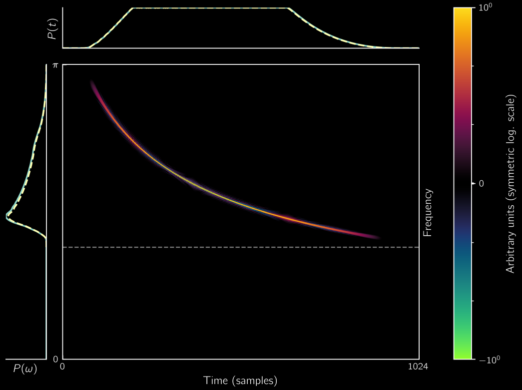
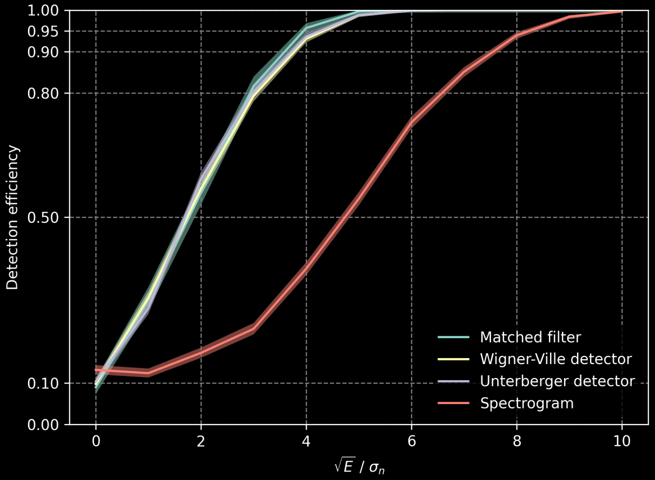
For radio astronomy signals the Unterberger distribution ($k = -1$) is the most suitable (provides perfect localization the signals having the group delay $1/\nu^2$).

Property of perfect localization (Unterberger distribution)

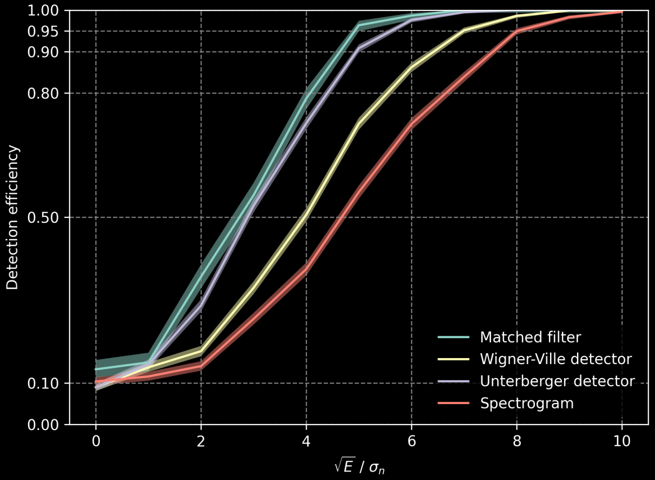




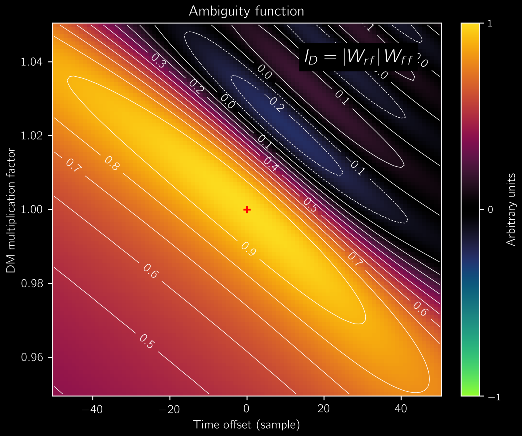
Sensitivity of the detectors

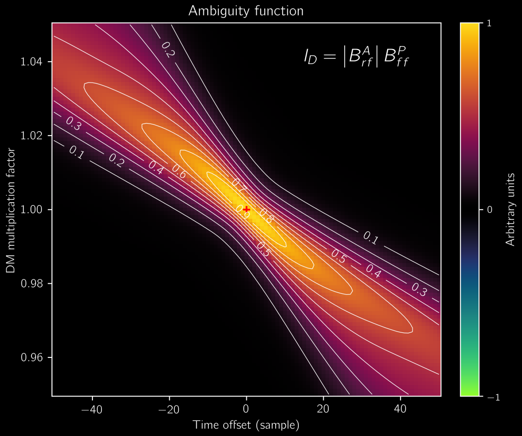


Sensitivity of the phase agnostic detectors ($\mathrm{Re}\{\,\cdot\,\} \to |\,\cdot\,|$)



Ambiguity functions for power-law chirps





Summary

- Bayesian signal detection provide large set of possible detection criteria (e.g. likelihood ratio detector and correlation sum/integral detectors)
- Optimal = maximizes the detection efficiency
- Optimal detector has a particular structure that can guide our choice of representations
- Spectrogram is a simple time-frequency distribution, but with a certain limitations
- There are alternatives to spectrogram (Wigner-Ville distribution, Unterberger distribution, *etc*)