

# Photo- and Hadrodisintegration constraints on massive relics decaying into neutrinos



Jonas is not talking about Dark Photons????



SAPIENZA  
UNIVERSITÀ DI ROMA

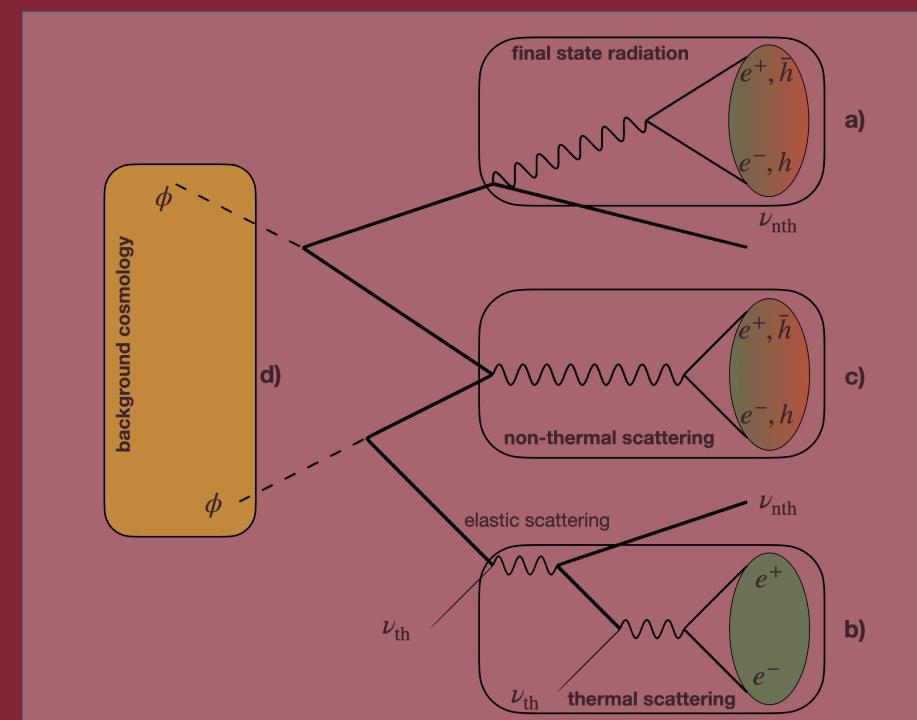


Jonas Frerick (jonas.frerick@uniroma1.it)

Based on 2505.01492

w/ Sara Bianco, Frederik Depta, Thomas Hambye,  
Marco Hufnagel & Kai Schmidt-Hoberg

Theory Seminar  
23.06.2025  
DESY Hamburg



Funded by  
the European Union



European Research Council

# What's on the menu today?

→ Motivation

*Observable effects of unstable relics*

*Effects of decays into neutrinos*

→ Interlude

*BBN in the SM*

*Photodisintegration & Hadrodisintegration*

→ Detailed treatment of neutrino injections

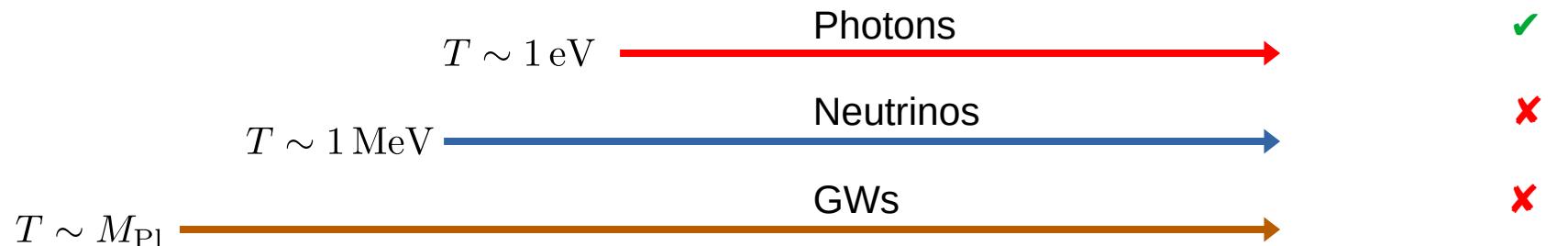
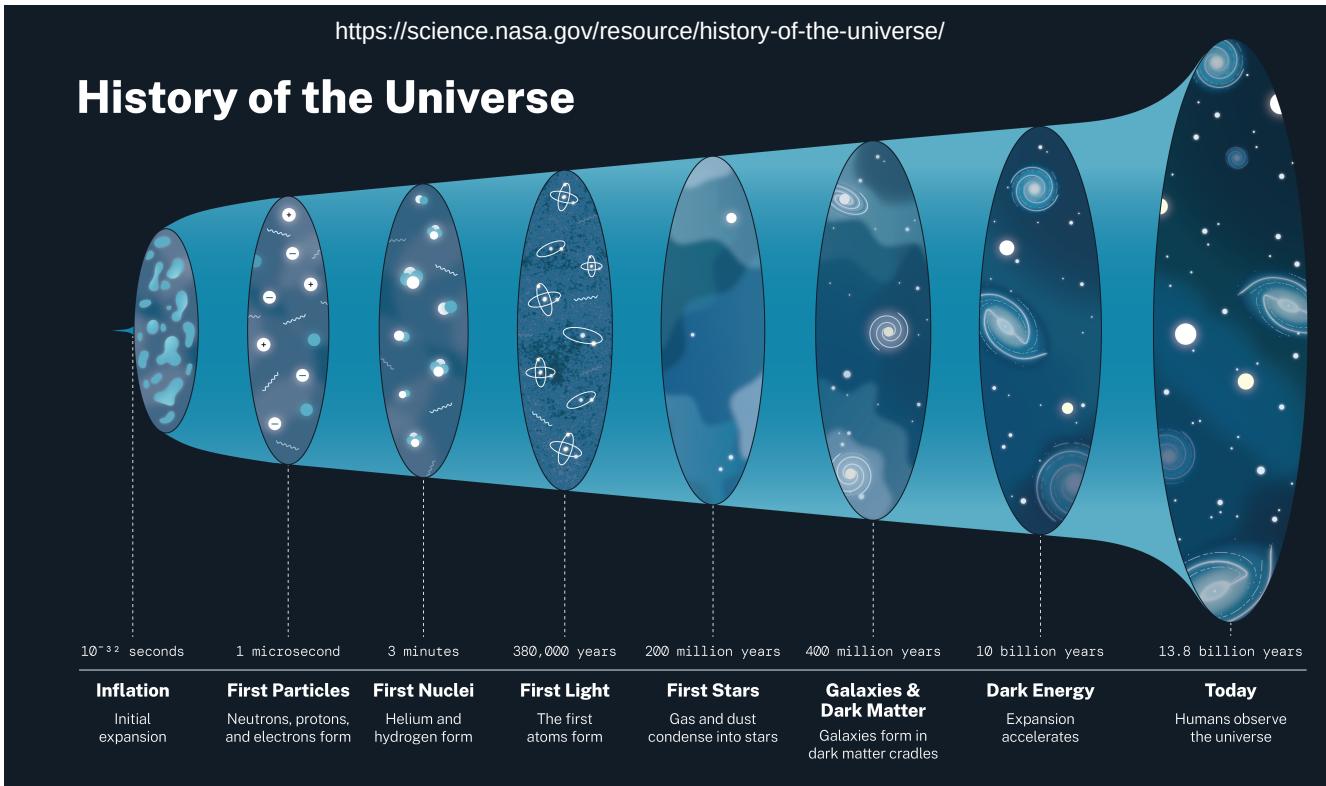
*Ingredients for a full treatment*

*How to treat the injections in practice*

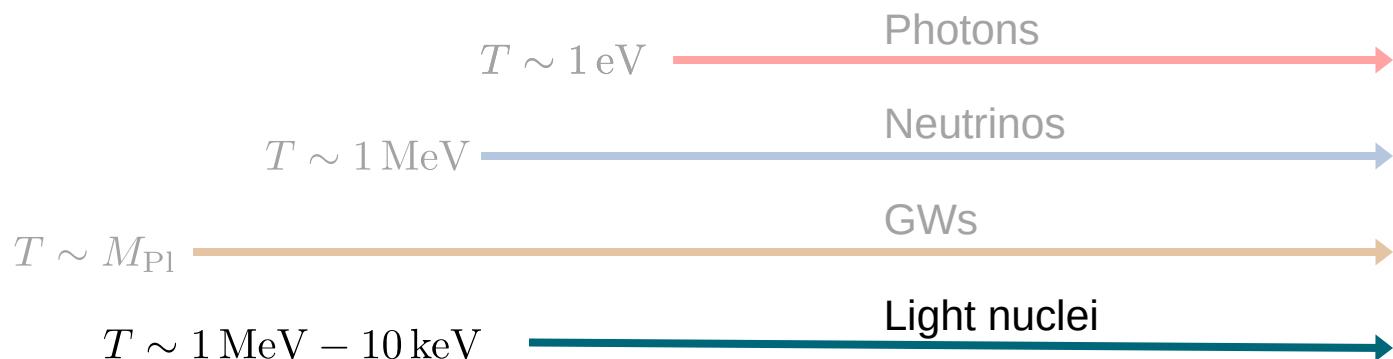
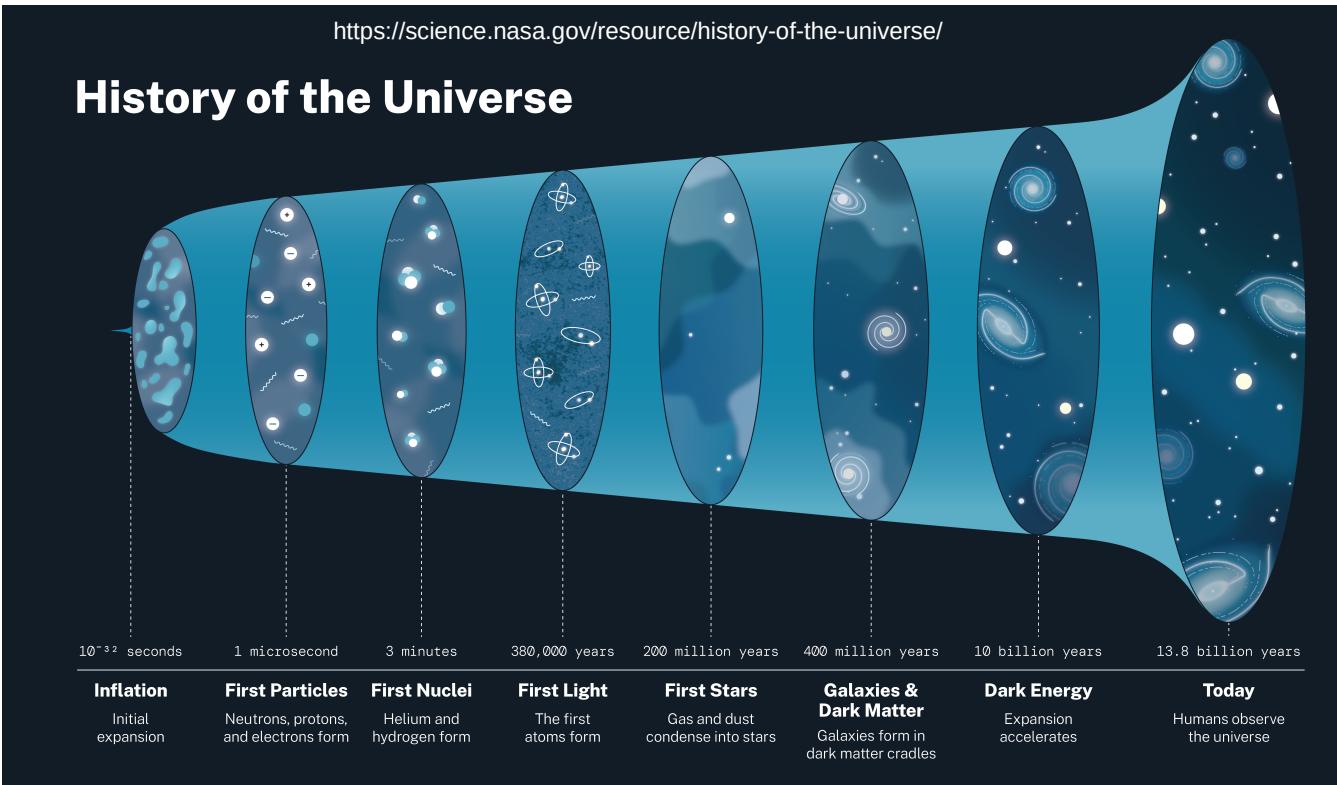
→ Results, Conclusions & Outlook



# Investigating unstable relics in the early Universe



# Investigating unstable relics in the early Universe



Light nuclei can be the strongest probe of new physics effects between MeV and eV temperatures

Detection:



# Introducing a new unstable (heavy) relic

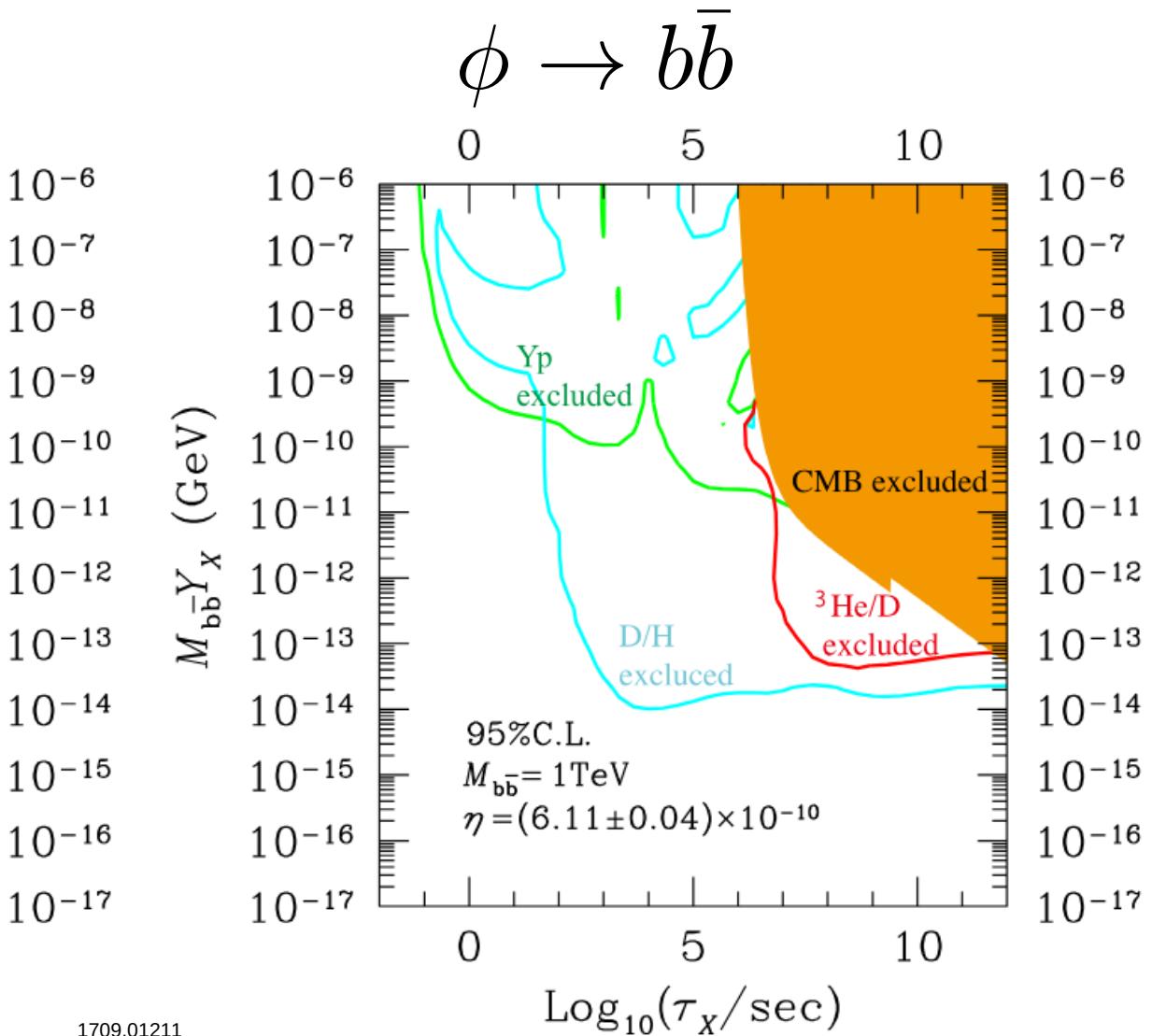
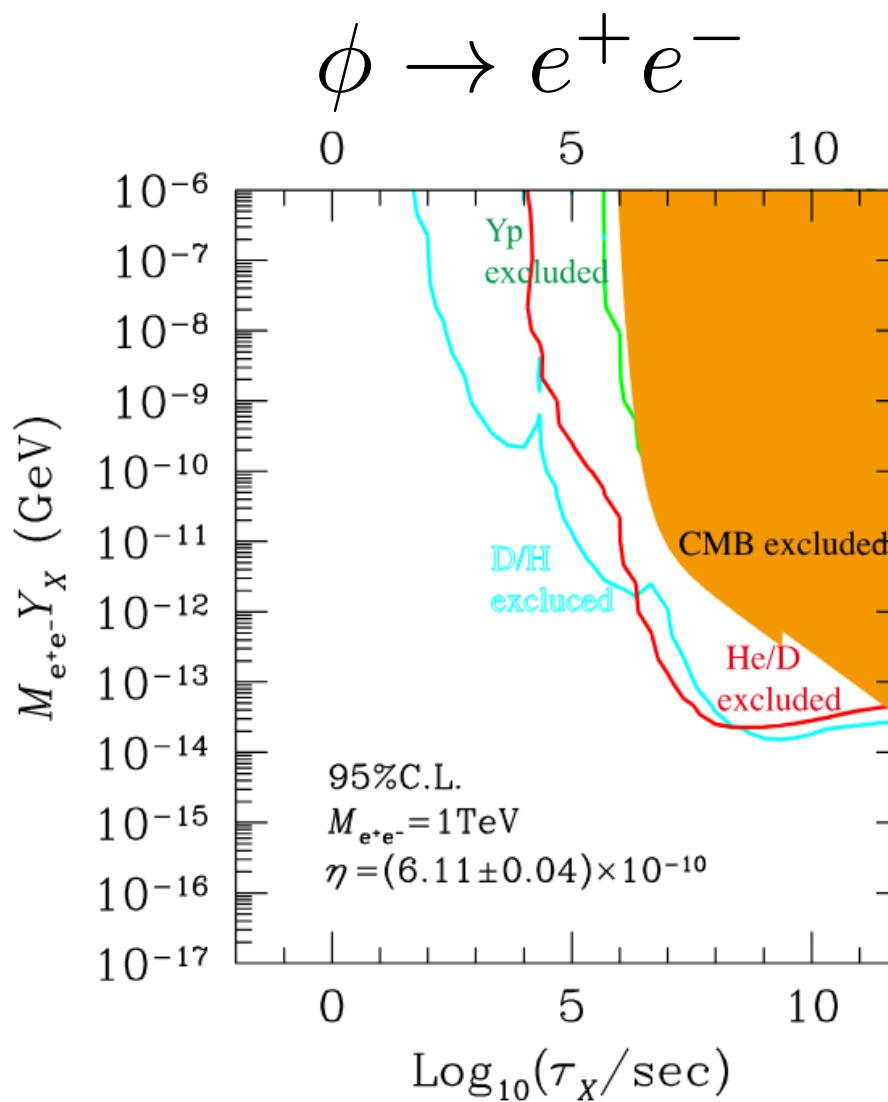
$$\phi \rightarrow \left\{ \begin{array}{l} \text{EM Particles} \\ \text{Hadrons} \\ \text{Neutrinos} \\ \text{Dark Sector} \end{array} \right\}$$

*Immediate destruction of the nuclei*

**???**

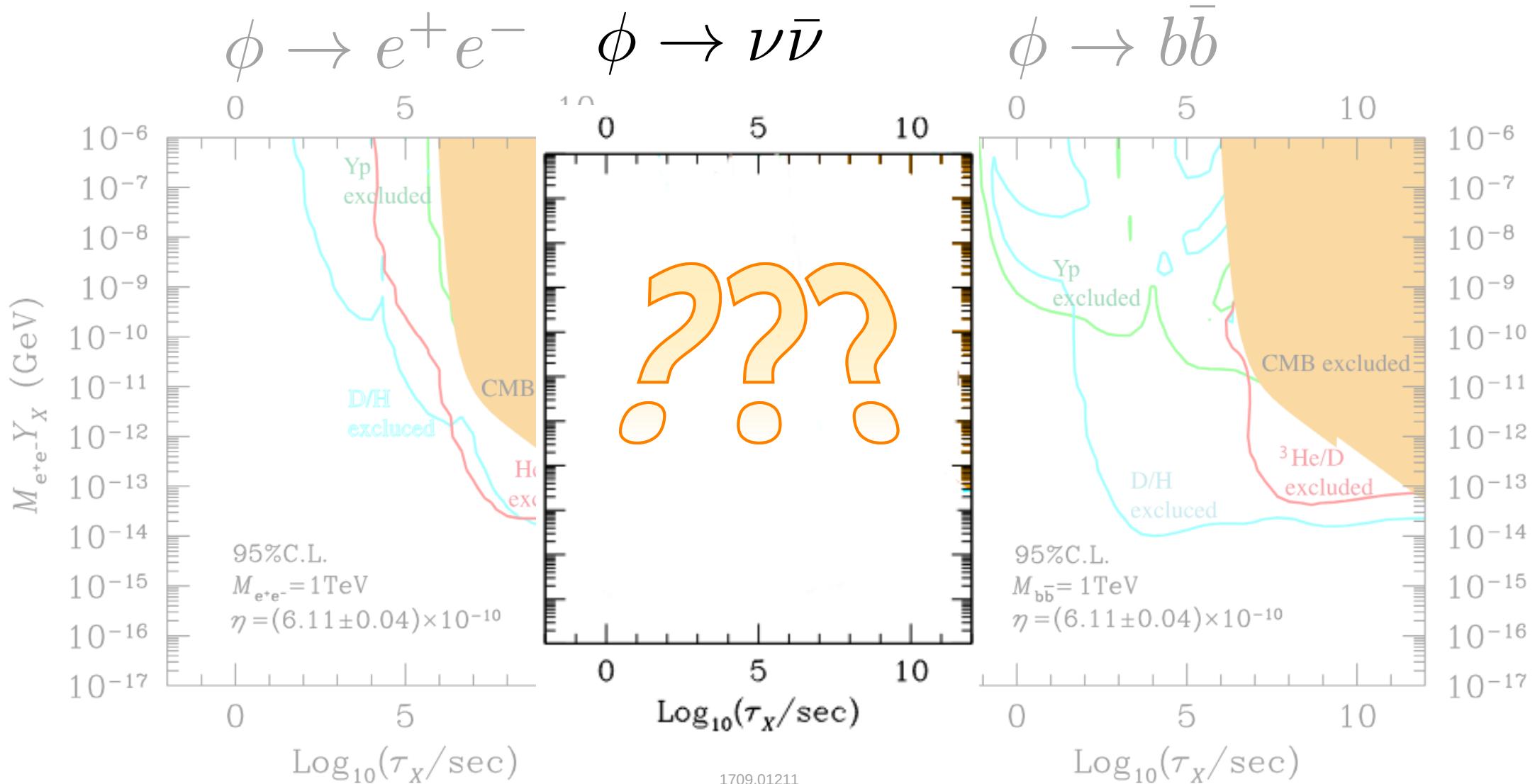
*Effects on the background cosmology*

# Can you hide an unstable Dark Sector in neutrinos?



1709.01211

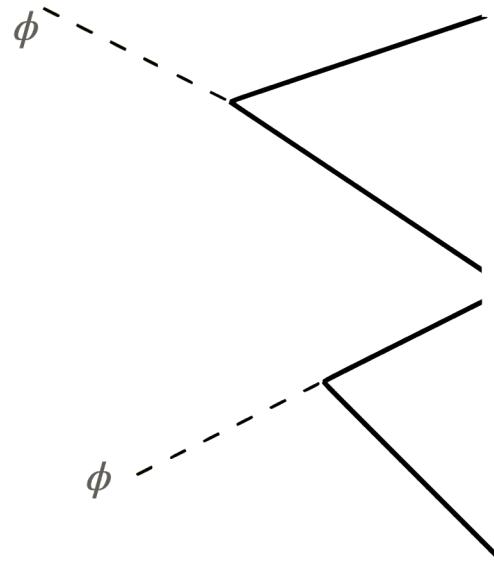
# Can you hide an unstable Dark Sector in neutrinos?



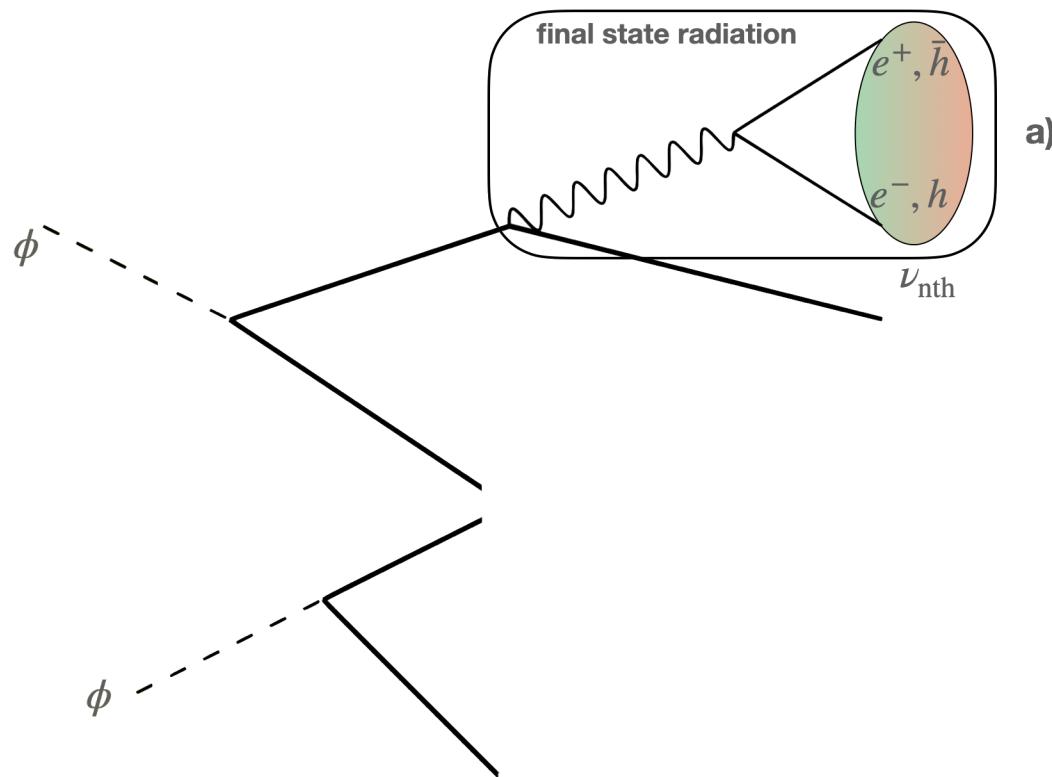
1709.01211

# Neutrino injections $\neq$ Injection of only neutrinos

How much harm can a neutrino do?



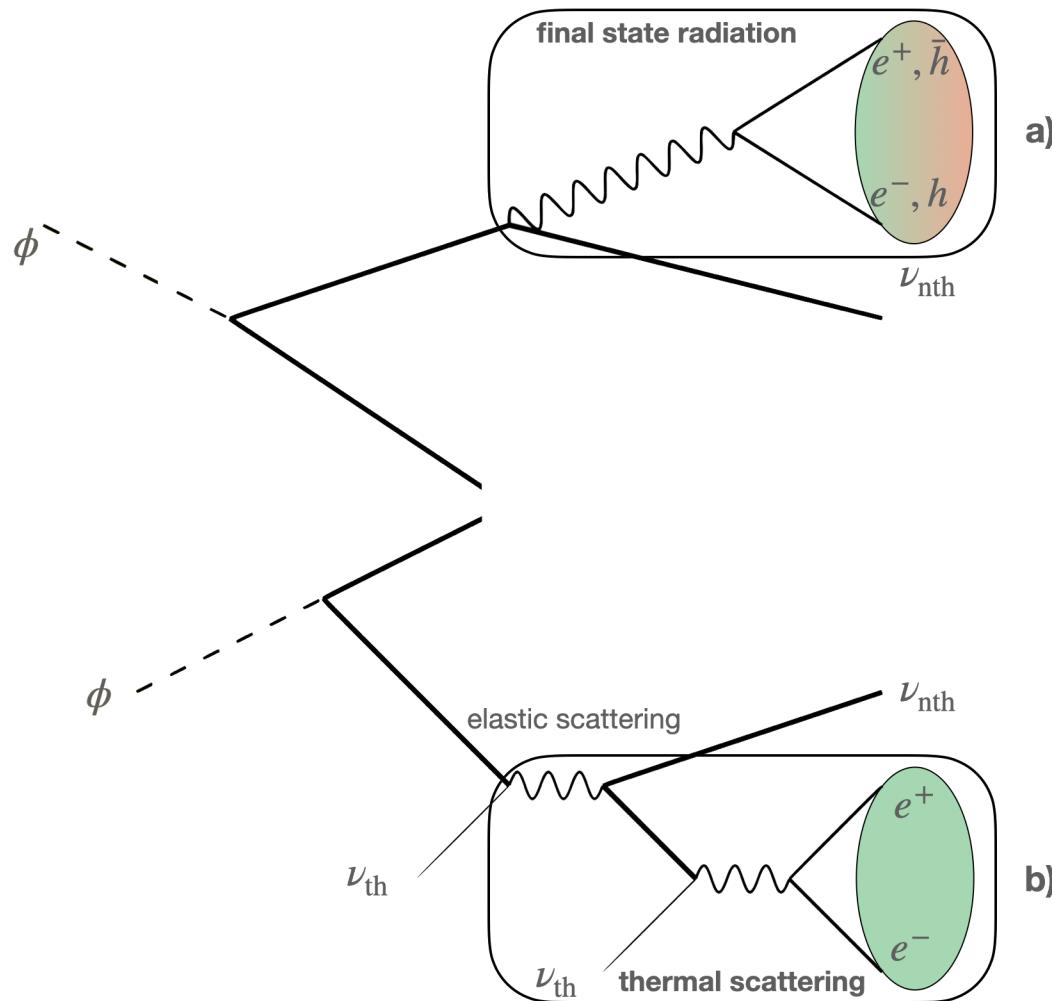
# Neutrino injections $\neq$ Injection of only neutrinos



How much harm can a neutrino do?

a) **Final-state radiation**, injects **EM** and **hadrons**

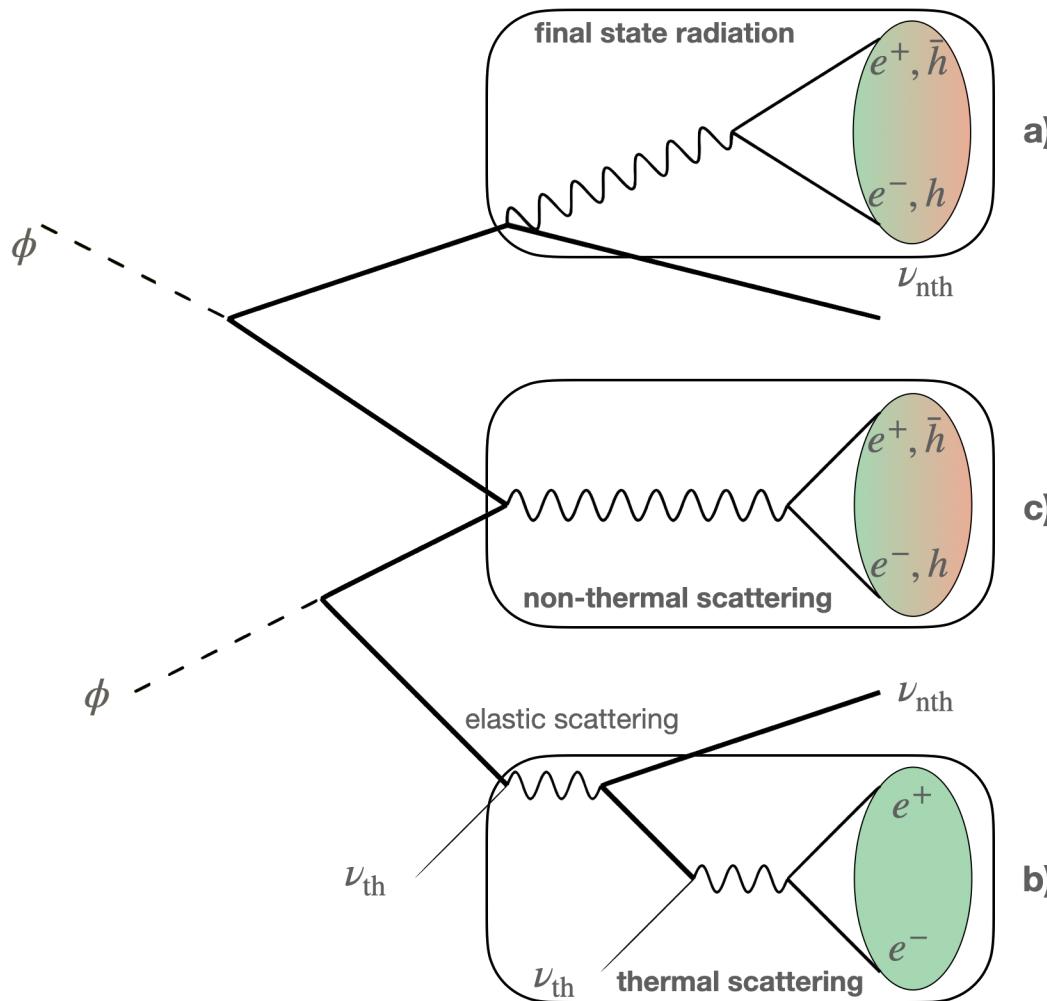
# Neutrino injections $\neq$ Injection of only neutrinos



How much harm can a neutrino do?

- a) **Final-state radiation**, injects **EM** and **hadrons**
- b) **Thermal scattering**, injects **EM**

# Neutrino injections $\neq$ Injection of only neutrinos

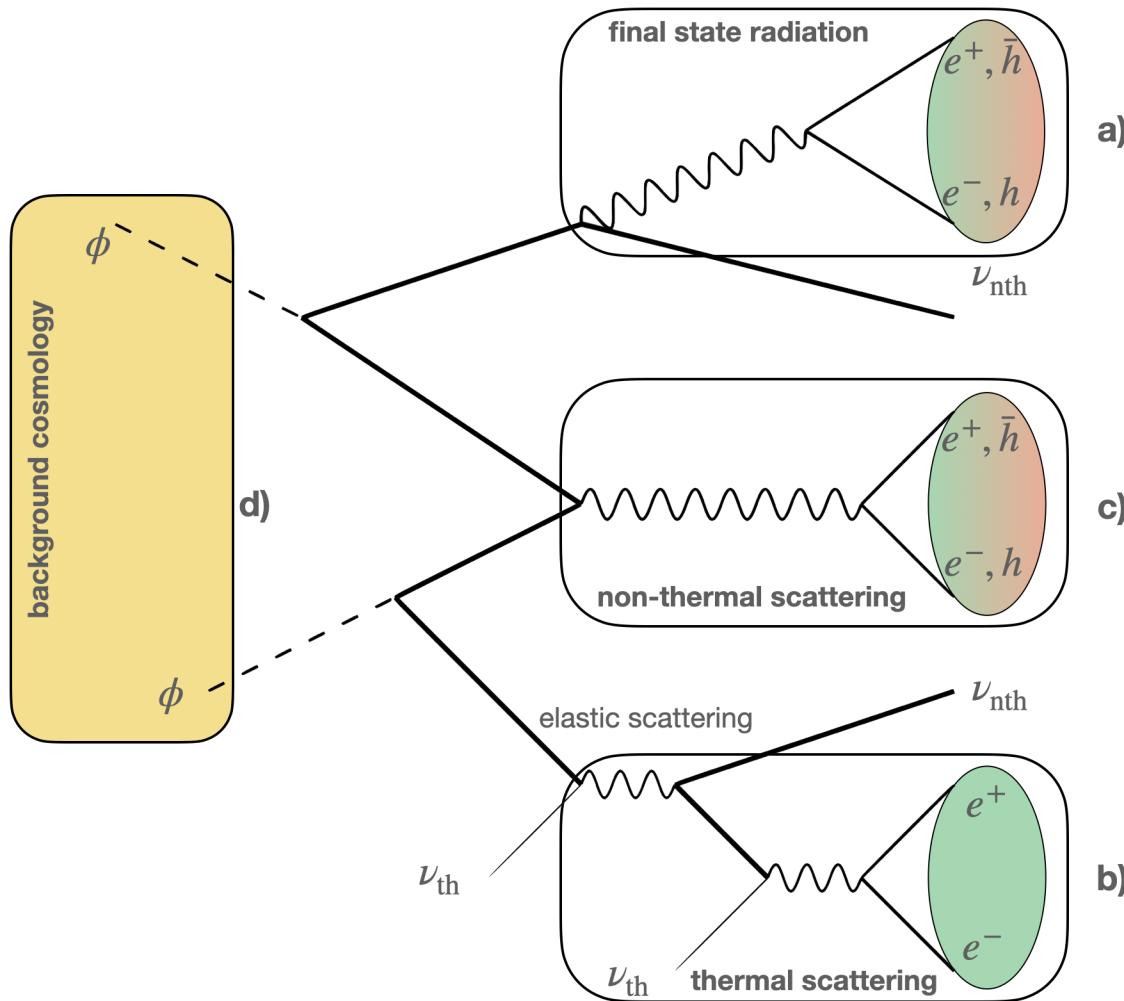


How much harm can a neutrino do?

- a) **Final-state radiation**, injects **EM** and **hadrons**
- b) **Thermal scattering**, injects **EM**
- c) **Non-thermal scattering**, injects **EM** and **hadrons**

**Neutrino cascade**

# Neutrino injections $\neq$ Injection of only neutrinos



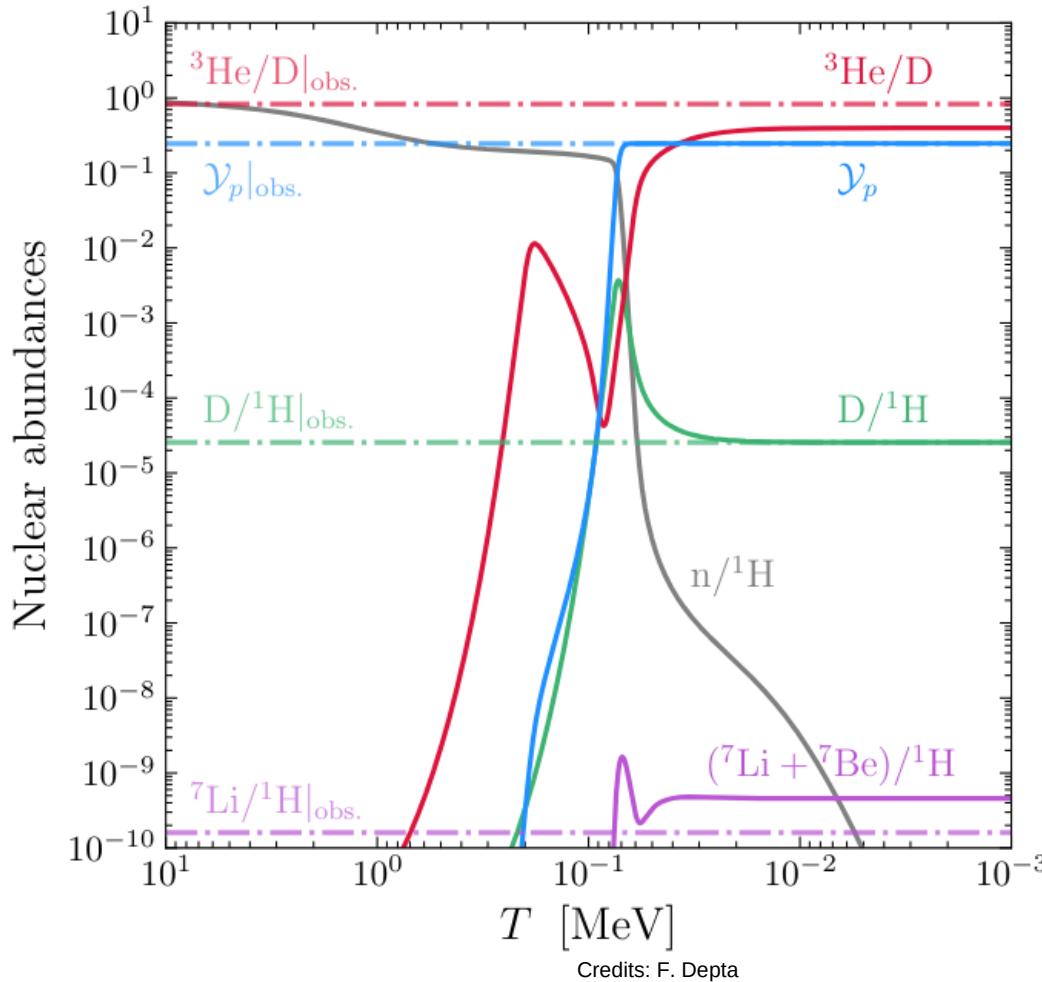
How much harm can a neutrino do?

- a) **Final-state radiation**, injects **EM** and **hadrons**
- b) **Thermal scattering**, injects **EM**
- c) **Non-thermal scattering**, injects **EM** and **hadrons**
- d) Changes in the **background cosmology**

# **Interlude: A brief summary of BBN and disintegration**

## Interlude: BBN data, photodisintegration & hadrodisintegration

We use a modified version of AlterBBN [1806.11095].



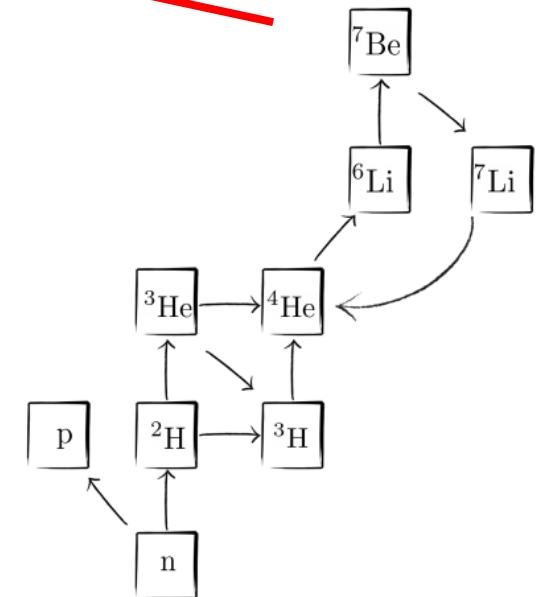
$$\mathcal{Y}_p = (2.45 \pm 0.03) \times 10^{-1}$$

$$\text{D}/{}^1\text{H} = (2.547 \pm 0.029) \times 10^{-5}$$

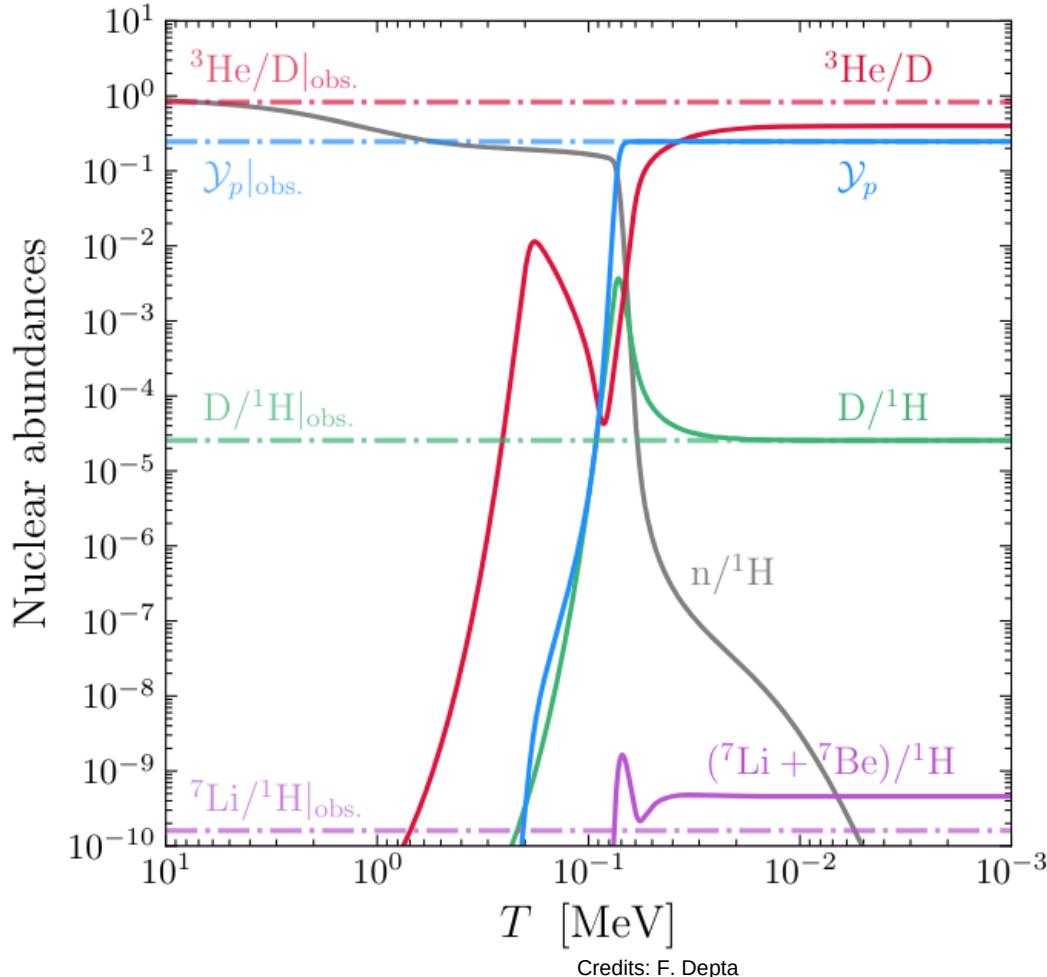
$${}^3\text{He}/\text{D} = (8.3 \pm 1.5) \times 10^{-1}$$

$${}^7\text{Li}/{}^1\text{H} = (1.6 \pm 0.3) \times 10^{-10}$$

Reaction chain of  
Standard BBN

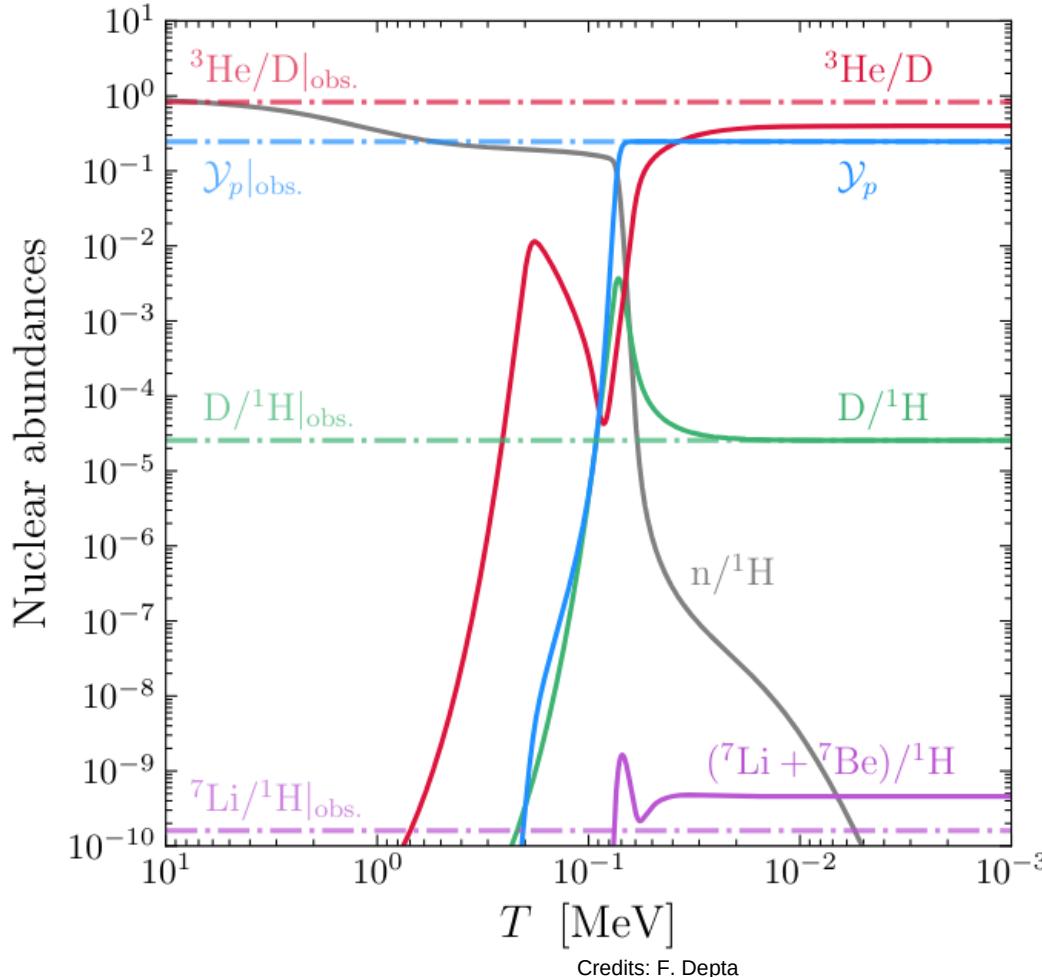


## Interlude: BBN data, photodisintegration & hadrodisintegration



**Effects of changes in the background cosmology**

# Interlude: BBN data, photodisintegration & hadrodisintegration

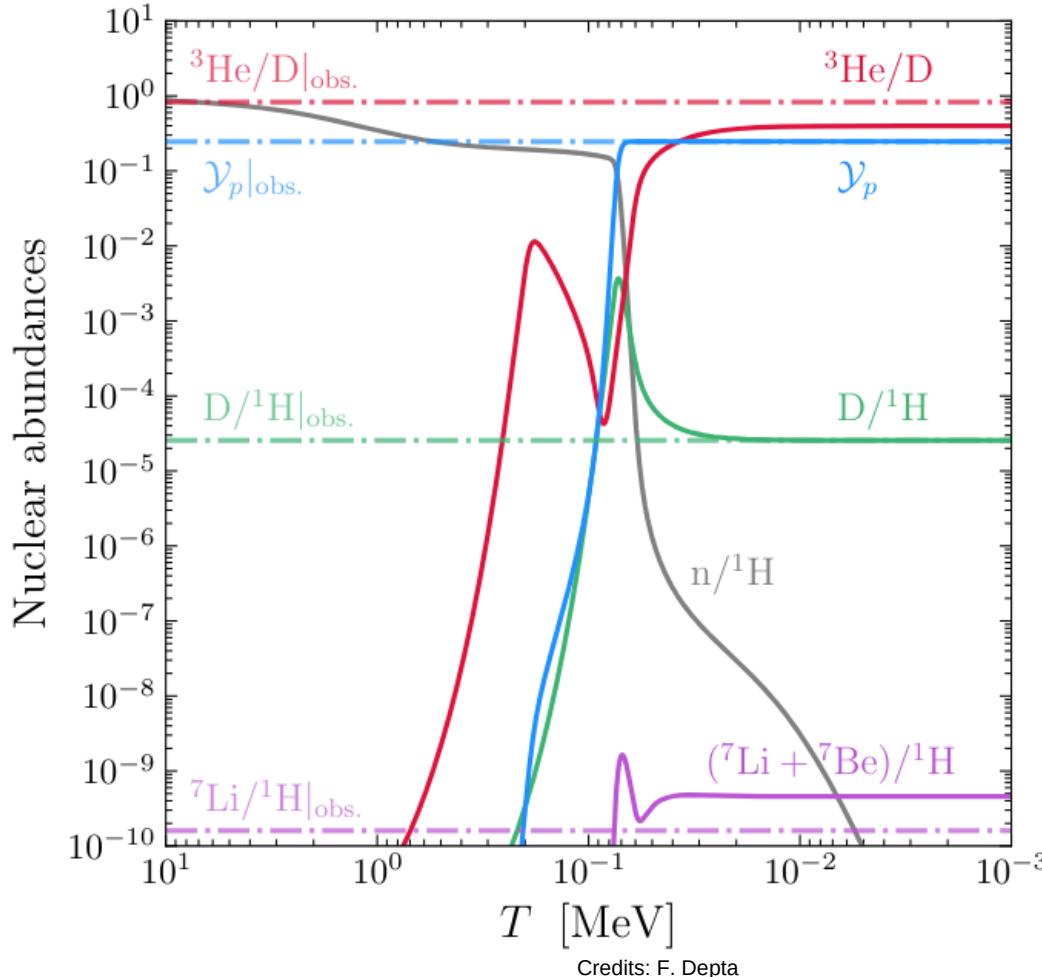


## Effects of changes in the background cosmology

- Modification of the **Hubble rate**

$$H(t) \sim \sqrt{\rho_{\text{SM}} + \rho_{\text{DS}}}$$

# Interlude: BBN data, photodisintegration & hadrodisintegration



## Effects of changes in the background cosmology

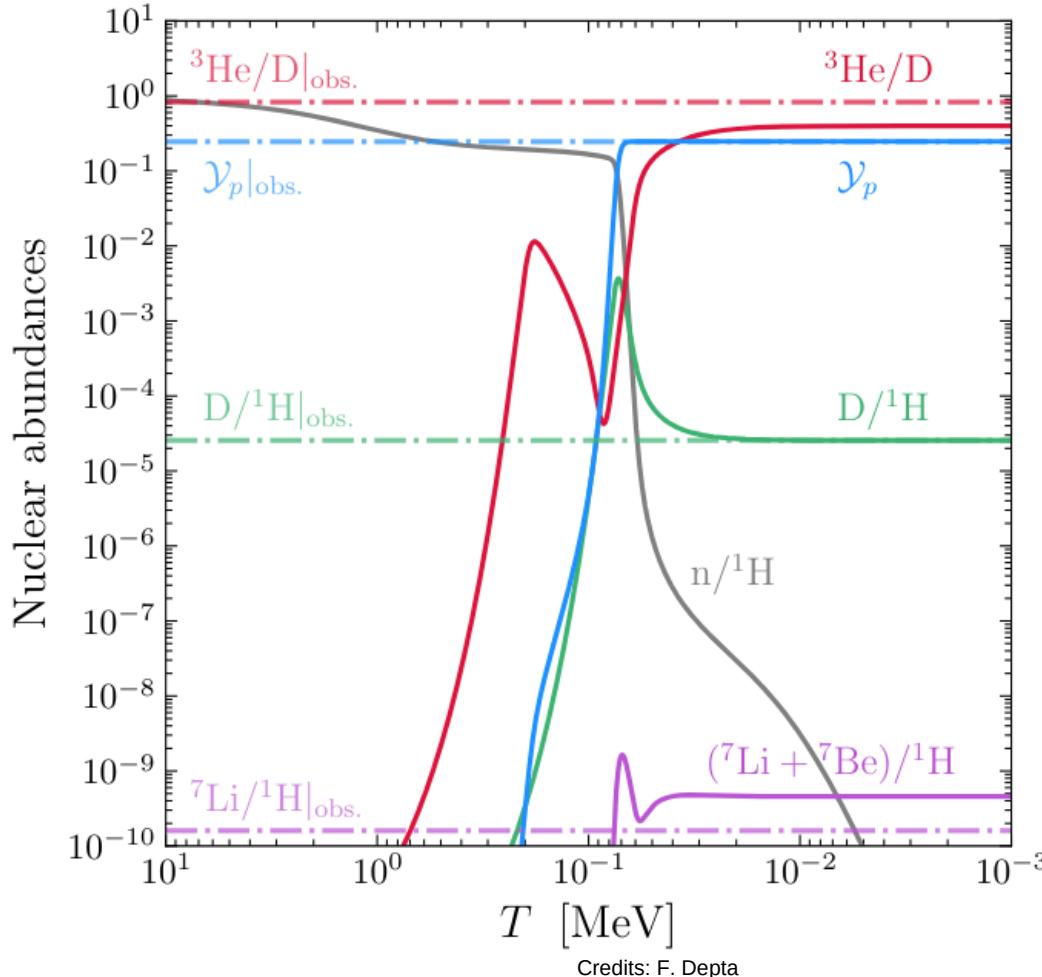
- Modification of the **Hubble rate**

$$H(t) \sim \sqrt{\rho_{\text{SM}} + \rho_{\text{DS}}}$$

- Modification of the **time-temperature relation**

$$\dot{T} = \frac{\dot{q}_{\text{SM}} - 3H(\rho_{\text{SM}} + P_{\text{SM}})}{d\rho_{\text{SM}}/dt}$$

# Interlude: BBN data, photodisintegration & hadrodisintegration



## Effects of changes in the background cosmology

- Modification of the **Hubble rate**

$$H(t) \sim \sqrt{\rho_{\text{SM}} + \rho_{\text{DS}}}$$

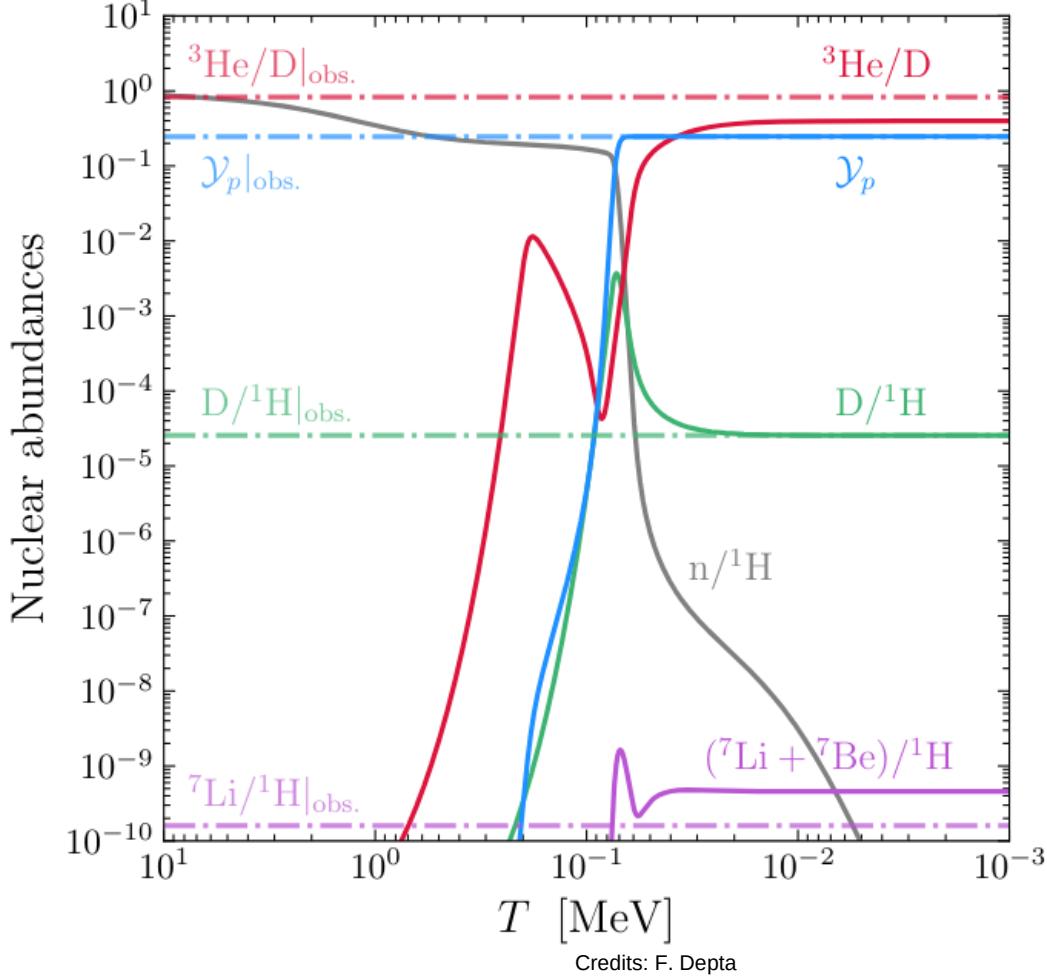
- Modification of the **time-temperature relation**

$$\dot{T} = \frac{\dot{q}_{\text{SM}} - 3H(\rho_{\text{SM}} + P_{\text{SM}})}{d\rho_{\text{SM}}/dt}$$

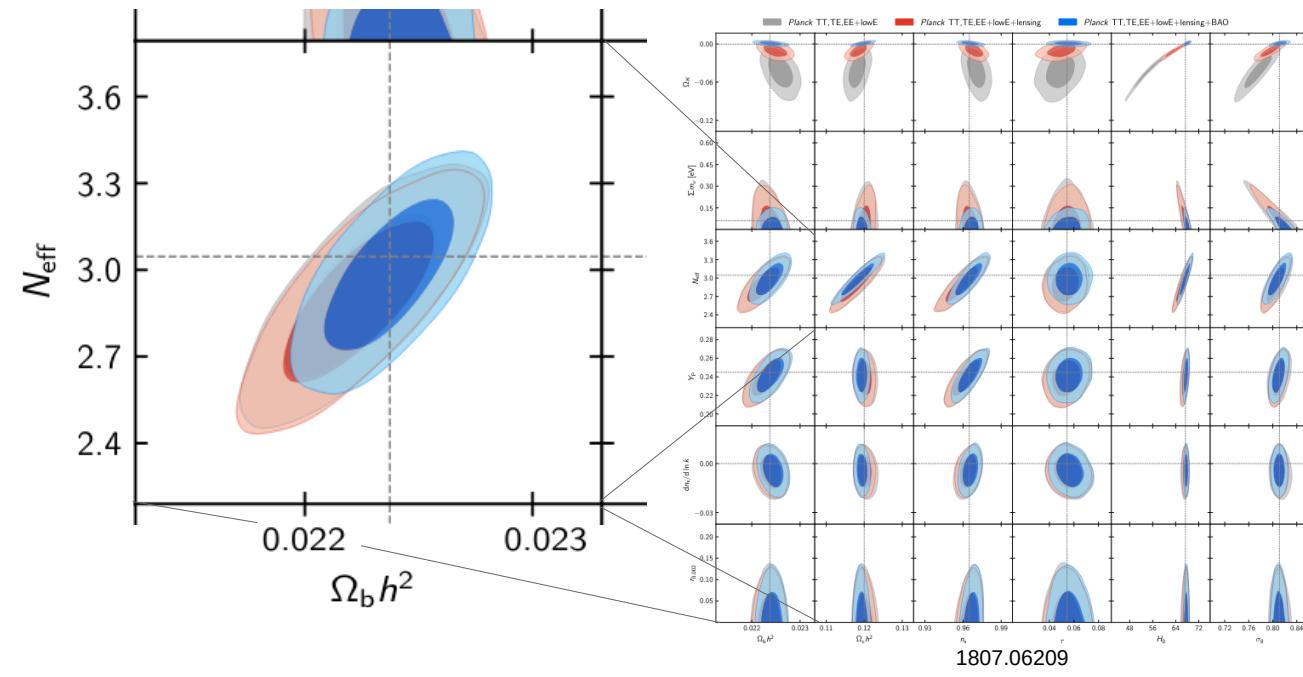
- Modification of the baryon-to-photon ratio

# Interlude: BBN data, photodisintegration & hadrodisintegration

Modification is driven by correlation to the **effective number of relativistic neutrinos**:



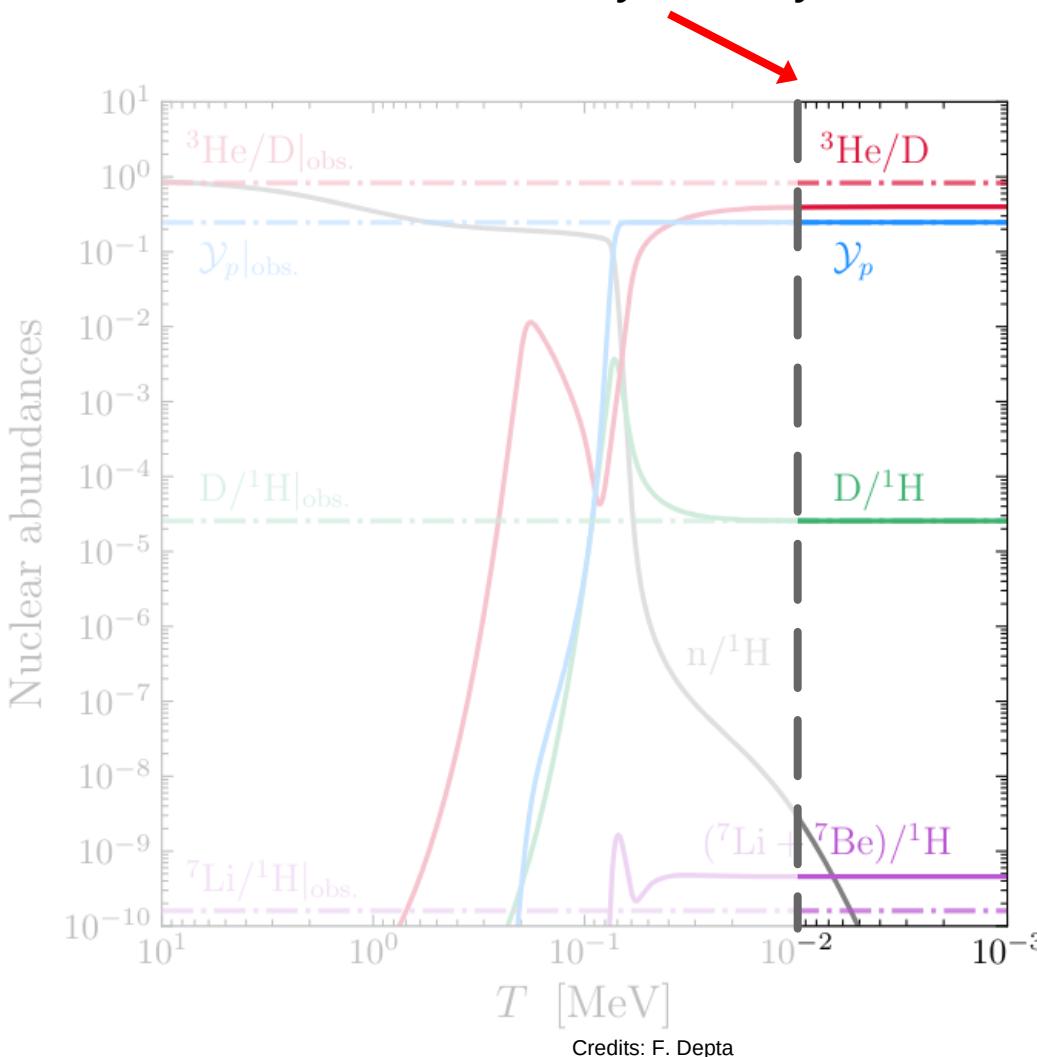
$$N_{\text{eff}} = \frac{\rho_{\nu}^{\text{th}}(t_{\text{rec}}) + \rho_{\nu}^{\text{n-th}}(t_{\text{rec}})}{2 \frac{7}{8} \frac{\pi^2}{30} \left(\frac{4}{11}\right)^{4/3} T(t_{\text{rec}})^4} \equiv [3 + \Delta(t_{\text{rec}})] \left(\frac{11}{4}\right)^{4/3} \left(\frac{T_{\nu}(t_{\text{rec}})}{T(t_{\text{rec}})}\right)^4.$$



$$\Rightarrow \eta = \bar{\eta} + r\sigma_{\eta} \frac{N_{\text{eff}} - \bar{N}_{\text{eff}}}{\sigma_{N_{\text{eff}}}}$$

## Interlude: BBN data, photodisintegration & hadrodisintegration

Abundances are effectively fixed by  $T \simeq 10 \text{ keV}$  ( $t \sim 10^4 \text{ s}$ )



**Photodisintegration**

=

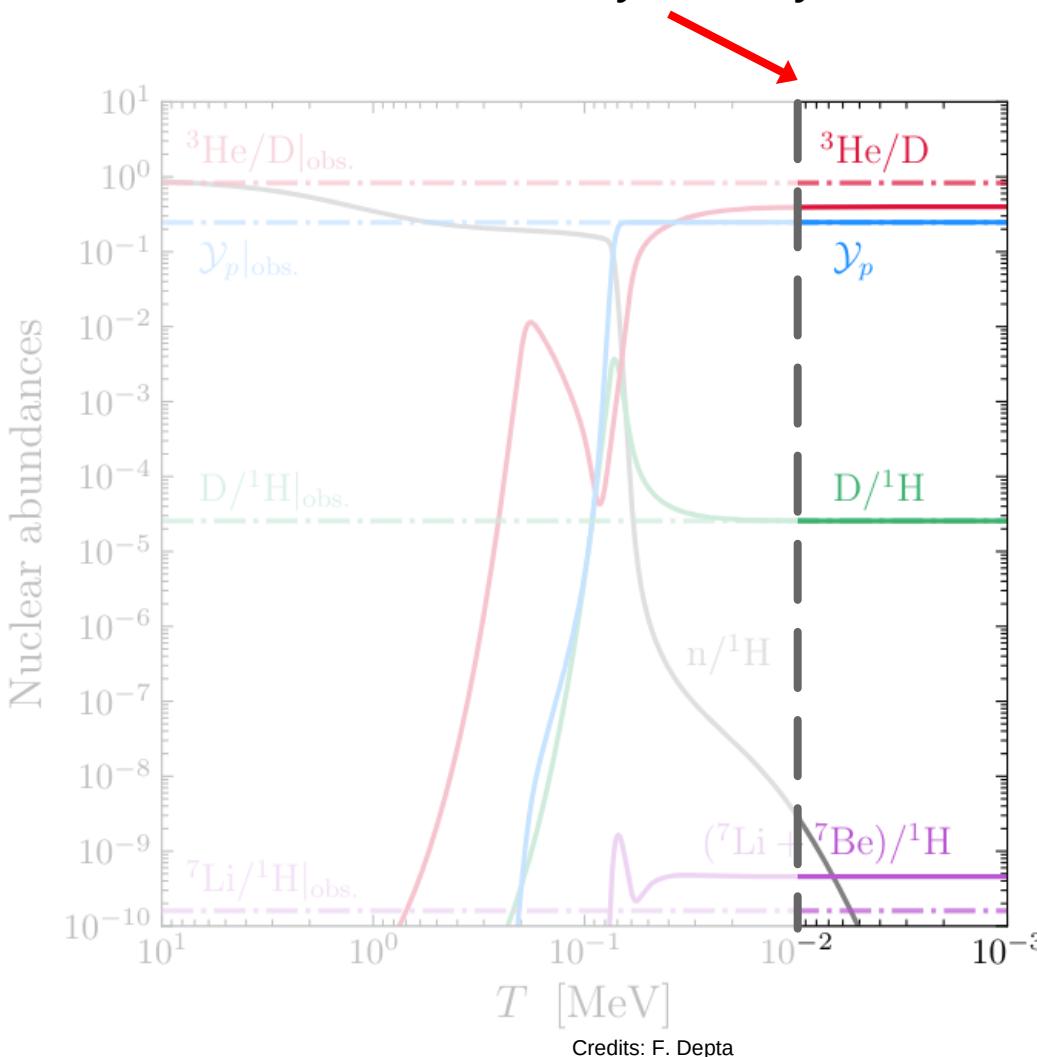
**late-time destruction** of the light elements by **EM** particles

$$\Rightarrow \tau_\phi \gtrsim 10^4 \text{ s}$$

allows factorisation of the processes

# Interlude: BBN data, photodisintegration & hadrodisintegration

Abundances are effectively fixed by  $T \simeq 10 \text{ keV}$  ( $t \sim 10^4 \text{ s}$ )



## Photodisintegration

=

**late-time destruction** of the light elements by **EM** particles

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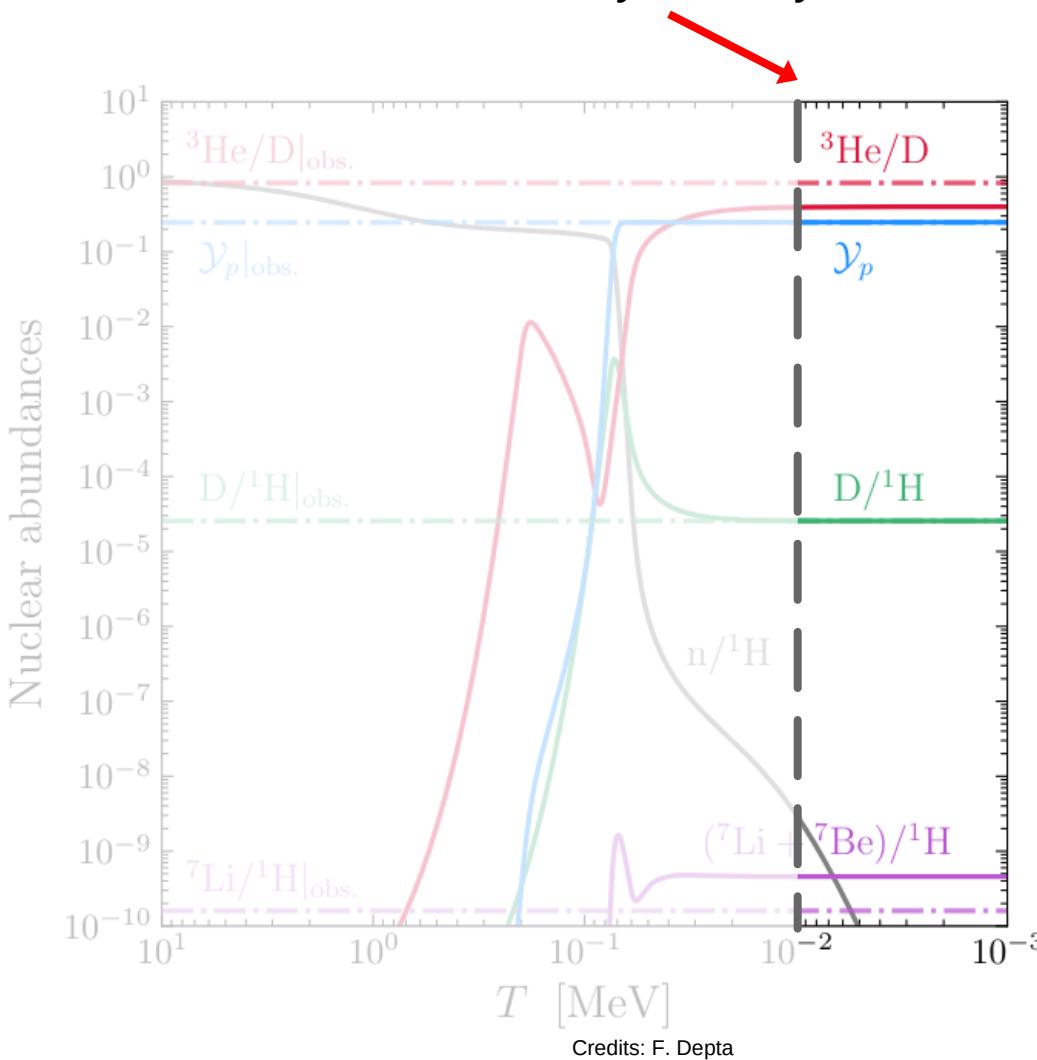
Rapid scattering on the CMB induces a **universal spectrum**

$$f_{\gamma, \text{univ}}(T, E) \sim \begin{cases} K_0(E/E_X)^{-3/2} & , E < E_X \\ K_0(E/E_X)^{-2} & , E_X < E < E_{ee}^{\text{th}} \\ 0 & , E > E_{ee}^{\text{th}} \end{cases}$$

$$E_X = m_e^2/(80T) , \quad E_{ee}^{\text{th}} = m_e^2/(22T) , \quad K_0 \propto E_0$$

# Interlude: BBN data, photodisintegration & hadrodisintegration

Abundances are effectively fixed by  $T \simeq 10 \text{ keV}$  ( $t \sim 10^4 \text{ s}$ )



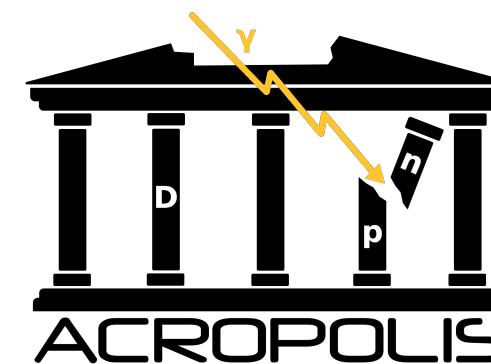
## Photodisintegration

=

late-time destruction of the light elements by EM particles

$$\Rightarrow \tau_\phi \gtrsim 10^4 \text{ s}$$

allows factorisation of the processes



Numerical Framework **ACROPOLIS**  
By Frederik, Marco and Kai  
(2011.06518)

<https://github.com/hep-mh/acropolis/tree/v2>

$$\left[ \frac{dn_X}{dt} \right]_{\text{photo}} = \sum_j n_j N_{j\gamma \rightarrow X} \int_0^\infty dE f_\gamma(E) \sigma_{j\gamma \rightarrow X}(E) - n_X \sum_{j'} \int_0^\infty dE f_\gamma(E) \sigma_{X\gamma \rightarrow j'}(E)$$

## Interlude: BBN data, photodisintegration & hadrodisintegration

Hadrodisintegration treatment à la [astro-ph/0408426] and [1709.01211]:

### Big-Bang Nucleosynthesis and Hadronic Decay of Long-Lived Massive Particles

Masahiro Kawasaki<sup>(a)</sup>, Kazunori Kohri<sup>(b)</sup> and Takeo Moroi<sup>(c)</sup>

### Revisiting Big-Bang Nucleosynthesis Constraints on Long-Lived Decaying Particles

Masahiro Kawasaki<sup>(a,b)</sup>, Kazunori Kohri<sup>(c,d)</sup>, Takeo Moroi<sup>(e,b)</sup>,  
and Yoshitaro Takaesu<sup>(e,f)</sup>

# Interlude: BBN data, photodisintegration & hadrodisintegration

Hadrodisintegration treatment à la [astro-ph/0408426] and [1709.01211]:

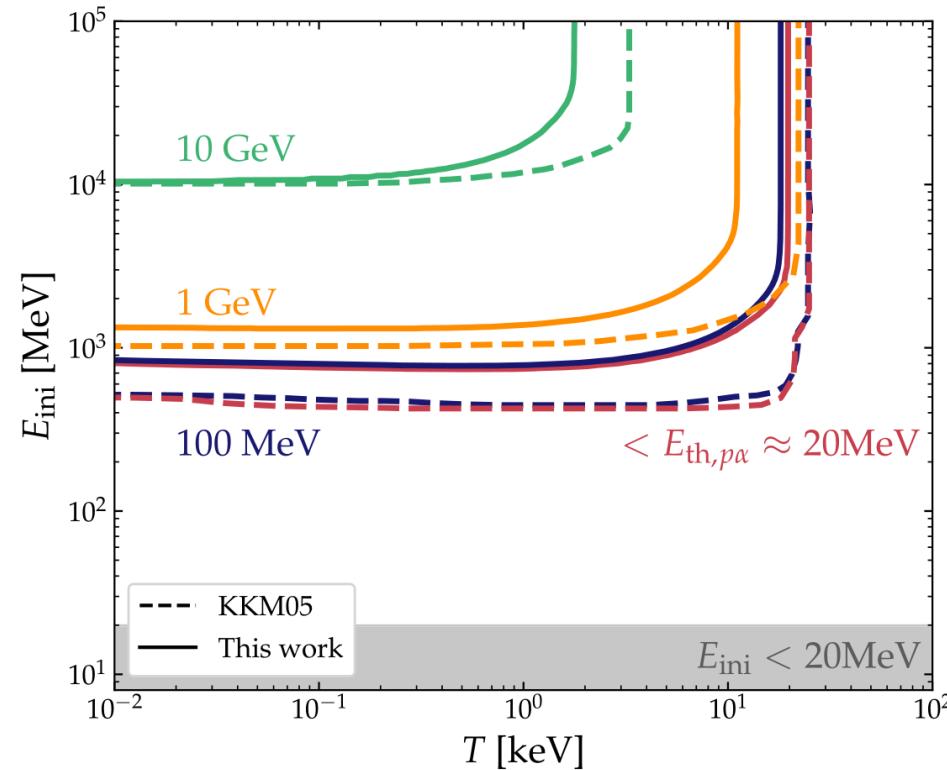
## I. Inject hadrons (in our case: neutrons and protons)

# Interlude: BBN data, photodisintegration & hadrodisintegration

Hadrodisintegration treatment à la [astro-ph/0408426] and [1709.01211]:

I. Inject hadrons (in our case: neutrons and protons)

**II. Apply energy loss formalism\***



\* We actually found a small discrepancy with previous literature results.

# Interlude: BBN data, photodisintegration & hadrodisintegration

Hadrodisintegration treatment à la [astro-ph/0408426] and [1709.01211]:

I. Inject hadrons (in our case: neutrons and protons)

II. Apply energy loss formalism

**III. Calculate nuclear scattering and determine final products**

Process	$i = n$	$i = p$	Reaction Type
(i, $p_{BG}$ ; 1)	$n + p_{BG} \rightarrow n + p$	$p + p_{BG} \rightarrow p + p$	elastic
(i, $p_{BG}$ ; 2)	$n + p_{BG} \rightarrow n + p + \pi$	$p + p_{BG} \rightarrow p + p + \pi$	inelastic
(i, $p_{BG}$ ; 3)	$n + p_{BG} \rightarrow n + n + \pi$	$p + p_{BG} \rightarrow p + n + \pi$	inelastic
(i, $p_{BG}$ ; 4)	$n + p_{BG} \rightarrow p + p + \pi$	$p + p_{BG} \rightarrow n + p + \pi$	inelastic
(i, $p_{BG}$ ; 5)	$n + p_{BG} \rightarrow p + p + \pi$	$p + p_{BG} \rightarrow n + n + \pi$	inelastic

Observation:

While PDI tends to destroy deuterium, HDI produces it from destroying helium-4.



Redistribute the kinetic energy

Process	$i = n$	$i = p$	Reaction Type
(i, $\alpha$ ; 1)	$n + \alpha_{BG} \rightarrow n + \alpha$	$p + \alpha_{BG} \rightarrow p + \alpha$	elastic
(i, $\alpha$ ; 2)	$n + \alpha_{BG} \rightarrow D + T$	$p + \alpha_{BG} \rightarrow D + {}^3\text{He}$	inelastic
(i, $\alpha$ ; 3)	$n + \alpha_{BG} \rightarrow 2n + {}^3\text{He}$	$p + \alpha_{BG} \rightarrow p + n + {}^3\text{He}$	inelastic
(i, $\alpha$ ; 4)	$n + \alpha_{BG} \rightarrow p + n + T$	$p + \alpha_{BG} \rightarrow 2p + T$	inelastic
(i, $\alpha$ ; 5)	$n + \alpha_{BG} \rightarrow n + 2D$	$p + \alpha_{BG} \rightarrow p + 2D$	inelastic
(i, $\alpha$ ; 6)	$n + \alpha_{BG} \rightarrow p + 2n + D$	$p + \alpha_{BG} \rightarrow 2p + n + D$	inelastic
(i, $\alpha$ ; 7)	$n + \alpha_{BG} \rightarrow 2p + 3n$	$p + \alpha_{BG} \rightarrow 3p + 2n$	inelastic
(i, $\alpha$ ; 8)	$n + \alpha_{BG} \rightarrow n + \alpha + \pi$	$p + \alpha_{BG} \rightarrow p + \alpha + \pi$	inelastic

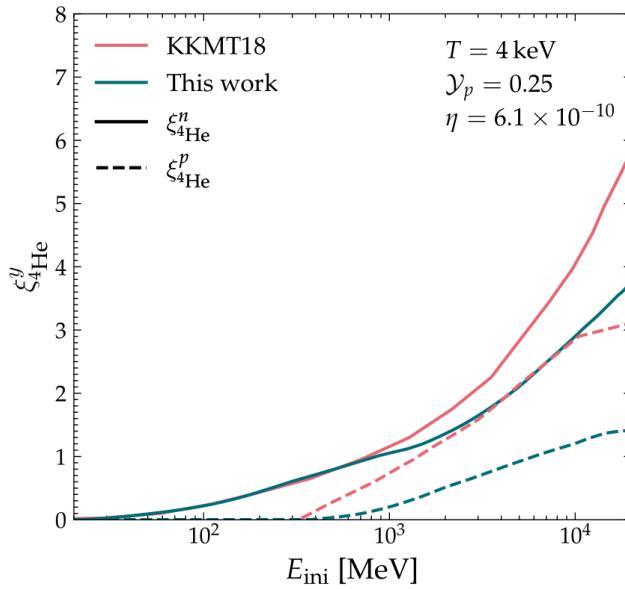


Disintegrate  ${}^4\text{He}$

# Interlude: BBN data, photodisintegration & hadrodisintegration

Hadrodisintegration treatment à la [astro-ph/0408426] and [1709.01211]:

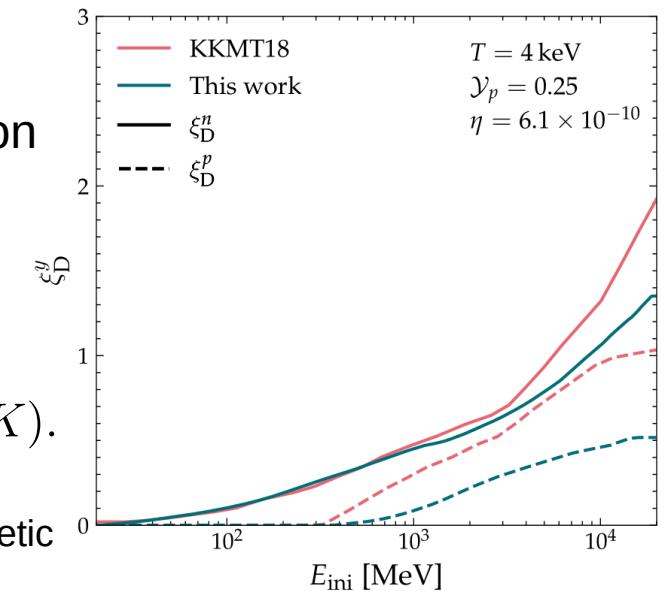
- I. Inject hadrons (in our case: neutrons and protons)
- II. Apply energy loss formalism
- III. Calculate nuclear scattering and determine final products
- IV. Repeat until all particles in the cascade have negligible kinetic energy**



Number of nuclei produced per injected nucleon

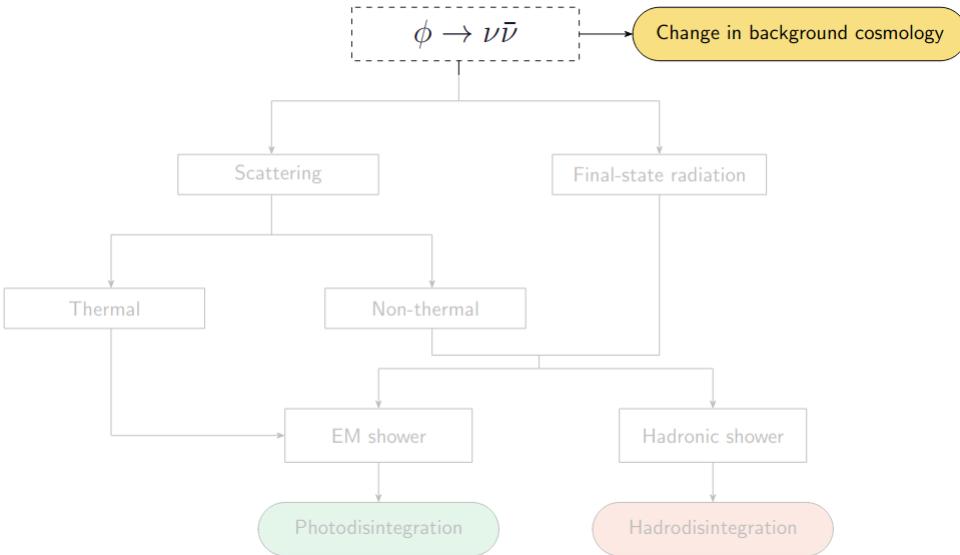
$$\left[ \frac{dn_X}{dt} \right]_{\text{hadro}} = \sum_{y=n,p} \int_0^\infty dK \xi_X^y[n_j](K) \frac{d^2 n_y^{\text{inj}}}{dtdK}(K).$$

Note: We approximate the injection spectrum as monoenergetic



# **Back to neutrinos: Handling the different effects in detail**

# Handling the contributions step-by-step



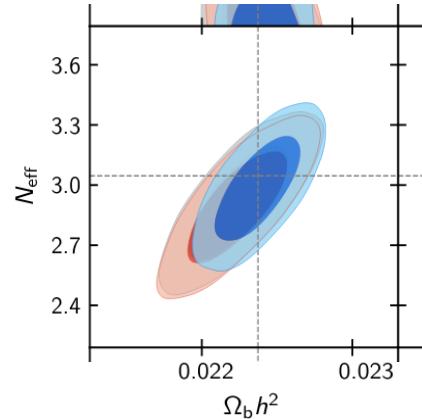
**The dominant effect:**

Additional energy density increases  $N_{\text{eff}}$

Change in  $\eta$  due to correlation with  $N_{\text{eff}}$

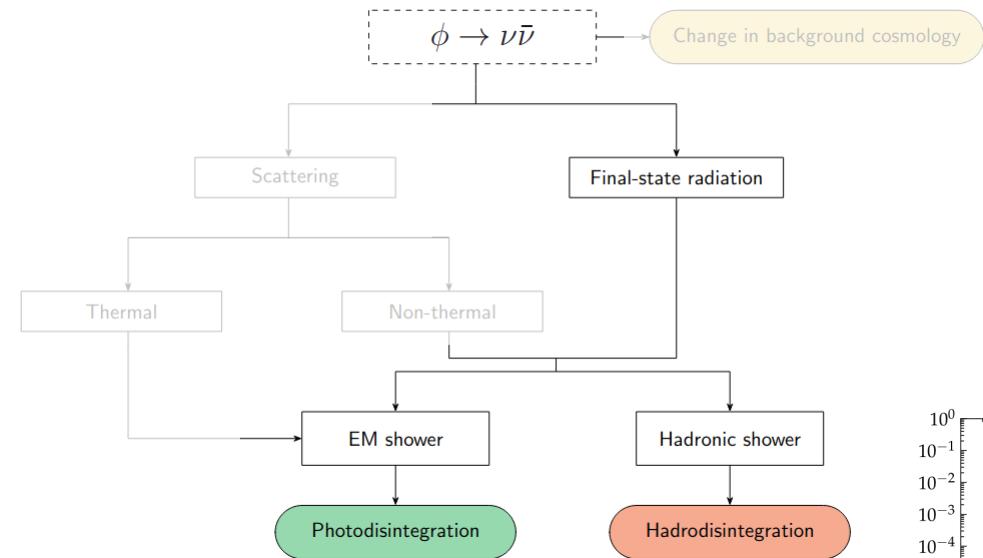
Initial conditions of BBN are varied due to  $\eta$

BBN is modified and we get different abundances

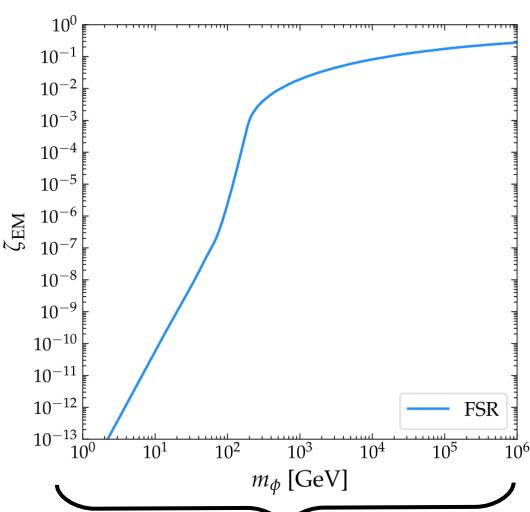


# Handling the contributions step-by-step

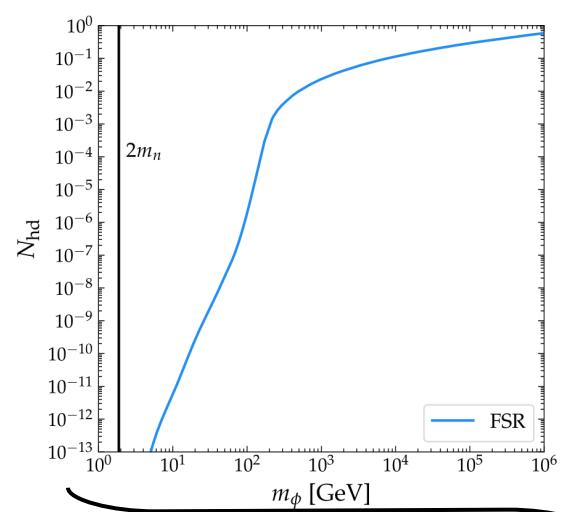
Injection rate is determined by the decay rate



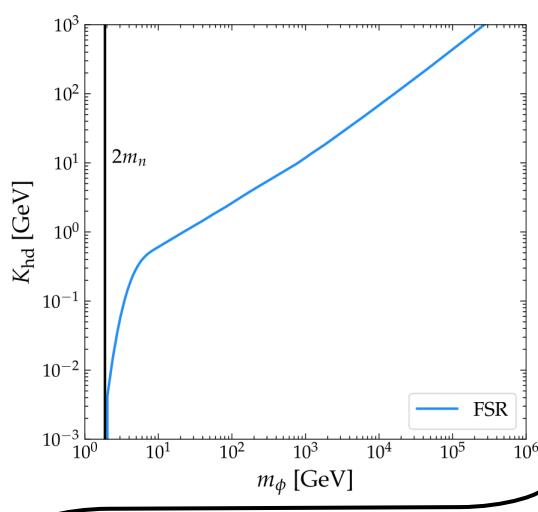
Effect of an individual injection determined by PYTHIA



PDI requires 1 parameter (universal spectrum)



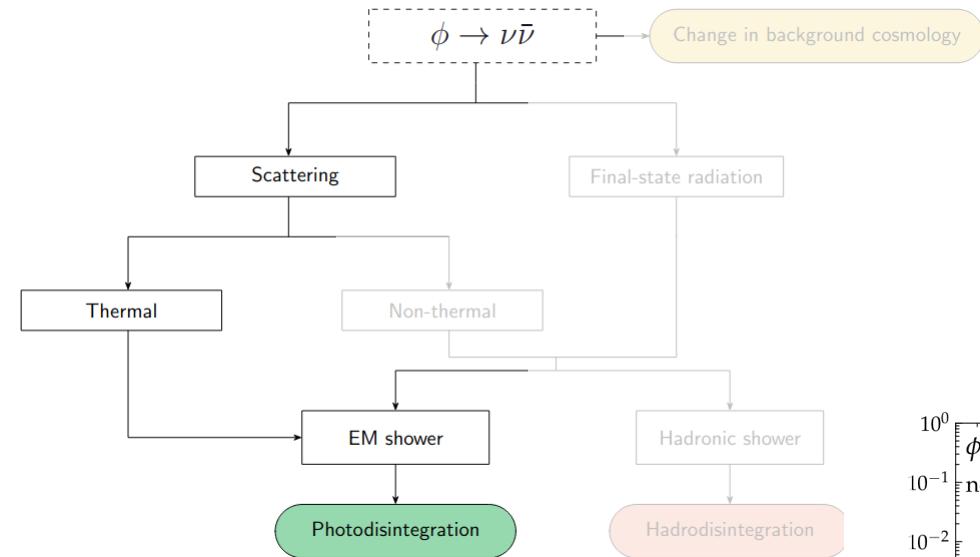
HDI requires 2 parameters



# Handling the contributions step-by-step

Injection rate given by thermal scattering rate

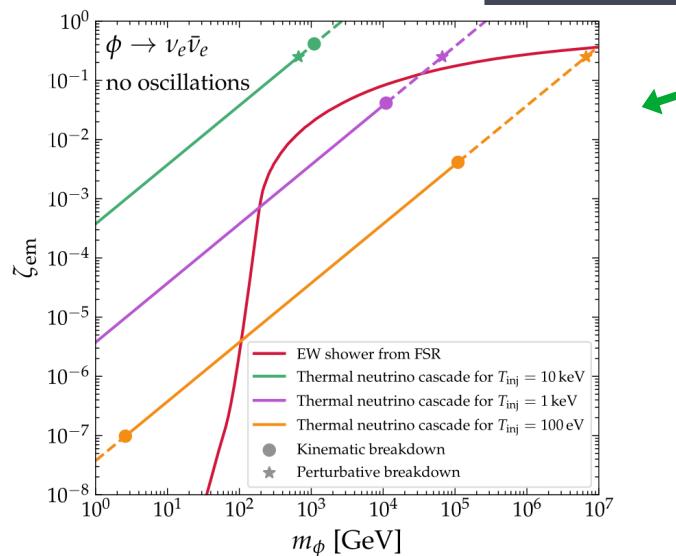
$$\Gamma_{ee}(T, E) \sim G_F^2 E T_\nu^4$$



Effect of an individual injection determined analytically

$$d\zeta_{\text{em}} \simeq \frac{E(t)}{E_{\text{inj}}} \Gamma_{ee}(t) dt$$

$$\Rightarrow \zeta_{\text{em}}(T_{\text{inj}}) = \frac{1}{4} \left( \frac{\Gamma_{ee}}{H} \right)_{\text{inj}}$$

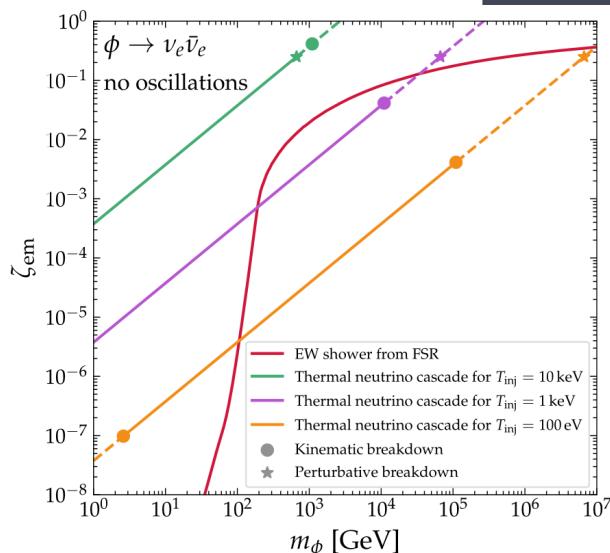
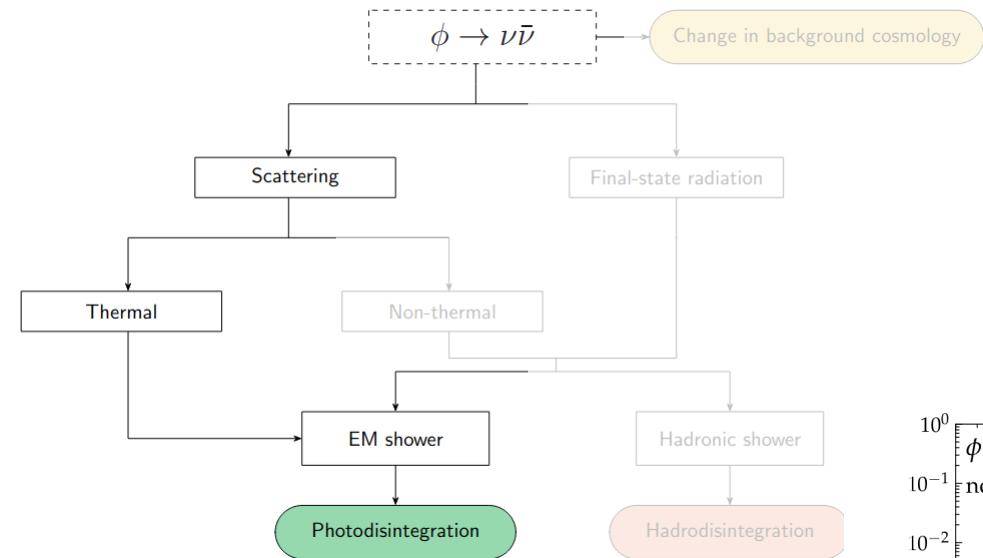


$$s \sim 4ET \ll 4m_n^2$$

# Handling the contributions step-by-step

Injection rate given by thermal scattering rate

$$\Gamma_{ee}(T, E) \sim G_F^2 E T_\nu^4$$



Effect of an individual injection determined analytically

$$d\zeta_{em} \simeq \frac{E(t)}{E_{inj}} \Gamma_{ee}(t) dt$$

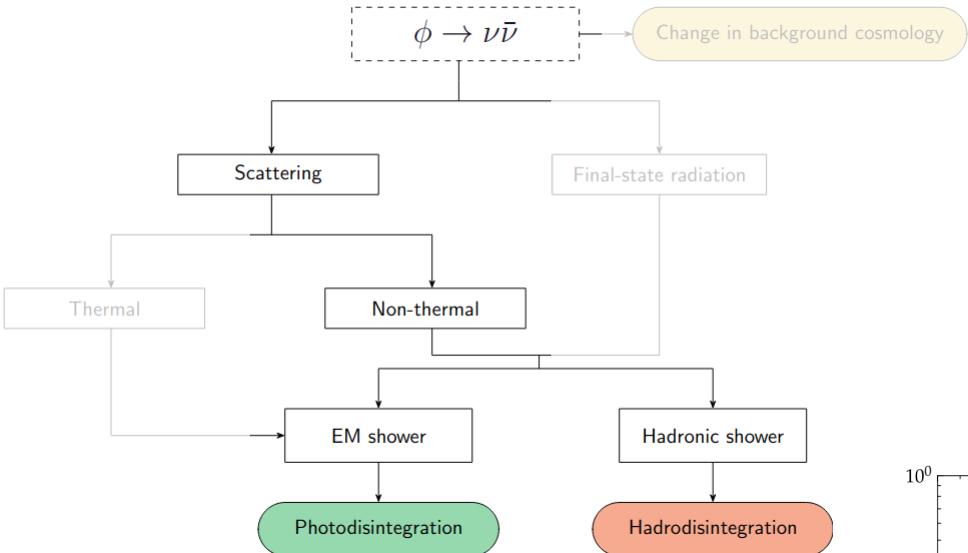
$$\Rightarrow \zeta_{em}(T_{inj}) = \frac{1}{4} \left( \frac{\Gamma_{ee}}{H} \right)_{inj}$$

“Elastic” scattering:

$$\begin{aligned} & \Gamma_{ee}(T, E_1) + \Gamma_{ee}(T, E_2) \\ &= \text{const.} \times G_F^2 T_\nu^4 (E_1 + E_2) \\ &= \Gamma_{ee}(T, E_1 + E_2) \end{aligned}$$

Redistribution of energy is irrelevant!

# Handling the contributions step-by-step



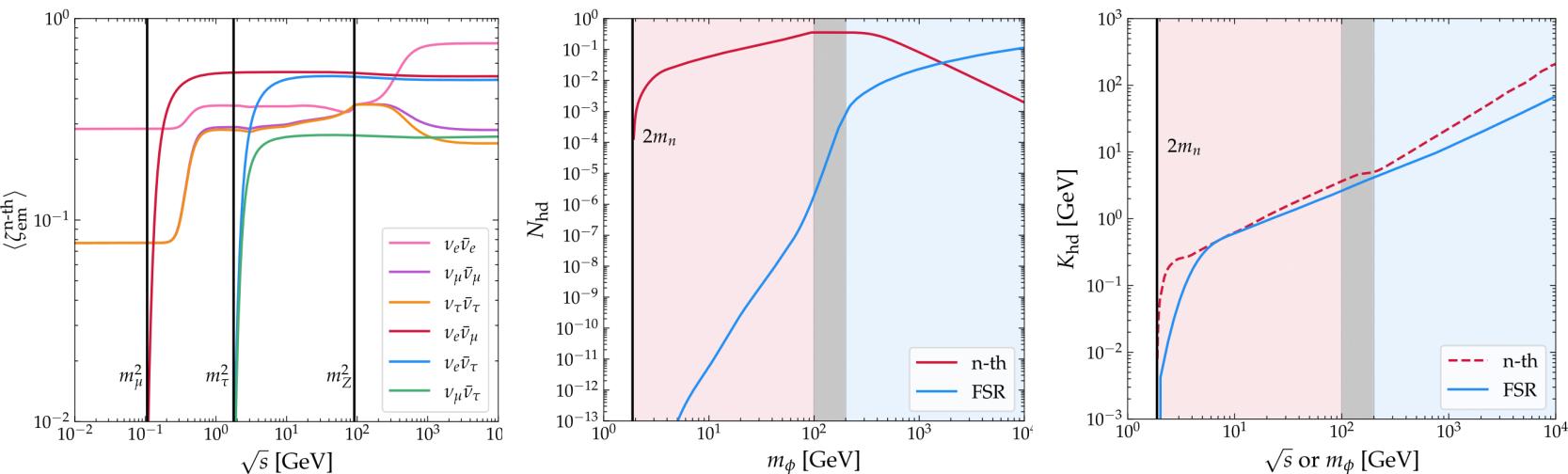
Injection rate given in general by

$$\Gamma_{\text{ann}}(T, E) = \frac{g_\nu}{16\pi^2 E^2} \int_0^\infty d\epsilon f_{\nu, \text{n-th}}(\epsilon) \int_0^{4E\epsilon} ds s \cdot \sigma_{\text{ann}}(s)$$

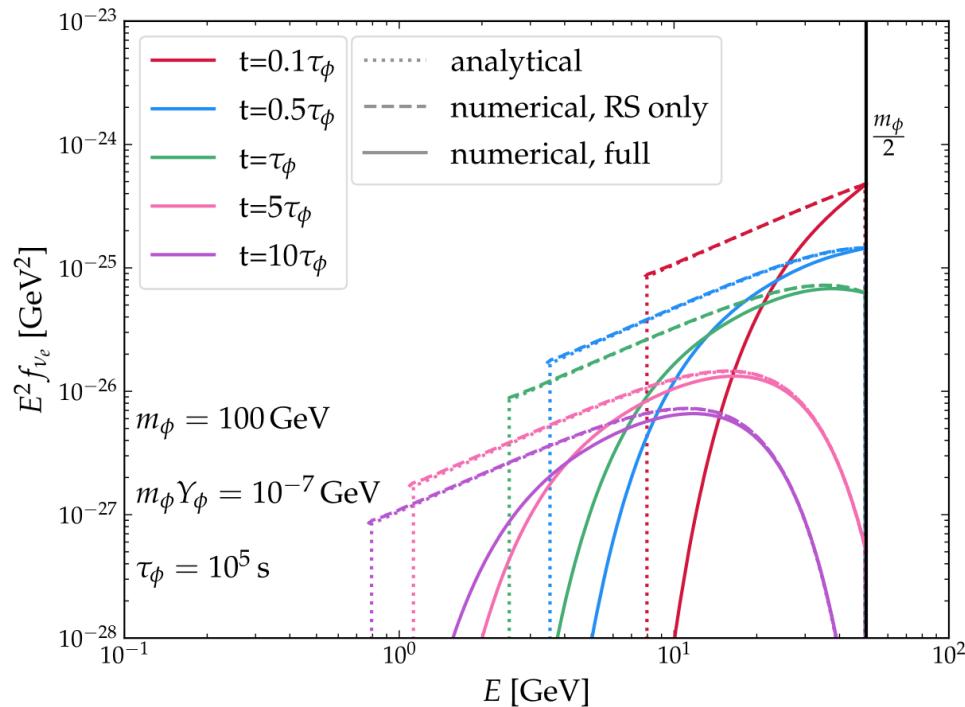
Requires knowledge of non-thermal spectrum!!!  
See following slides for exact methodology



Effects of individual event determined by PYTHIA



# How to track the neutrino cascade



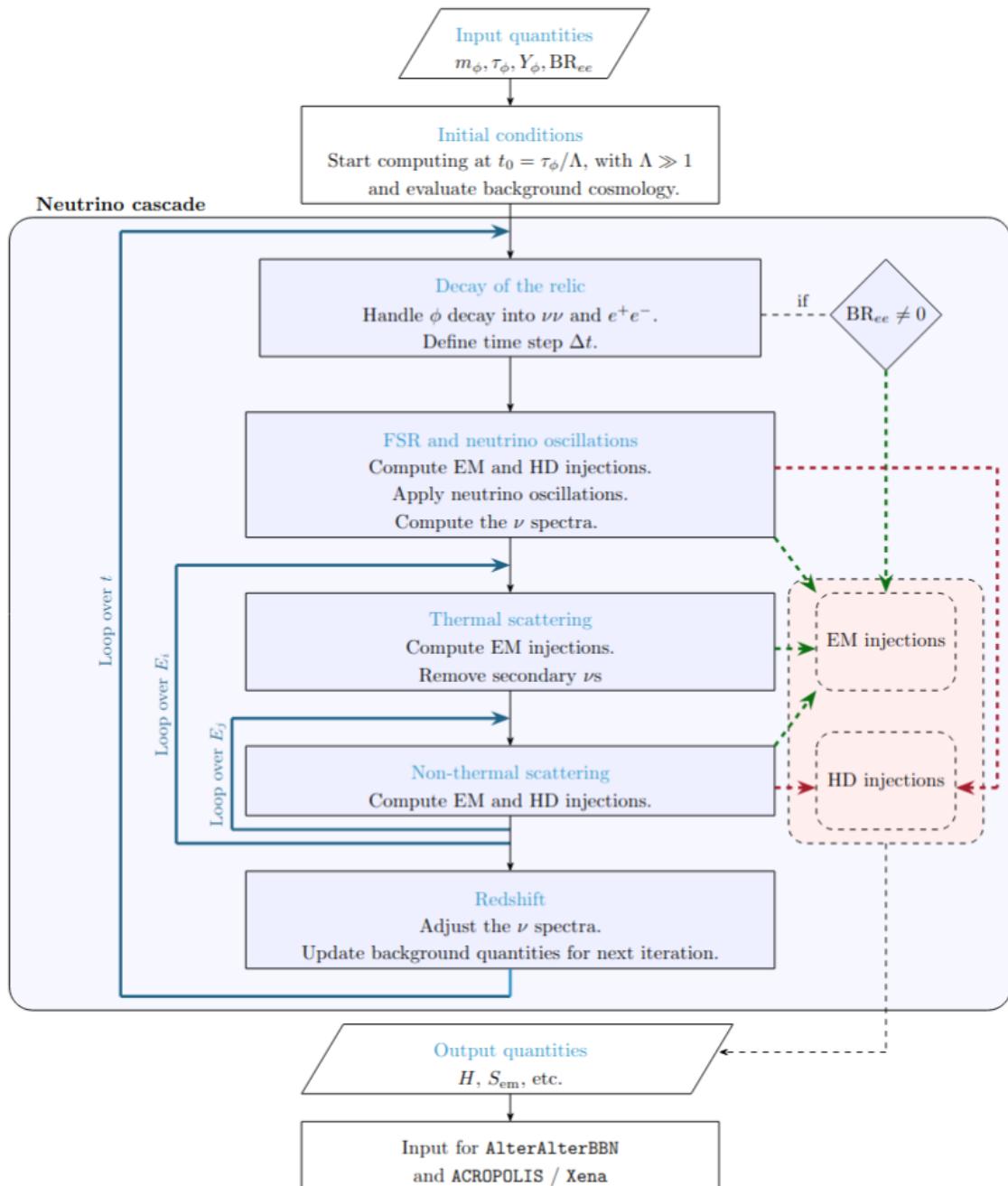
Is the Hubble rate dominant

Yes

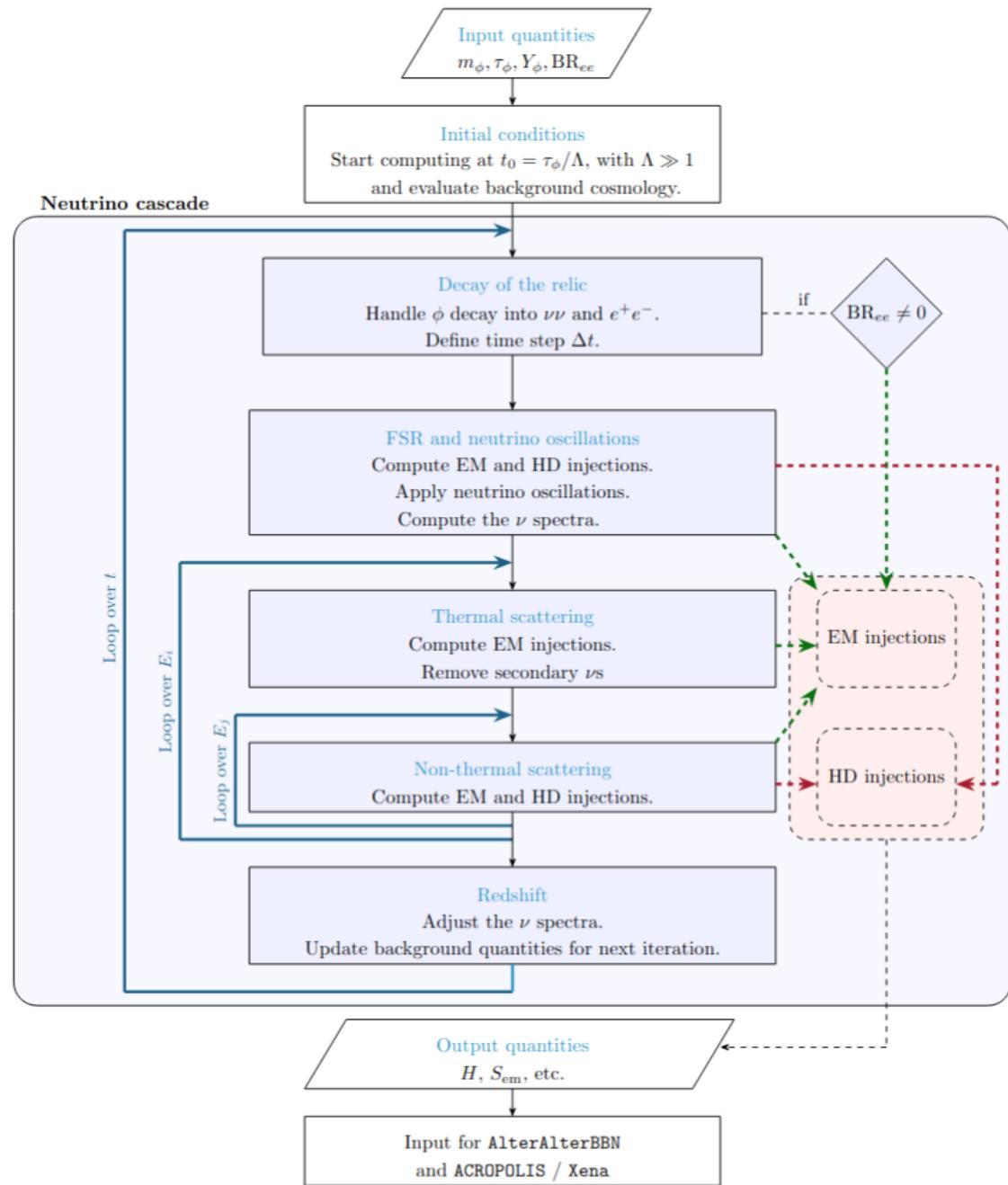
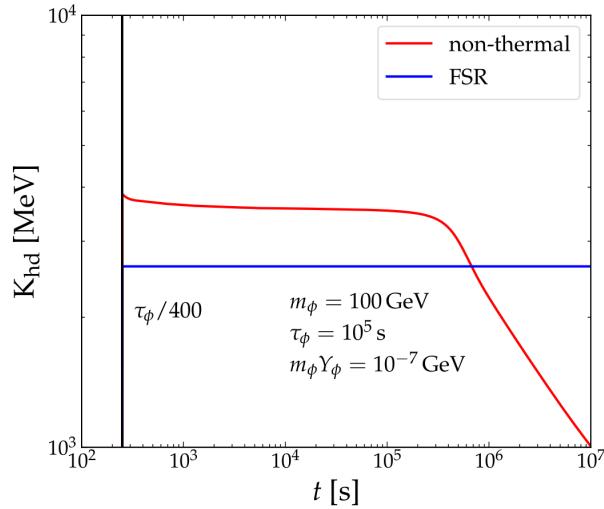
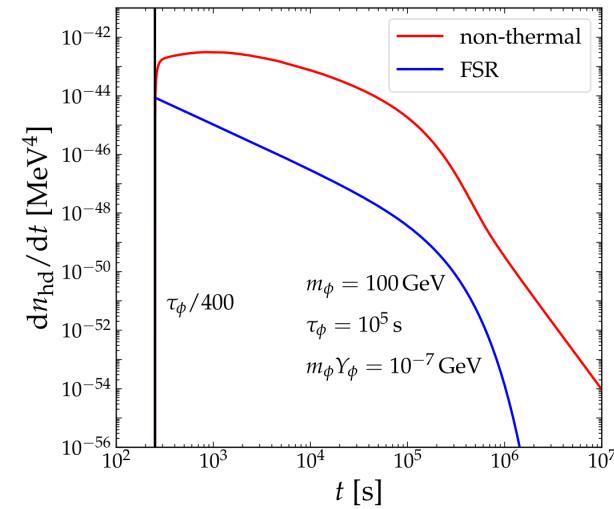
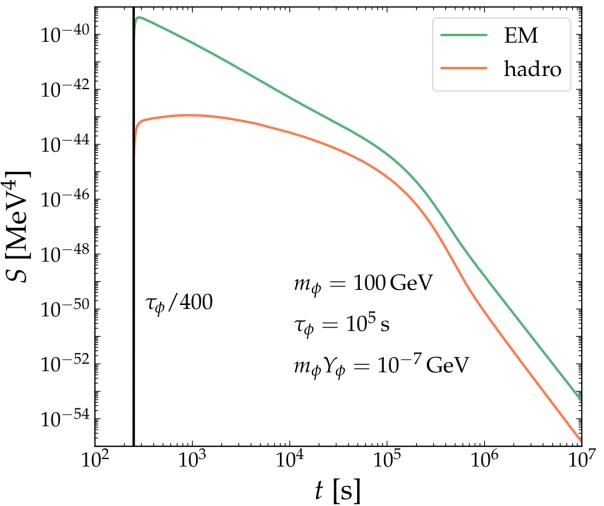
Spectrum mainly redshifts

No

Elastic scattering depletes the spectrum  
Approximation: No redistribution of energy

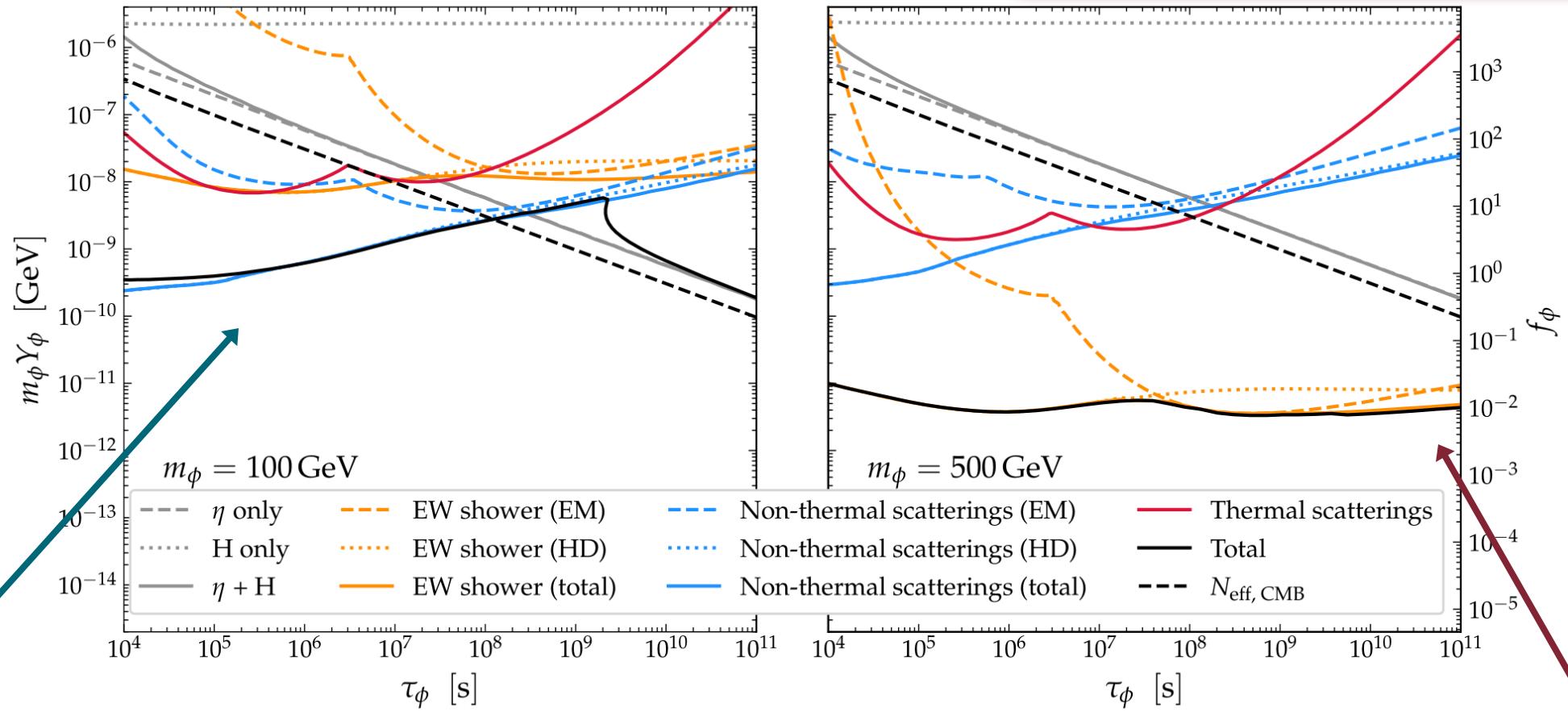


# How to track the neutrino cascade



# Detailed discussion of the resulting constraints

We observe both synergy effects and cancelations!



Below the EW scale:  
Domination of non-thermal scattering (light) and  
indirect effects (heavy)

Above the EW scale:  
Complete domination of EW shower from FSR

# Scanning the full parameter space

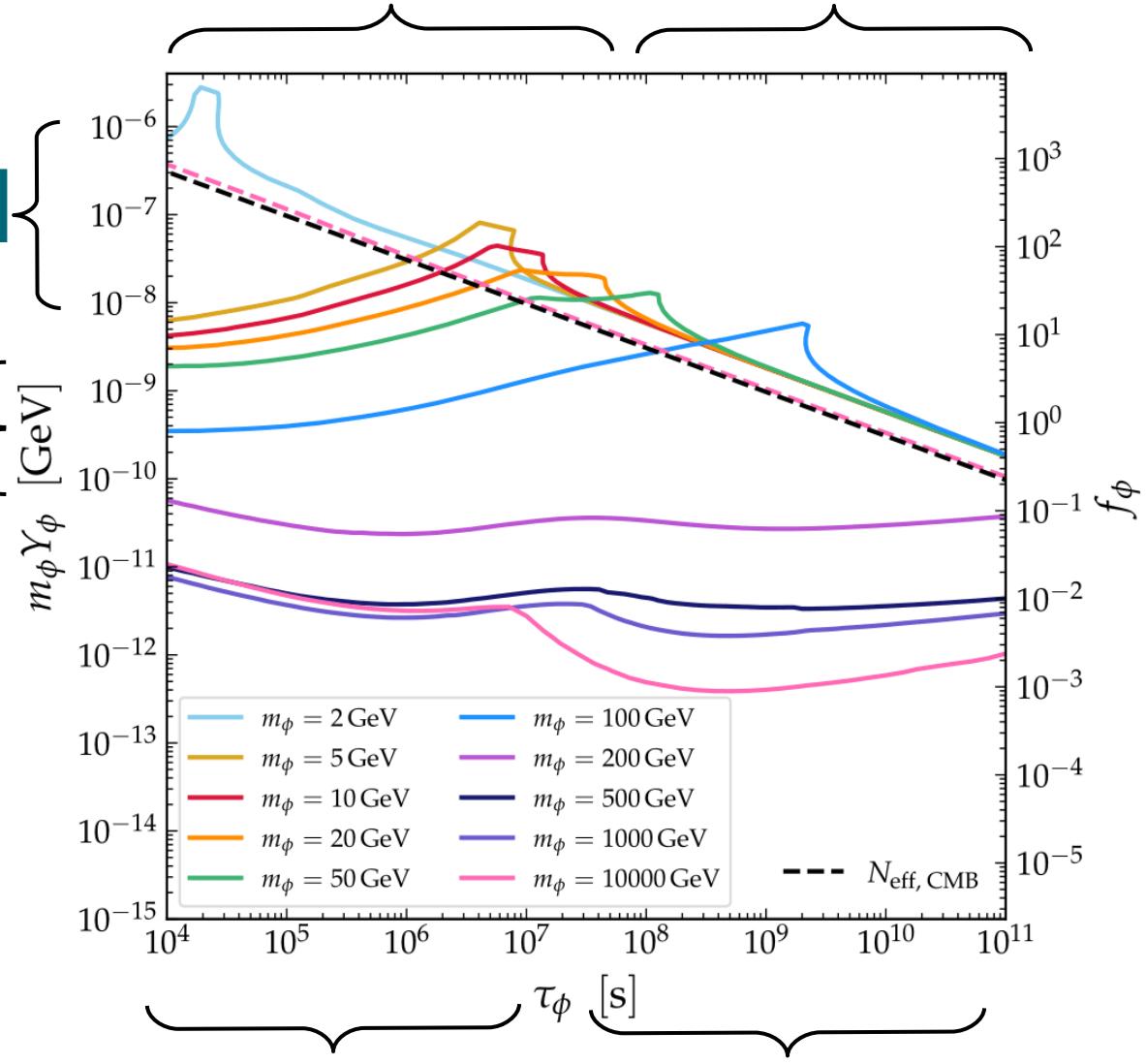
Jump across nucleon mass threshold

Unlocking of  $Z$  resonance  
Dominance of FSR begins

All 4 effects play an important role in some part of the parameter space, however the thermal scattering only due to the depletion of the spectrum.

Non-thermal domination

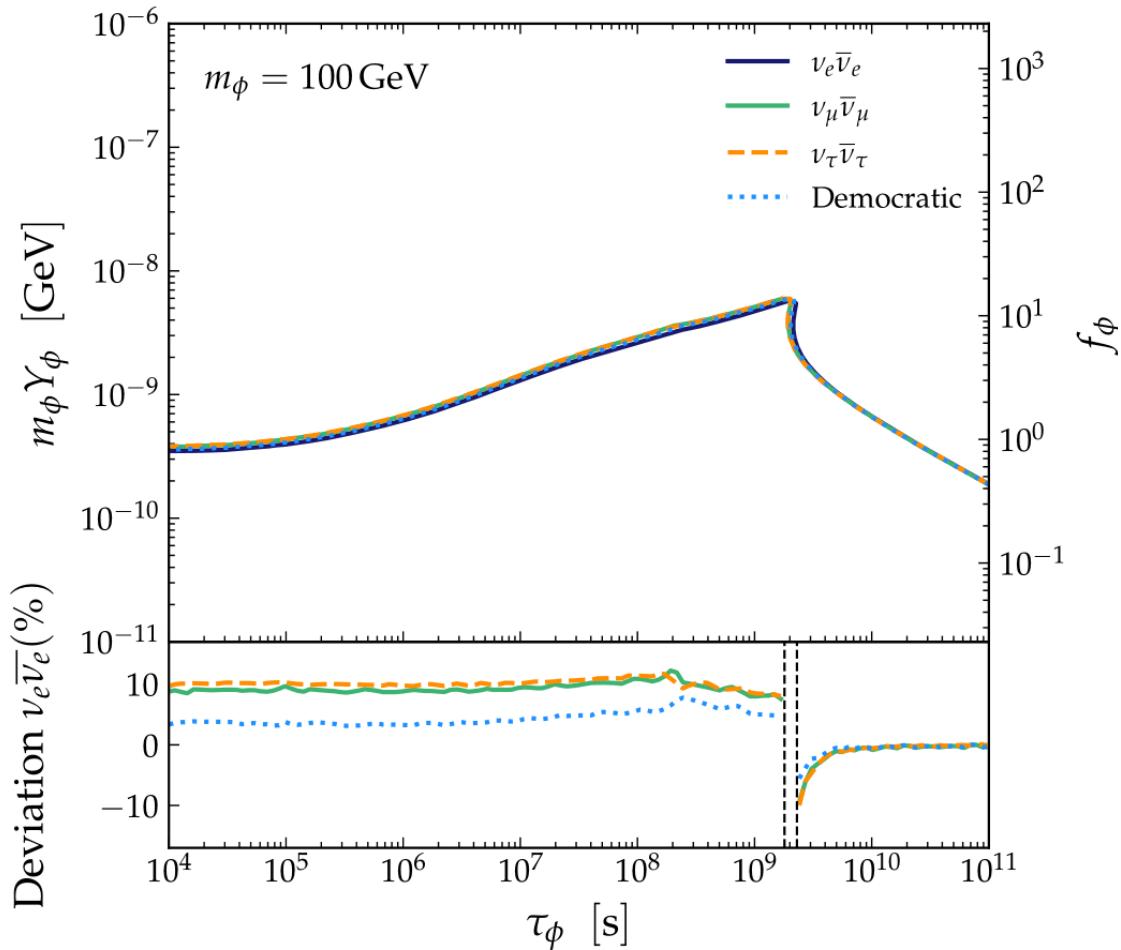
Cosmo domination



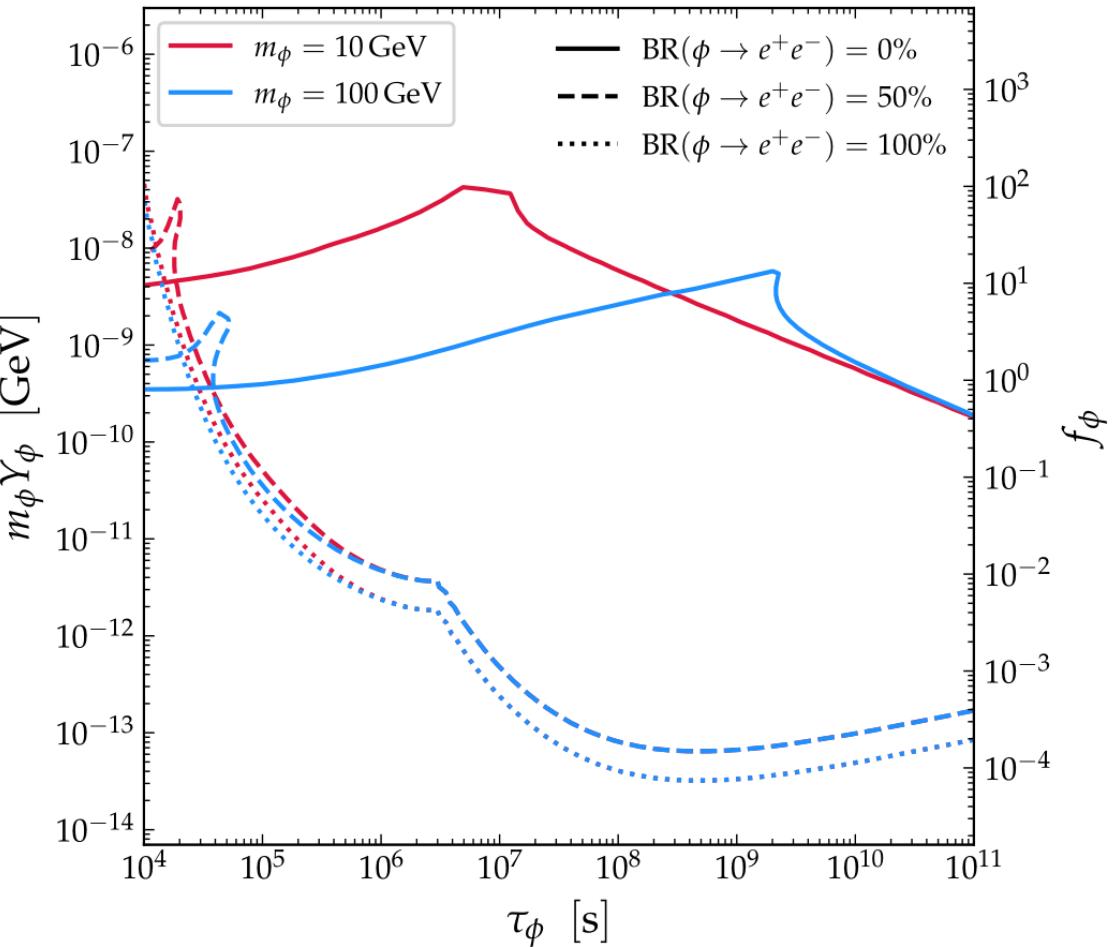
Hadronic domination

EM & mixed contribution

# How broadly applicable are these results?



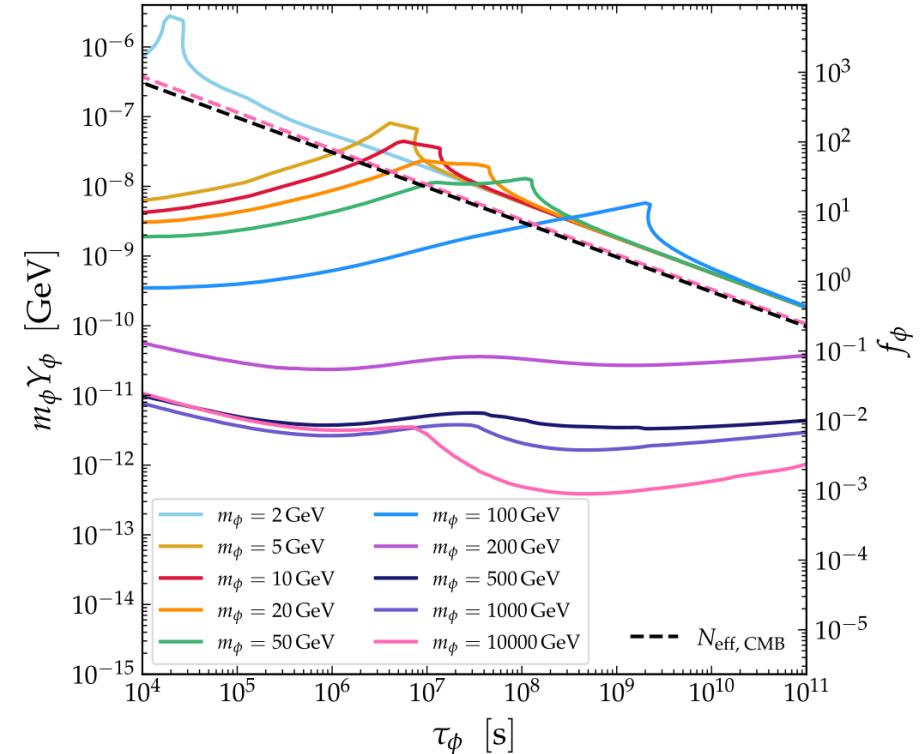
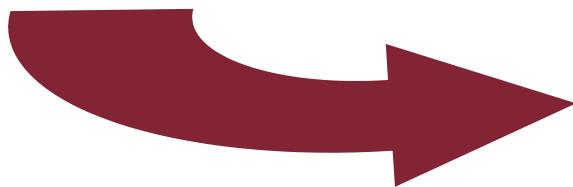
Results largely independent of initial  
neutrino composition ✓



Results likely still relevant for loop  
suppressed decay into electron ✓

# Conclusions & Outlook

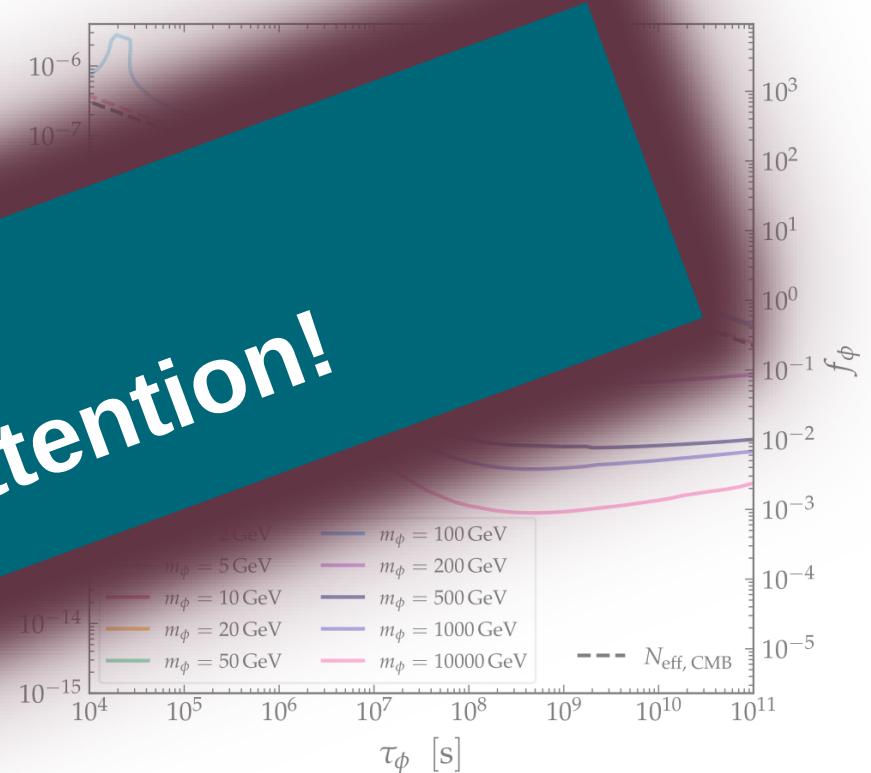
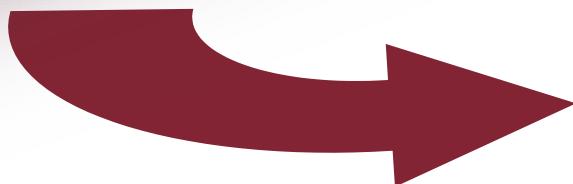
- Thorough study of neutrino injections after BBN
  - Surprisingly strong limits, especially from hadrons
  - Combination of improved literature results and new methods
  - Breakdown of many assumptions for low lifetimes requires more care (stay tuned!)
  - This work is part of an upcoming review to summarise the contributions of Marco and Frederik to the field
- ... and of course check the paper for (many) more details.

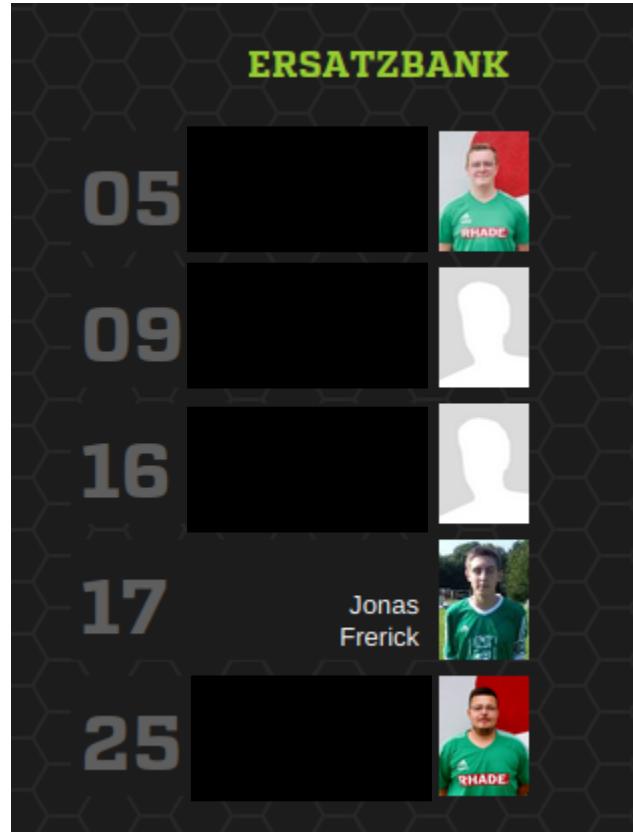


## Conclusions & Outlook

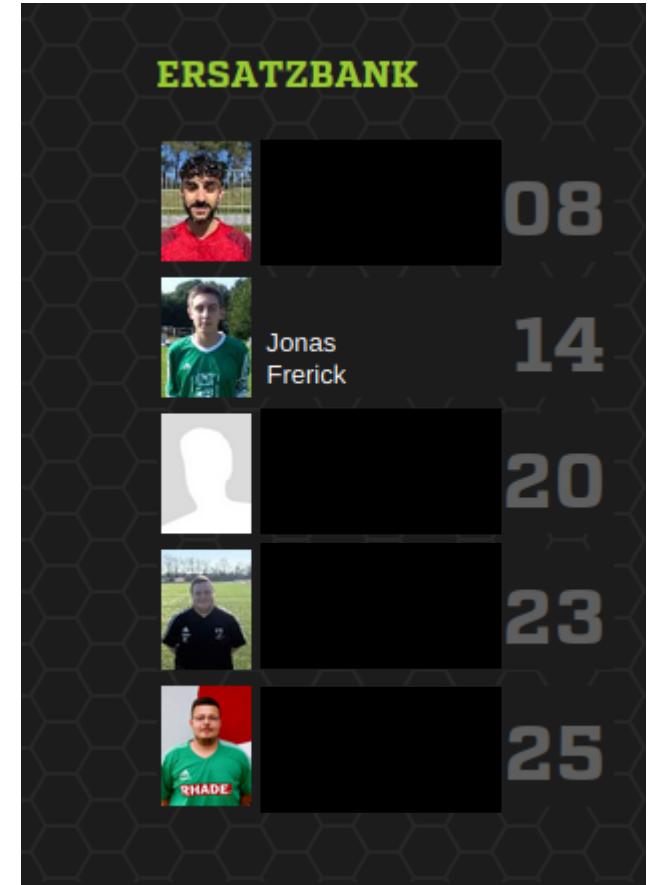
- Thorough study of neutrino injections after BBN
- Surprisingly strong limits, especially from baryon-to-photon conversion
- Combination of improved literature and new constraints
- Breakdown of many constraints due to lack of care (stay tuned)
- The results are conservative and can easily surpass the current limits in the field
- ... and much more!

That's it!  
Thank you for your attention!



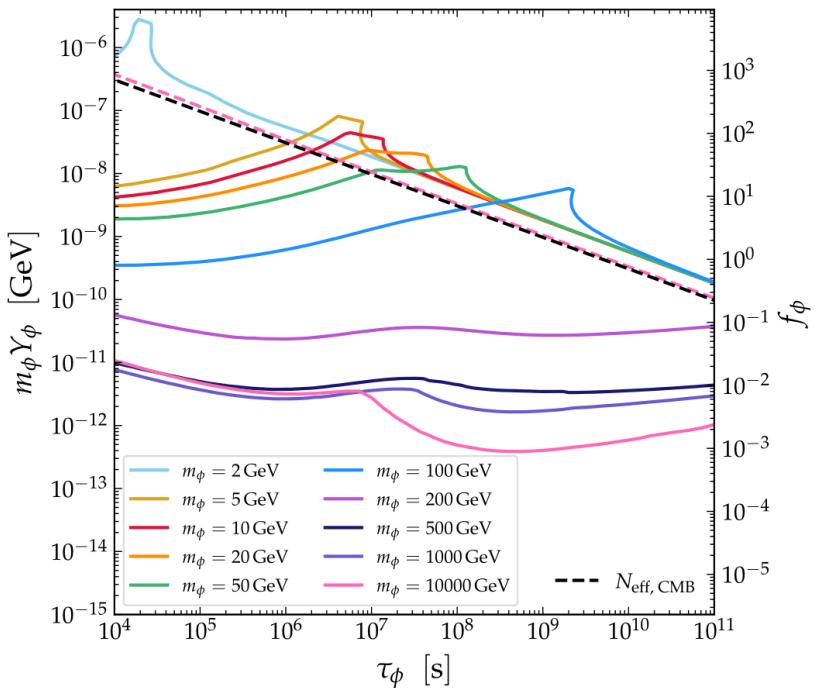


# Back-up

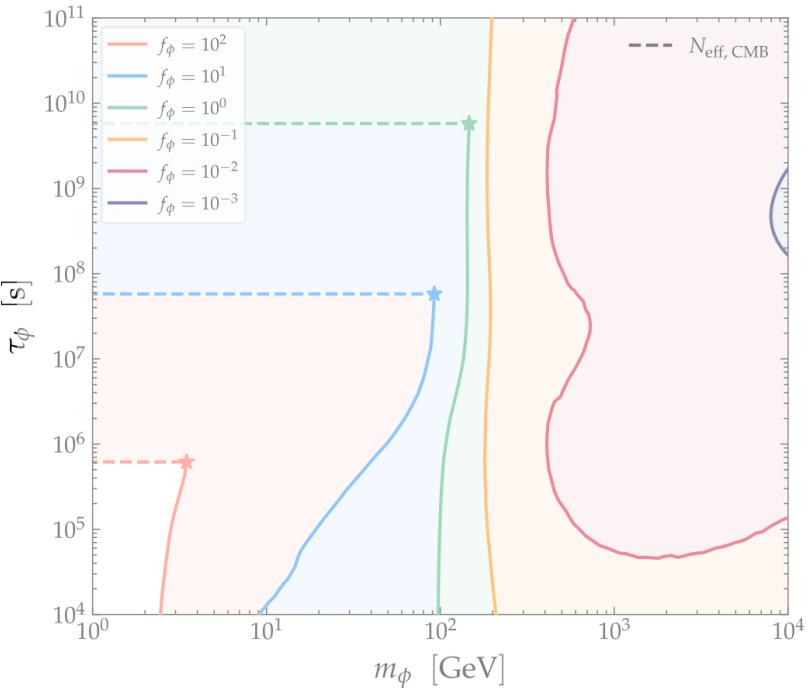


# All three parameter planes

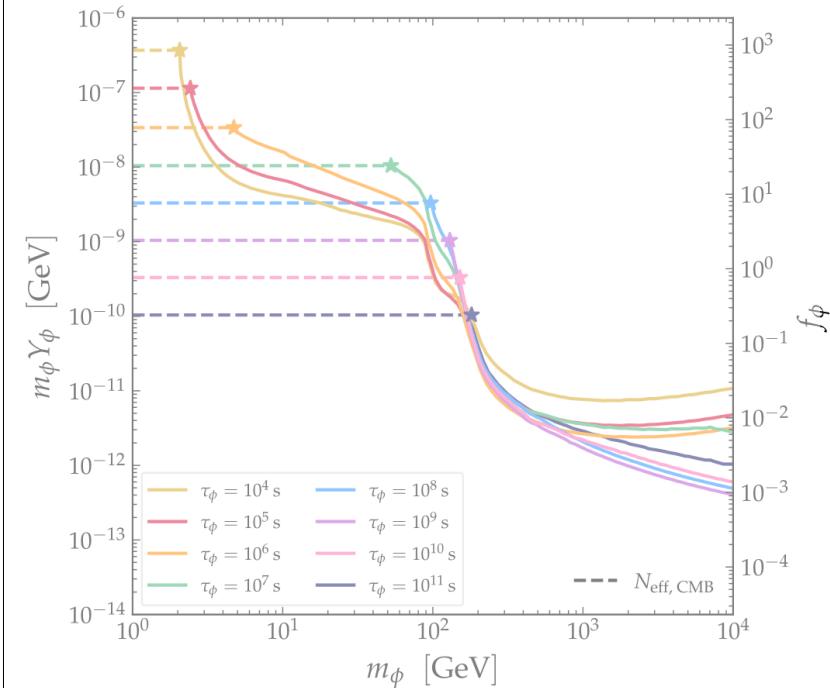
$m_\phi$



$f_\phi$

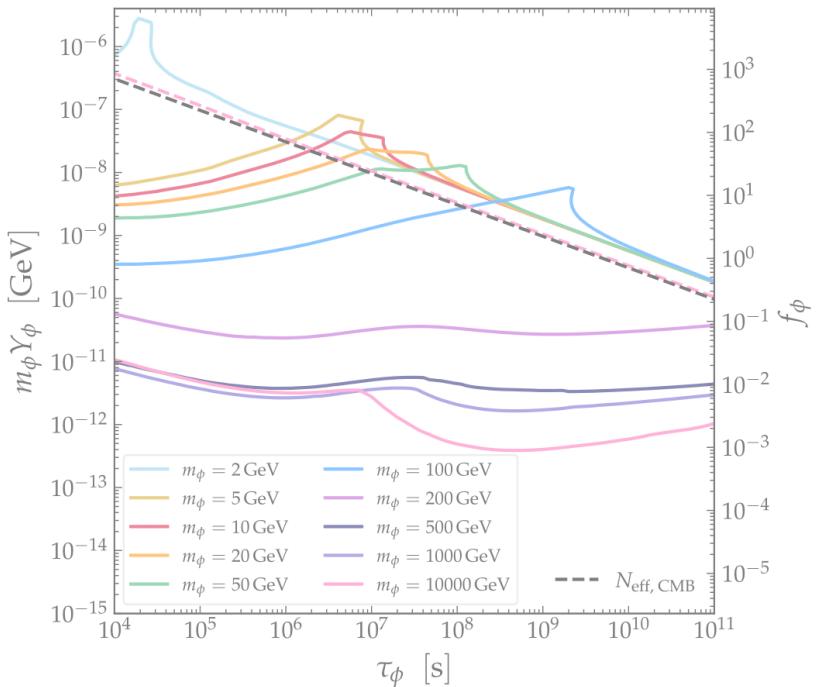


$\tau_\phi$

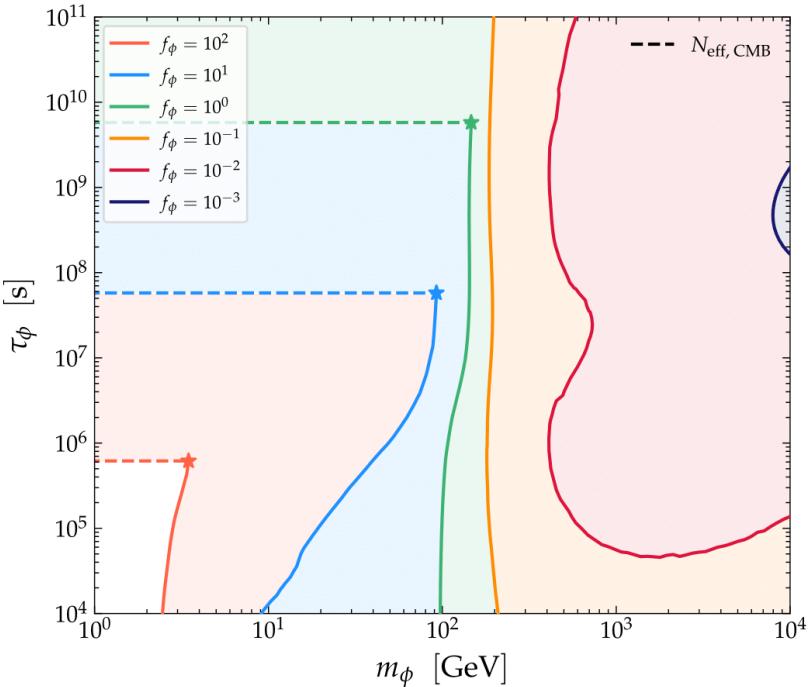


# All three parameter planes

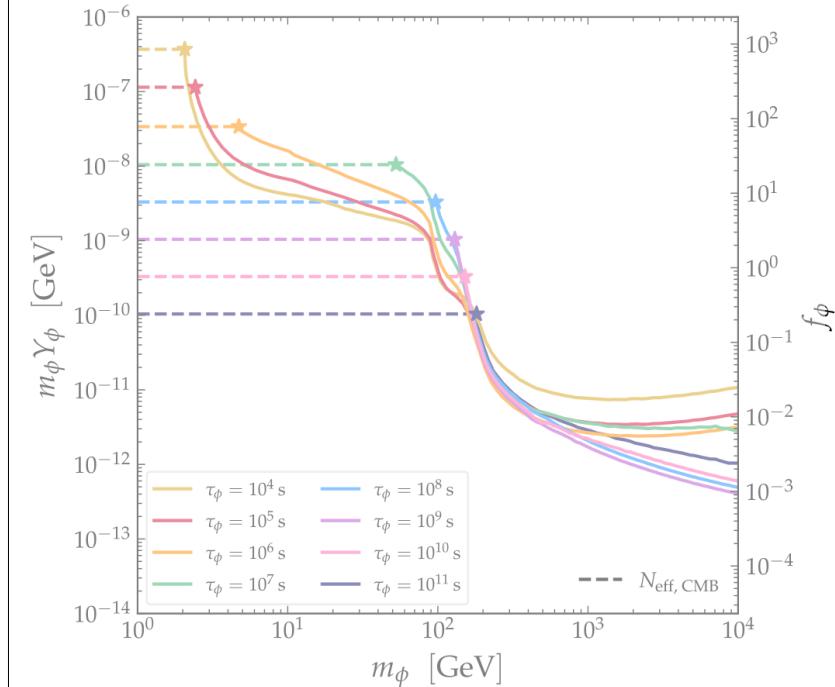
$m_\phi$



$f_\phi$

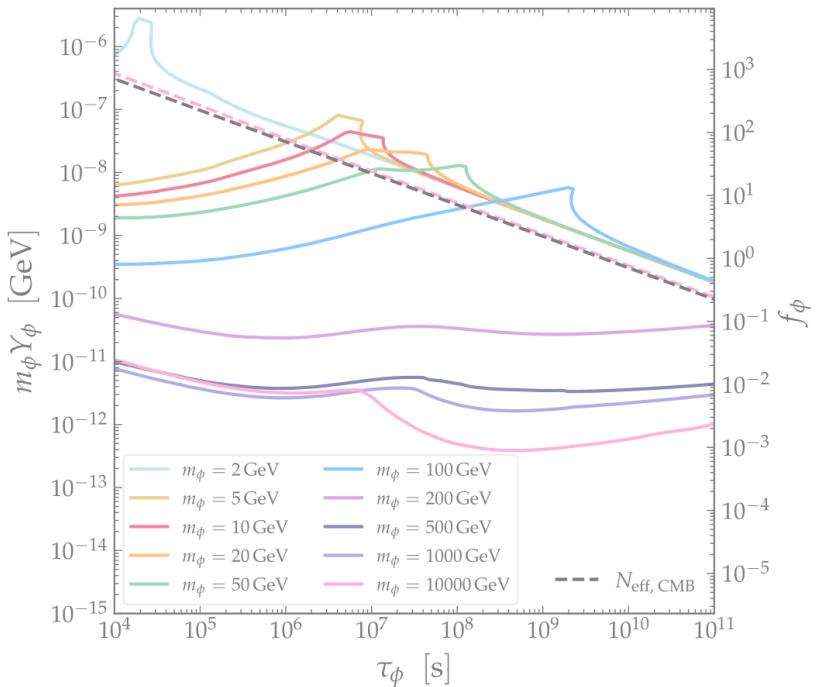


$\tau_\phi$

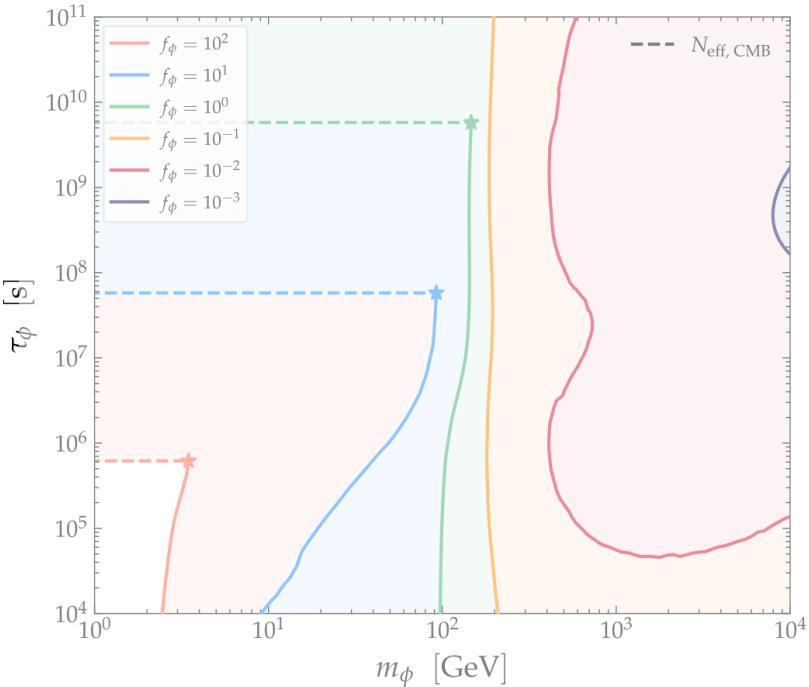


# All three parameter planes

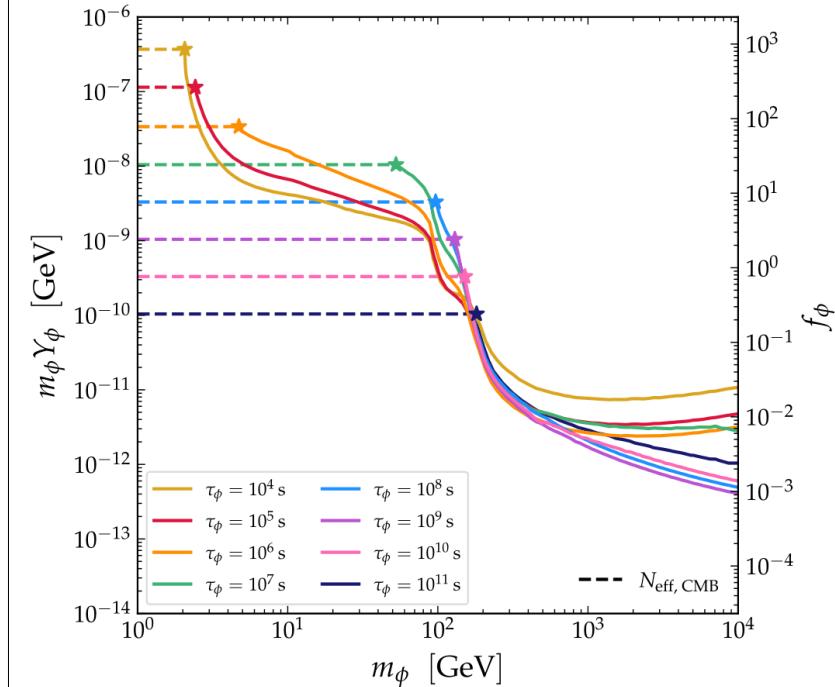
$m_\phi$



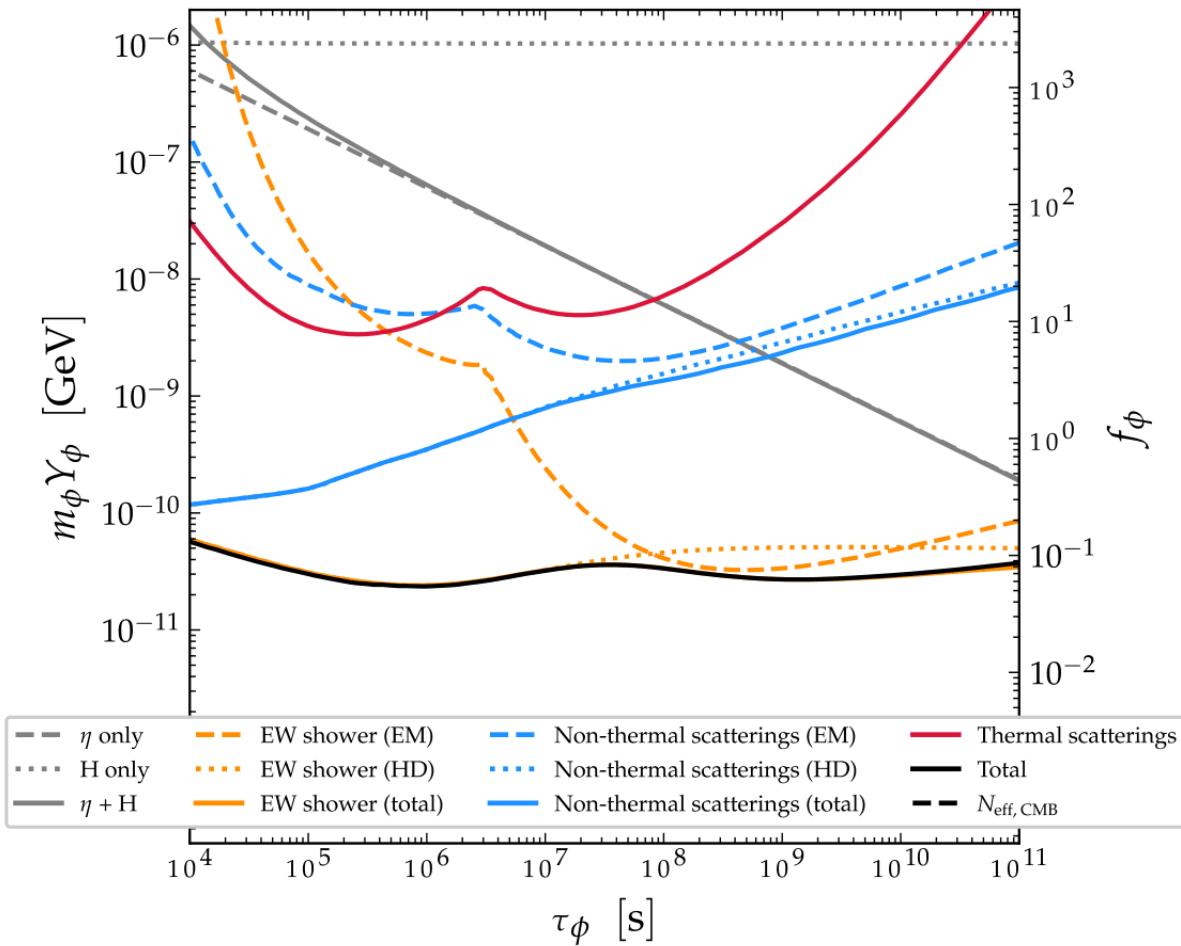
$f_\phi$



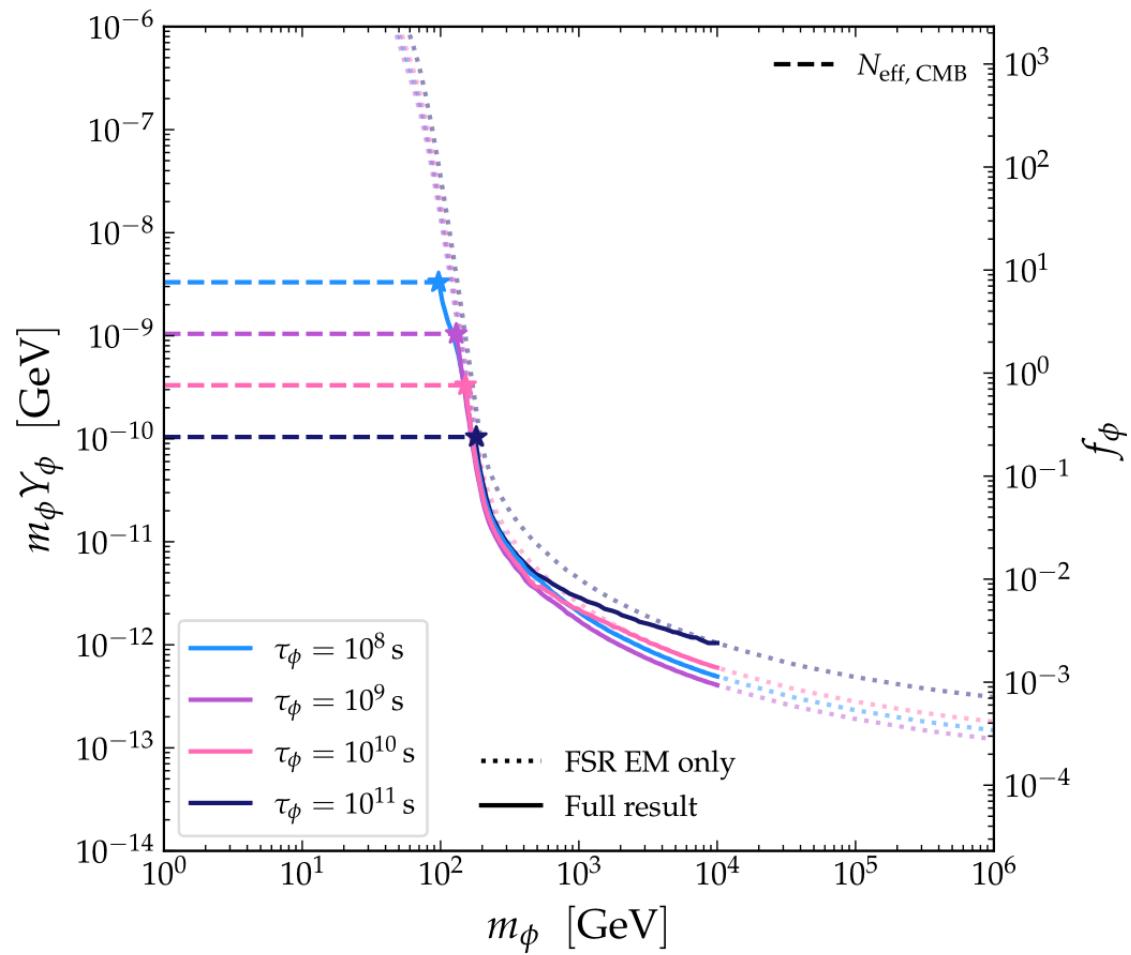
$\tau_\phi$



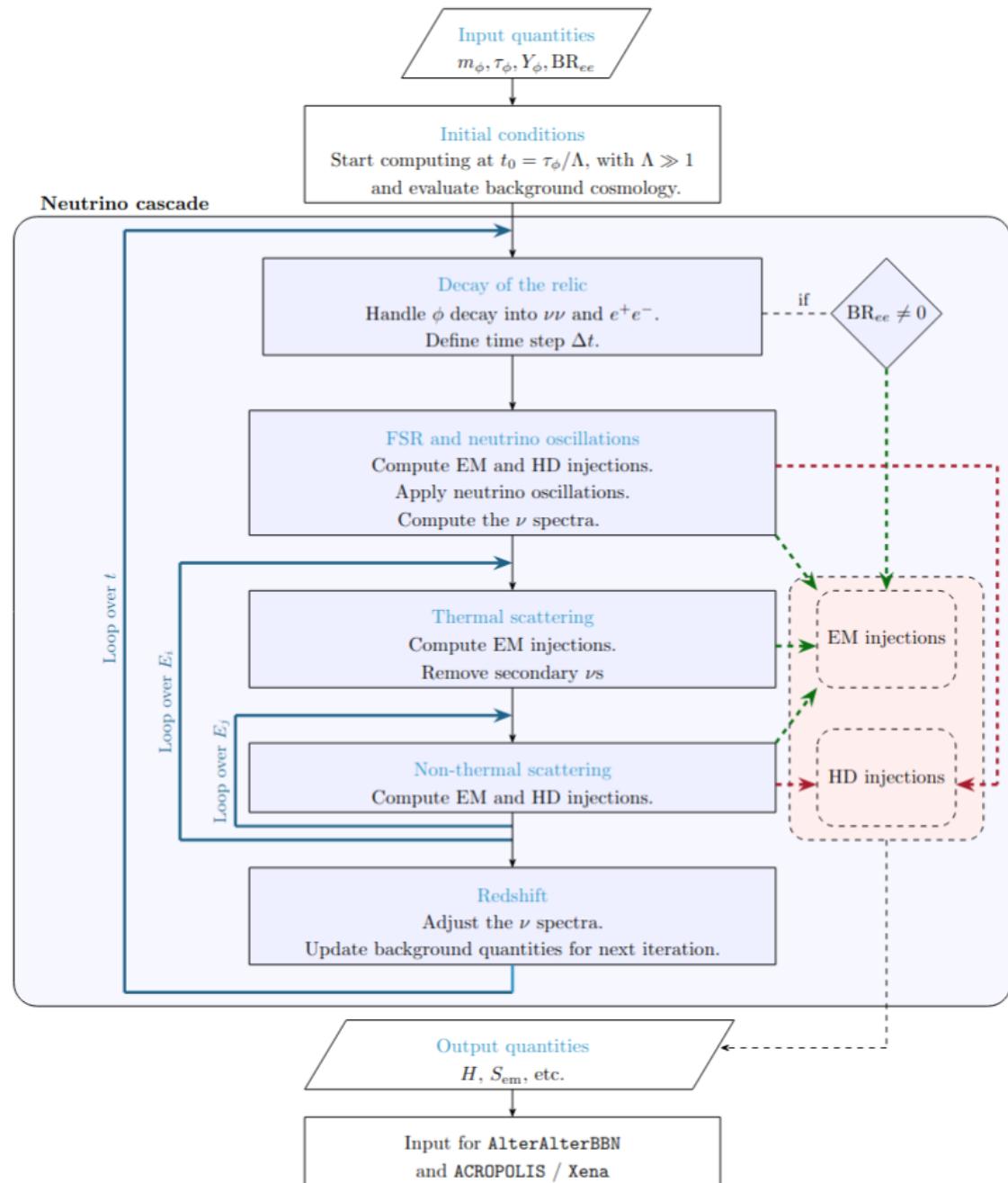
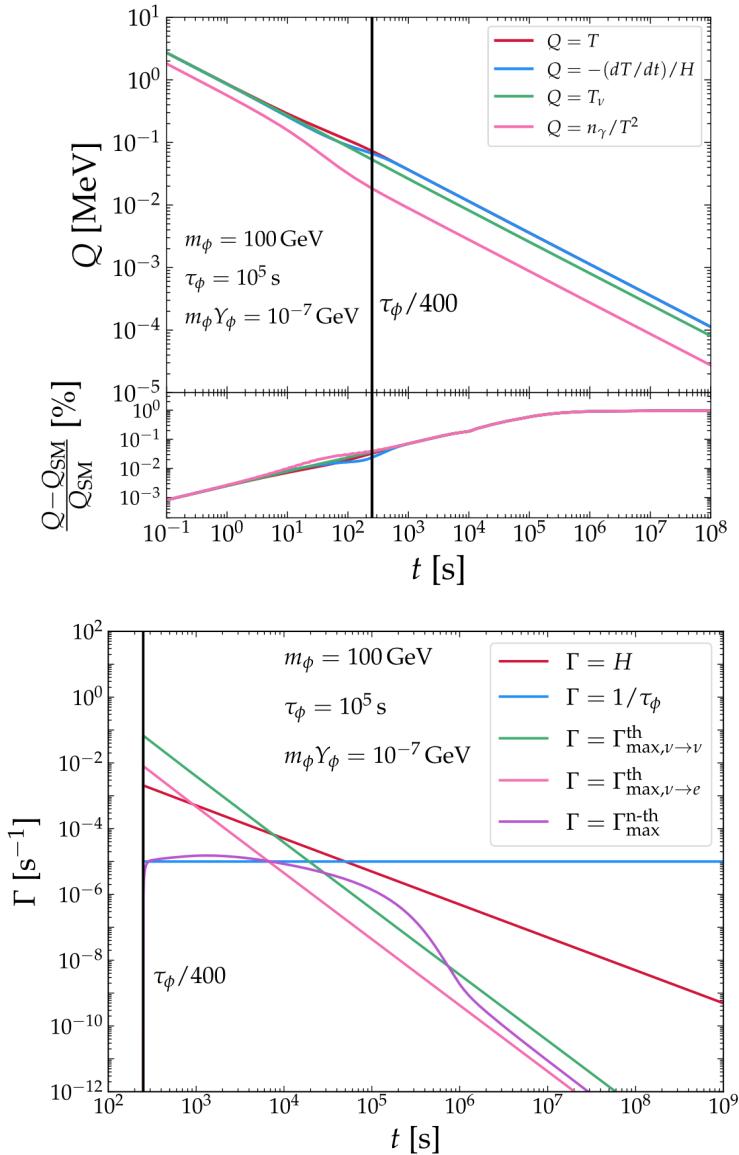
# “Intermediate” mass result: 200GeV



# Asymptotic behaviour

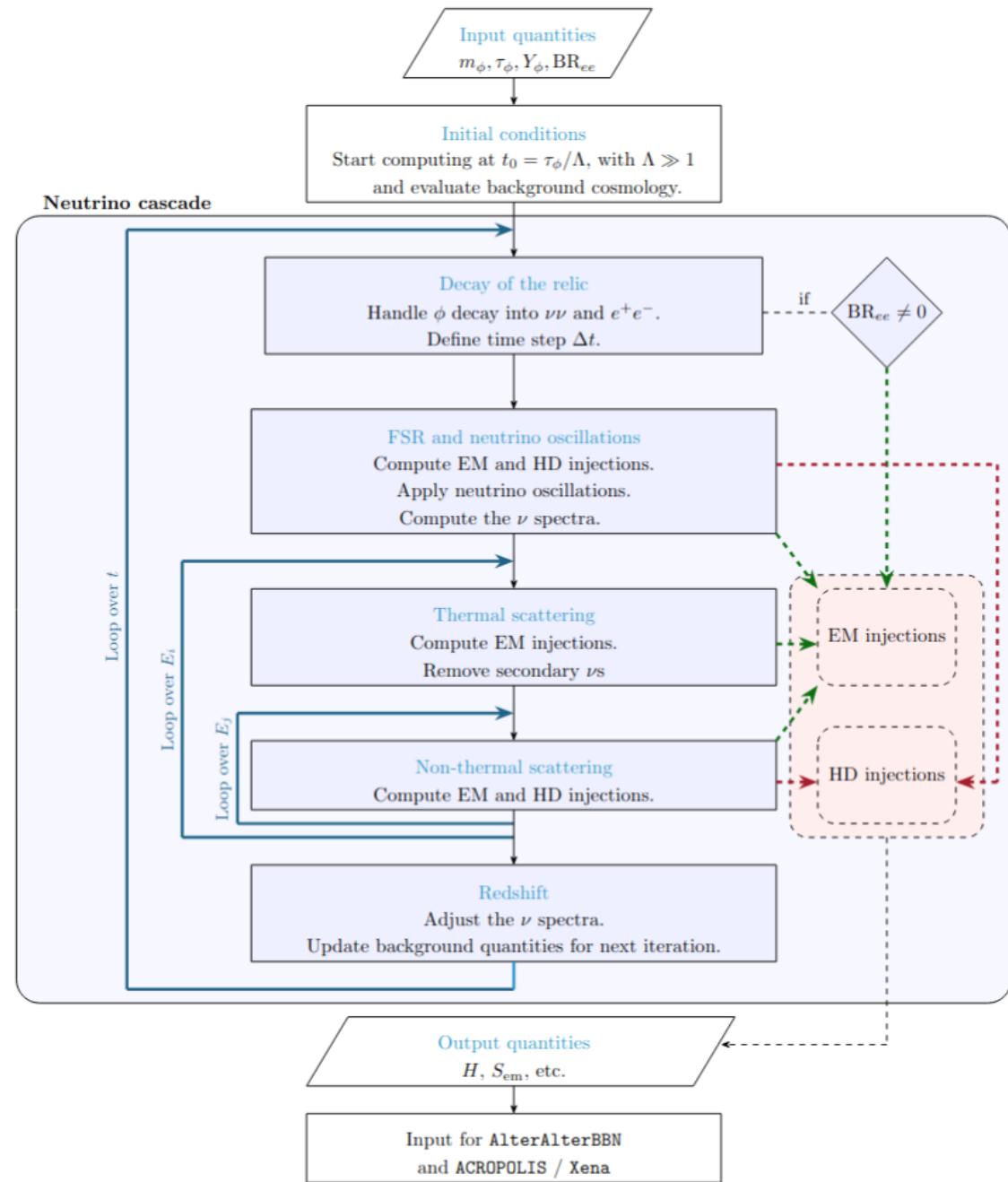


# How to track the neutrino cascade

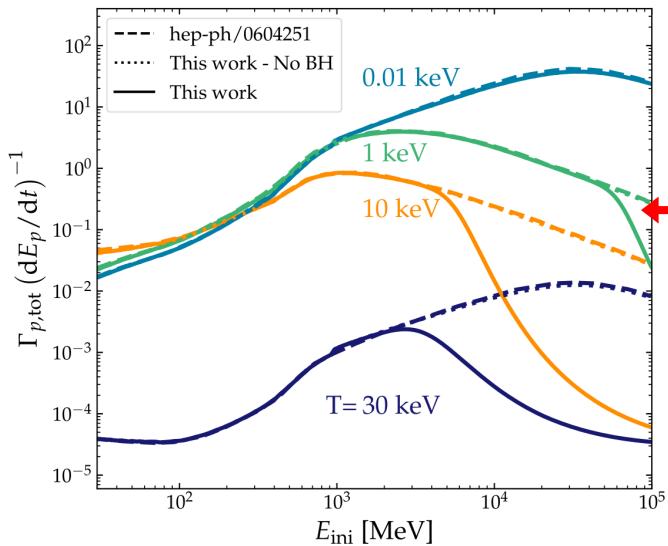


# How to track the neutrino cascade

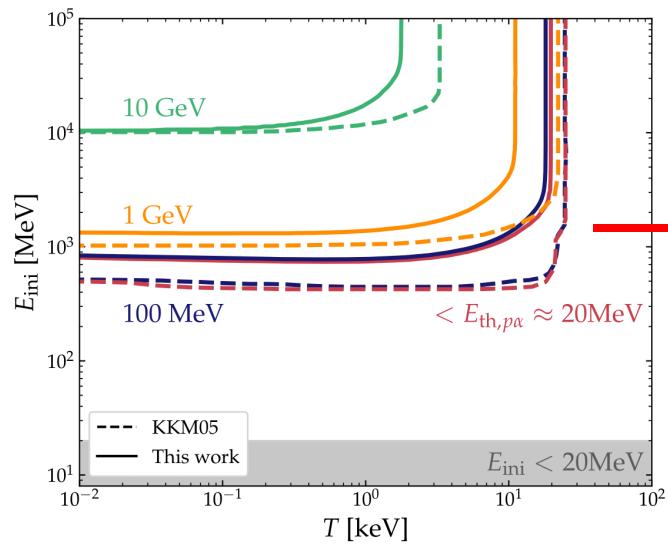
Variable	Description	Unit
<b>Input parameters</b>		
$Y_\phi$	Abundance of $\phi$	-
$m_\phi$	Mass of $\phi$	MeV
$\tau_\phi$	Lifetime of $\phi$	s
$\text{BR}_{ee}$	Branching ratio into $e^+e^-$	-
<b>One-dimensional output parameters</b>		
$f_\phi$	Fractional abundance of $\phi$ , $\frac{\Omega_\phi}{\Omega_{\text{DM}}}$	-
$N_{\text{eff}}$	Effective number of neutrinos	-
$\eta$	Baryon-to-photon ratio	-
<b>Two-dimensional output parameters</b>		
$t$	Time	s
$T$	Temperature	MeV
$\frac{dT}{dt}$	Time-temperature relation	MeV <sup>2</sup>
$T_\nu$	Neutrino temperature	MeV
$H$	Hubble rate	MeV
$n_\gamma$	Photon number density	MeV <sup>3</sup>
$S_{\text{em}}$	Electromagnetic source term, eq. (4.11)	MeV <sup>4</sup>
$S_{\text{hd}}$	Hadronic source term	MeV <sup>4</sup>
$\frac{dn_{\text{hd}}^{\text{n-th}}}{dt}$	Injected hadrons per time from non-th. scattering, eq. (4.14)	MeV <sup>4</sup>
$K_{\text{hd}}^{\text{n-th}}$	Avg. kinetic energy of hadrons from non-th. scattering, eq. (4.14)	MeV
$\frac{dn_{\text{fsr}}^{\text{hd}}}{dt}$	Injected hadrons per time interval from FSR, eq. (4.14)	MeV <sup>4</sup>
$K_{\text{hd}}^{\text{fsr}}$	Avg. kinetic energy of hadrons from FSR, eq. (4.14)	MeV



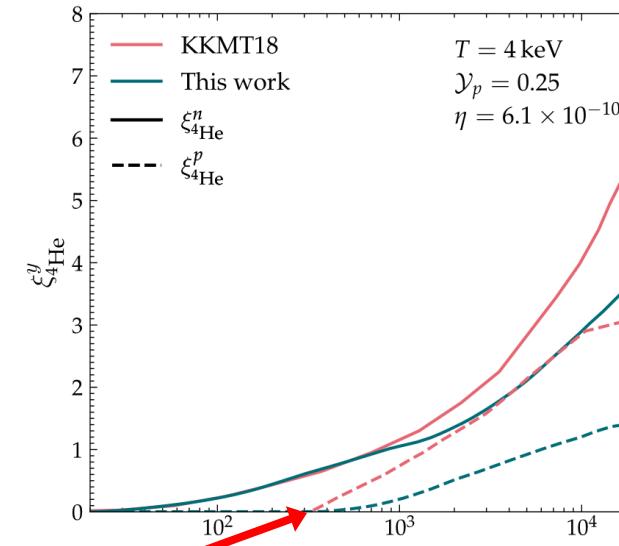
# Details on hadrodisintegration & discrepancy with the literature



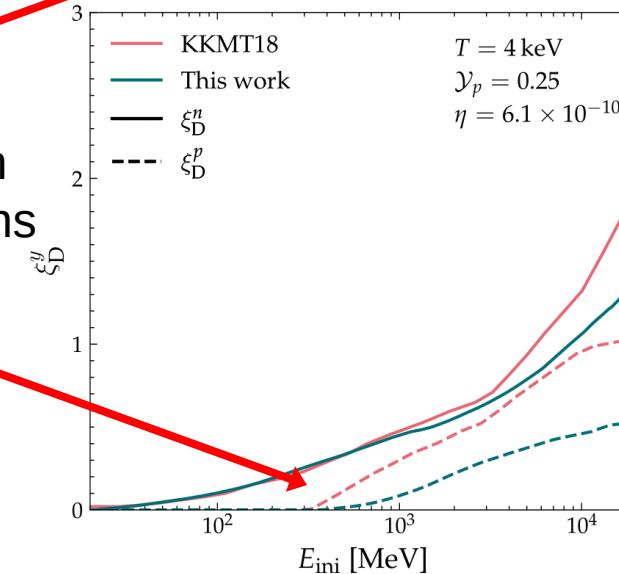
Cross check with the literature



Related to difference in stopping power of protons



Note:  
Discrepancy in neutrons due to interconversions



# Model building

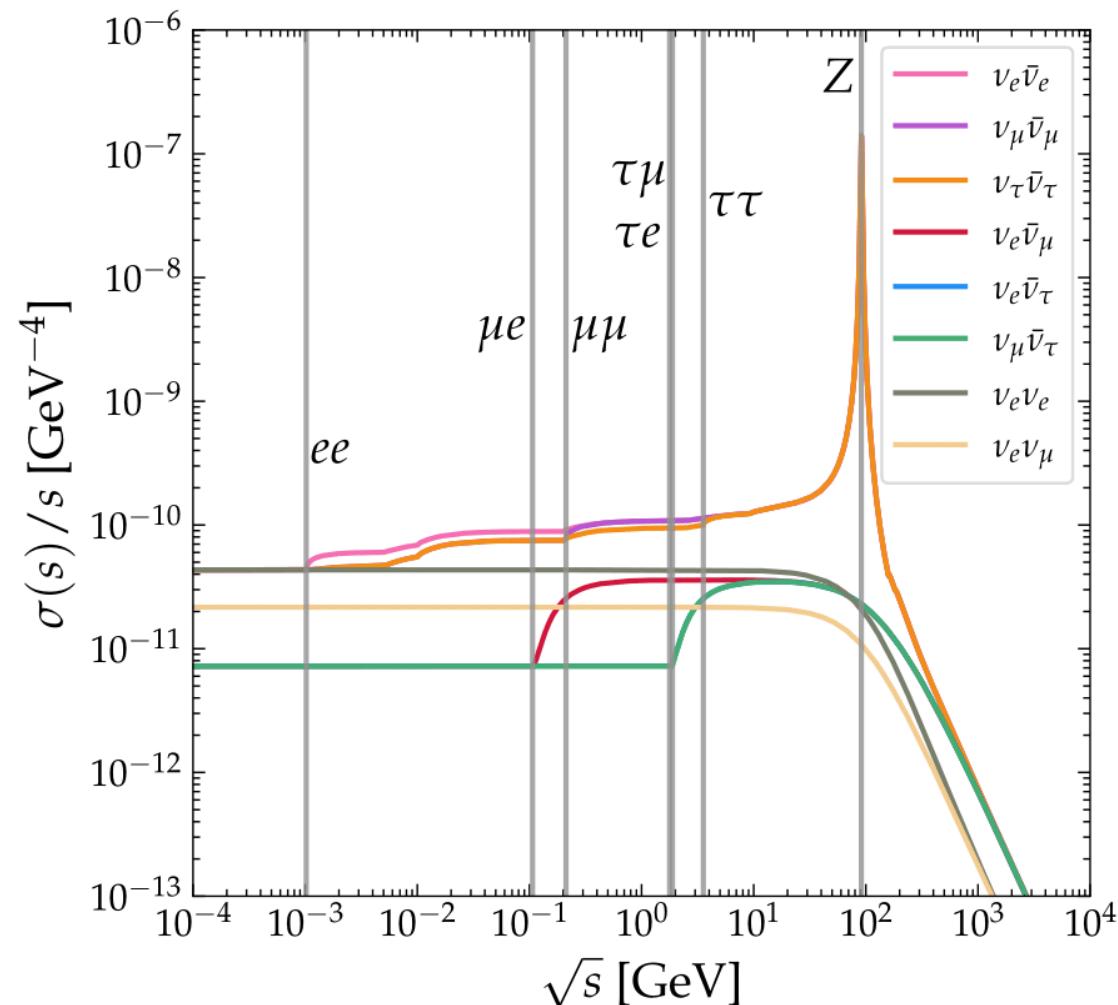
- Majoron: with loop suppressed decay into electrons
- Gauge boson coupled to two sterile neutrinos + seesaw mixing
- Neutral component of a scalar hypercharge 2 triplet
- Decay into one neutrino and one DS state is potentially also relevant

$$y_N \phi \overline{N^c} N$$

$$\Delta = (\delta^0, \delta^+, \delta^{++})$$

$$\Delta LL$$

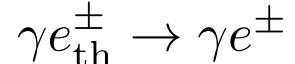
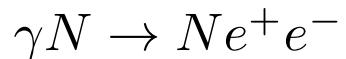
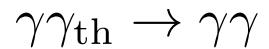
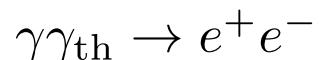
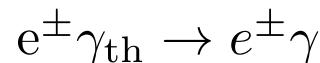
# The full EW cross sections



# Details on photodisintegration processes

## Important energy loss/redistribution processes

- Inverse Compton scattering
- Double-photon pair creation
- Photon-photon scattering
- Bethe-Heitler pair creation
- Compton scattering



## Disintegration processes

	$E_r^{\text{th}}$ [MeV]
D + $\gamma \rightarrow n + p$	2.22
$^3\text{H} + \gamma \rightarrow n + \text{D}$	6.26
$^3\text{H} + \gamma \rightarrow 2n + p$	8.48
$^3\text{He} + \gamma \rightarrow p + \text{D}$	5.49
$^3\text{He} + \gamma \rightarrow n + 2p$	7.72
$^4\text{He} + \gamma \rightarrow p + ^3\text{H}$	19.81
$^4\text{He} + \gamma \rightarrow n + ^3\text{He}$	20.58
$^4\text{He} + \gamma \rightarrow 2\text{D}$	23.85
$^4\text{He} + \gamma \rightarrow n + p + \text{D}$	26.07
$^6\text{Li} + \gamma \rightarrow n + p + ^4\text{He}$	3.70
$^6\text{Li} + \gamma \rightarrow X + ^3A$	15.79
$^7\text{Li} + \gamma \rightarrow ^3\text{H} + ^4\text{He}$	2.47
$^7\text{Li} + \gamma \rightarrow n + ^6\text{Li}$	7.25
$^7\text{Li} + \gamma \rightarrow 2n + p + ^4\text{He}$	10.95
$^7\text{Be} + \gamma \rightarrow ^3\text{He} + ^4\text{He}$	1.59
$^7\text{Be} + \gamma \rightarrow p + ^6\text{Li}$	5.61
$^7\text{Be} + \gamma \rightarrow n + 2p + ^4\text{He}$	9.30
	$\tau_r$ [s]
$n \rightarrow p + e^- + \bar{\nu}_e$	$8.802 \times 10^2$
$^3\text{H} \rightarrow ^3\text{He} + e^- + \bar{\nu}_e$	$5.61 \times 10^8$

# Neutrino oscillations

$$t_{\text{osc}} \sim \frac{T}{\Delta m^2} \sim 10^{-5} \text{ s} \left( \frac{T}{\text{MeV}} \right) \quad 1812.05605$$

Neutrino oscillation time scale much higher than other time scales in our setup!

Therefore, we can assume instantaneous oscillations using the following mixing matrix

$$(\mathbf{M}_{\text{osc}})_{\alpha\beta} = \sum_{i=1}^3 |U_{\alpha i}|^2 |U_{\beta i}|^2 \approx \begin{pmatrix} 0.55 & 0.17 & 0.28 \\ 0.17 & 0.45 & 0.37 \\ 0.28 & 0.37 & 0.35 \end{pmatrix}$$