Non-singular solutions to the Boltzmann equation with a fluid *Ansatz*

arXiv:2412.09266

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Modelling the evolutions of bubbles in cosmological First Order Phase Transitions

SYNERGIES TOWARDS THE FUTURE STANDARD MODEL

DESY Theory Workshop
23 - 26 September 2025 at DESY Hamburg, Germany

Why Cosmological First Order Phase Transitions (FOPT)?

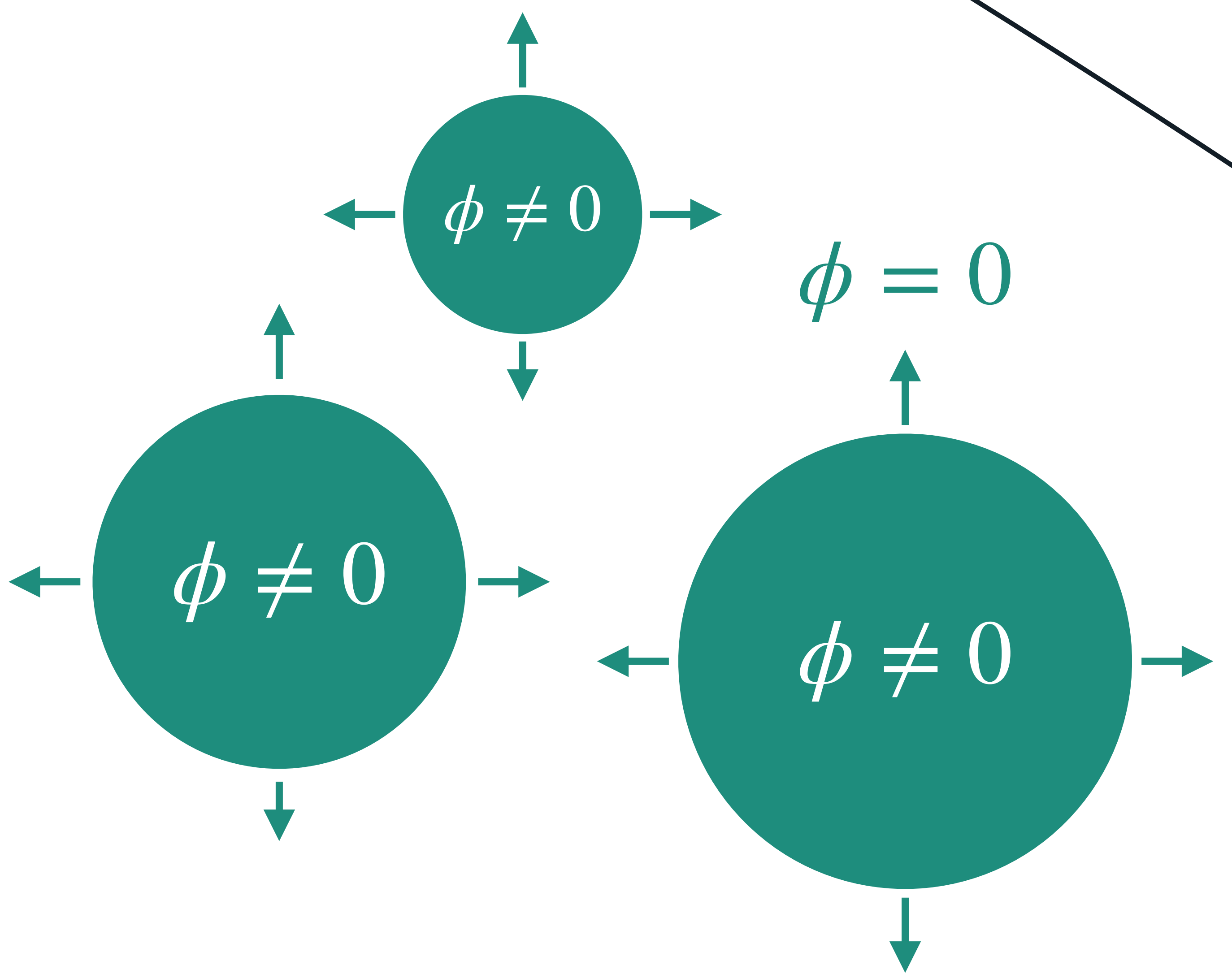
Bubble nucleation

Baryogenesis

Primordial magnetic fields

Stochastic background of gravitational waves

Particle Dark Matter



All these processes depend crucially on the velocity of the expanding bubble wall, ξ_w

The wall velocity ξ_w

The Klein-Gordon equation for the background field

The parameter ξ_w is closely connected to the friction on the expanding wall ϕ

$$\Box\phi + \frac{dV_0}{d\phi} + \sum_i \frac{dm_i^2}{d\phi}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_i} f_i(p^\mu, x^\mu) = 0$$

- V_0 is the zero-temperature potential
- f_i is the distribution function of the $i-$th particle
- m_i, E_i are the mass and energy of the $i-$th particle
- $\mathscr{F}$ is the free-energy of the system

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$$\Box \phi + \frac{\partial \mathcal{F}}{\partial \phi} - \mathcal{K}(\phi) = 0$$

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Equilibrium back reaction

Out of equilibrium friction

The wall velocity ξ_w

Solving the Boltzmann Equation (BE)

The dynamics of the particles in the plasma can be described by the BE

$$p^\mu \partial_\mu f_i\left(x^\mu, p^\mu\right) + \frac{1}{2}\partial_\mu m^2 \partial_{p^\mu} f_i\left(x^\mu, p^\mu\right) + \mathscr{C}_i = 0,$$

The source term drives particles out of equilibrium

Collision terms, couples the different species in the plasma

[1407.3132] Konstandin et al.

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We transform this integro-differential equation into a set of ODEs by:

1. *Linearisation* and *Ansatz* on distributions: $f_i(p^\mu, x^\mu) = f_{\mathrm{eq,i}} + \delta f_i(p^\mu, x^\mu)$

2.Taking momenta: $\int \frac{d^3p}{(2\pi)^3 E}, \int \frac{d^3p}{(2\pi)^3 E} p_\mu u^\mu, \int \frac{d^3p}{(2\pi)^3 E} p_\mu \bar{u}^\mu, \ldots$

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The wall velocity $\xi_{\rm w}$

Back to the friction

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fluid Ansatz $\delta f = f' \delta_p \, , \; \delta_p = \delta\mu + p^\mu(\delta u_\mu - u_\mu \delta T/T)$

BEs for the 3 fluctuations

$\delta\mu$, chemical pot. fluctuations

δu_μ, velocity fluctuations

δT, temperature fluctuations

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Usual approach:

$T_i(z) = \bar{T}_{bg} + \delta T_i(z),$

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Inserting the fluctuations in the linearised version of the Klein-Gordon equation we can solve for the wall velocity ξ_{w} and width L_{w}

$\delta\mu$, chemical pot. fluctuations

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A new way for solving Boltzmann Equations

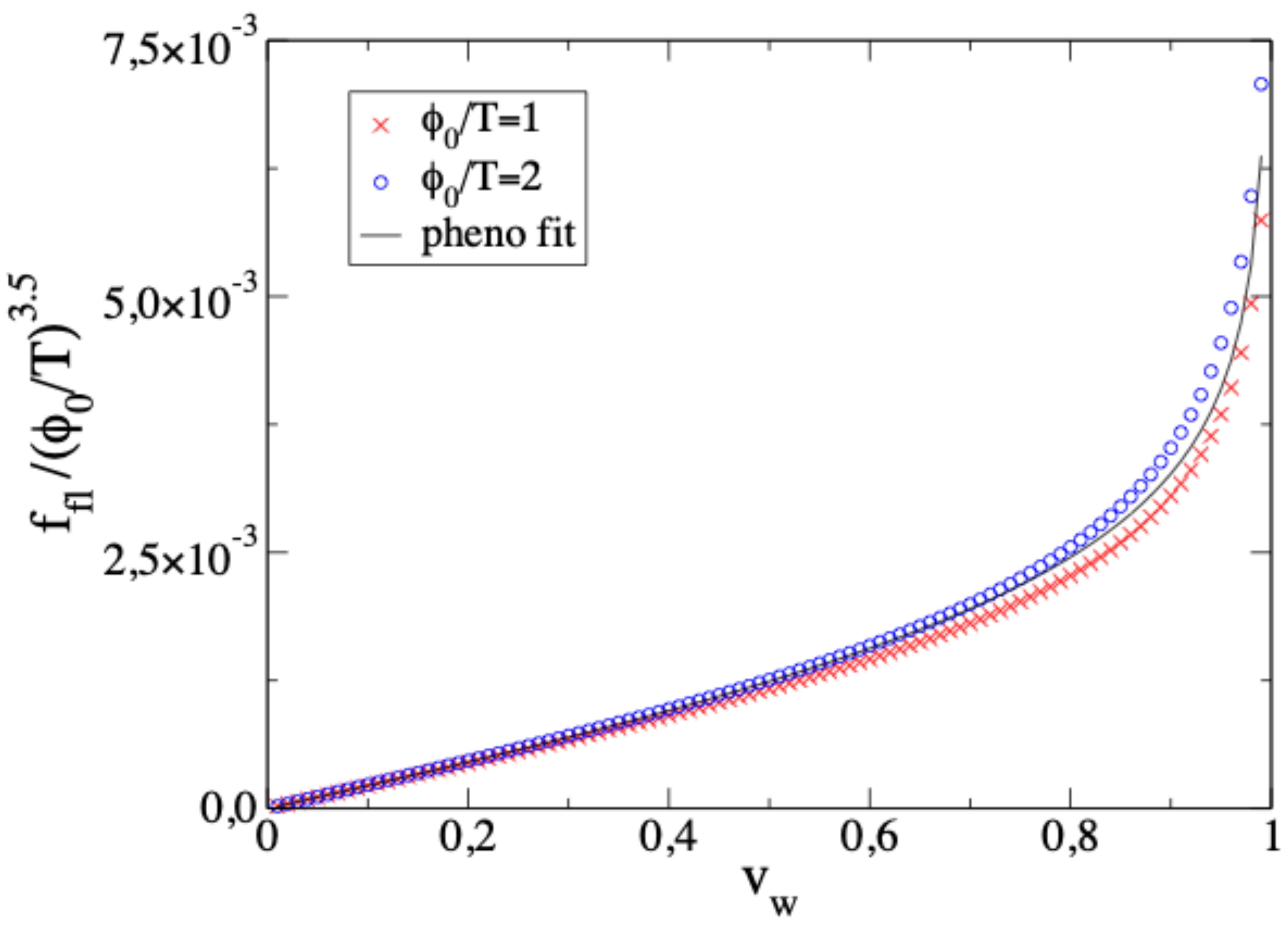
Once obtained the fluctuations $\delta \mathrm{T_i}$, $\delta \mathrm{v_i}$ and $\delta \mu_i$, we go back to the KG equation and solve for ξ_w and L_w:

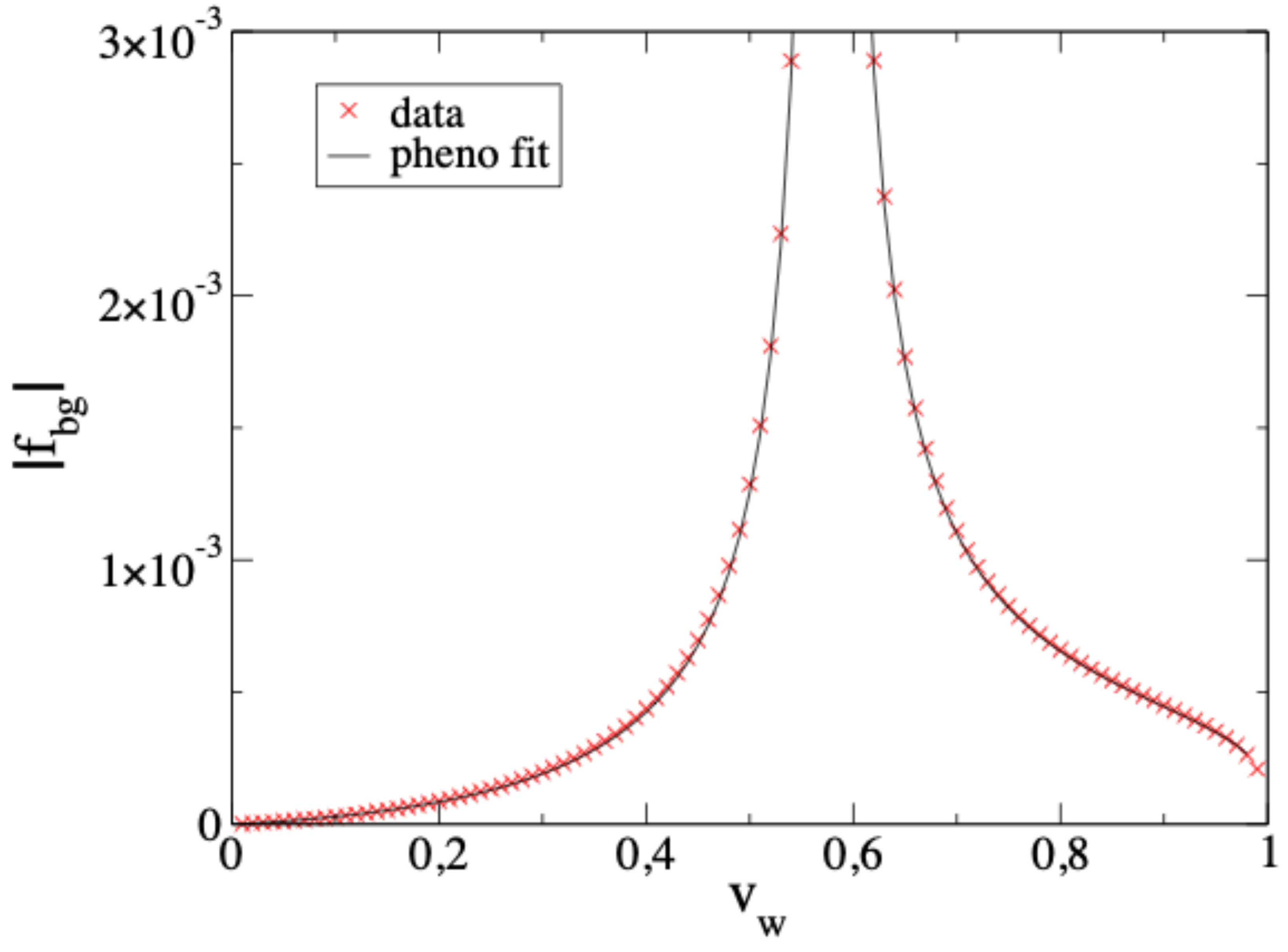
$$-\phi'' + \frac{\partial \mathcal{F}}{\partial \phi} + \sum_i \frac{dm_i^2}{d\phi} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_i} \delta f_i(p,x) = 0.$$

$$\int dz \,(\text{KG eq.}) \times \phi' = 0 \quad \longrightarrow \quad \frac{\mathcal{F}\,|_-^+}{T_+^4} - \frac{1}{T_+^4}\int dz \left(\partial_z T_{\rm bg}\right) \frac{\partial \mathcal{F}}{\partial T} + f_{\rm fl} + f_{\rm light} = 0$$

$$\int dz \,(\text{KG eq.}) \times \phi' \left(2\phi - \phi_0\right) = 0 \quad \longrightarrow \quad \frac{2}{15(T_+L_w)^2}\left(\frac{\phi_0}{T_+}\right)^3 + \frac{W}{T_+^5} + g_{\rm fl} + g_{\rm light} = 0$$

A “sonic boom” in the friction

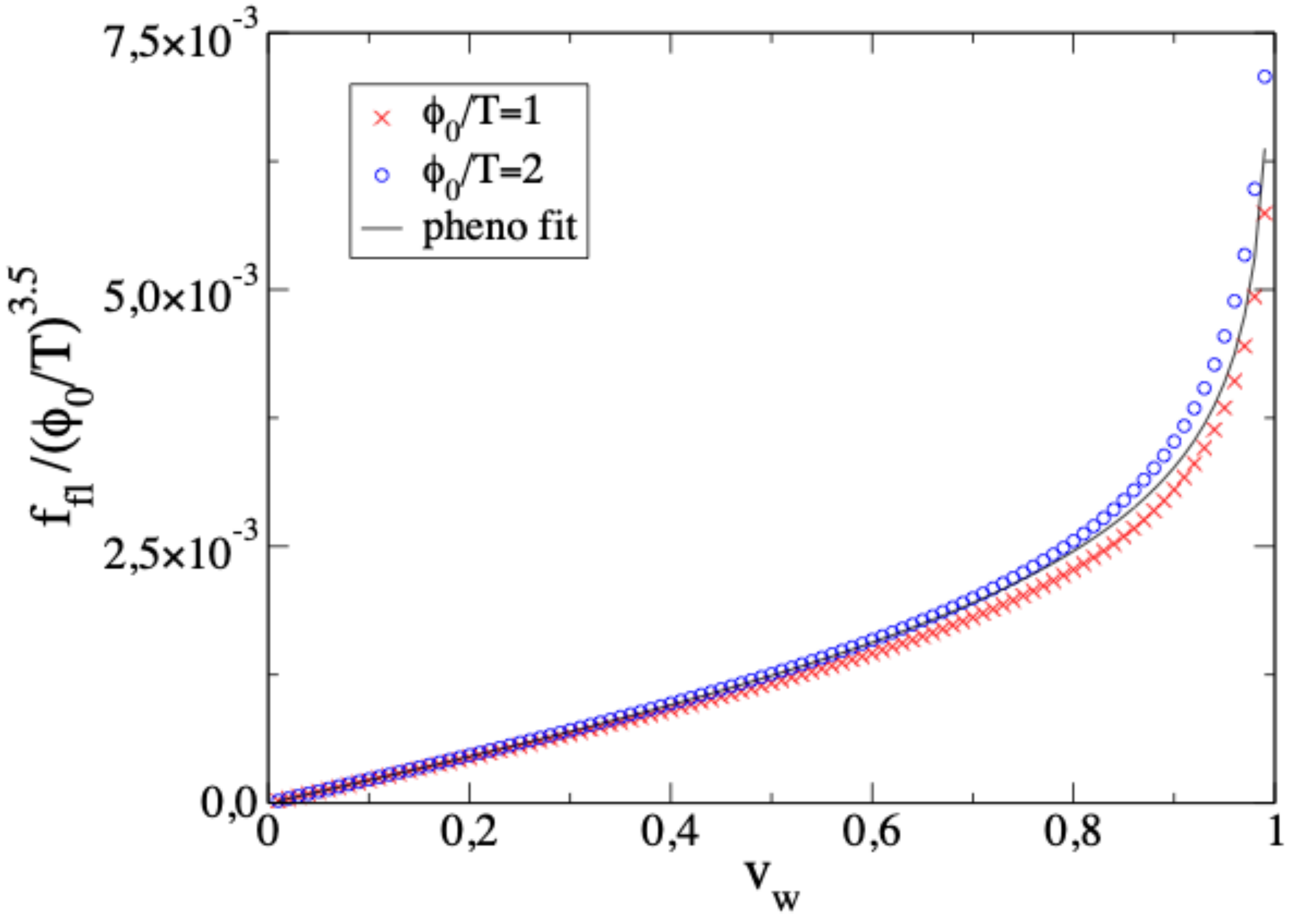


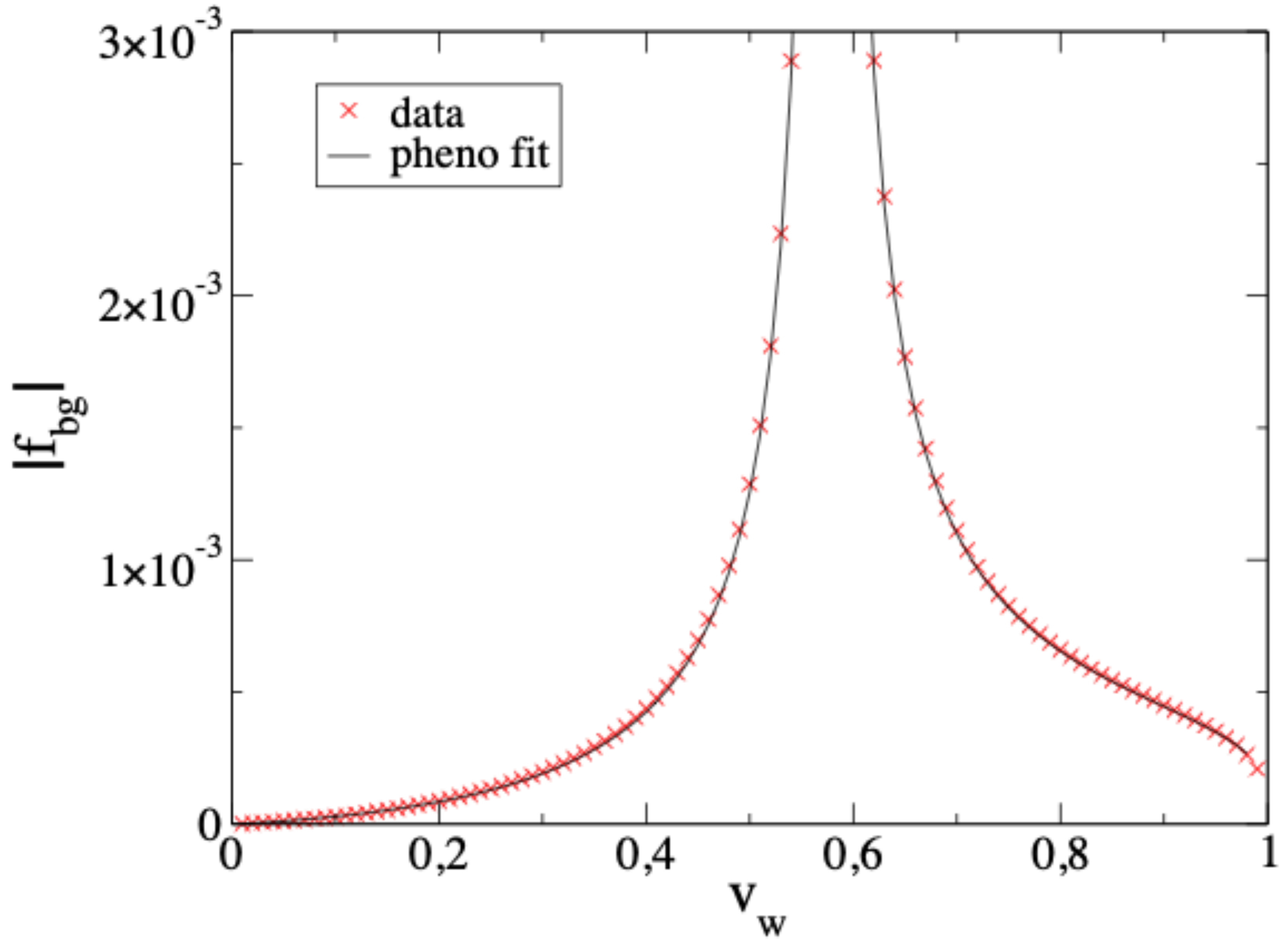


[1407.3132] Konstandin et al.

A divergent friction coming from massless particles is found at the speed of sound!

A “sonic boom” in the friction





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Is it physical?

The origin of the singularity

Hints from hydrodynamics

Hydrodynamics tells us that macroscopic quantities change across a phase transition front to satisfy $\partial_\mu T^{\mu\nu} = 0$. This gives us the conditions

[1004.4187] J.R. Espinosa et al.

$$\gamma_+^2 v_+^2 \omega_+ - \mathcal{F}_+ = \gamma_-^2 v_-^2 \omega_- - \mathcal{F}_-$$
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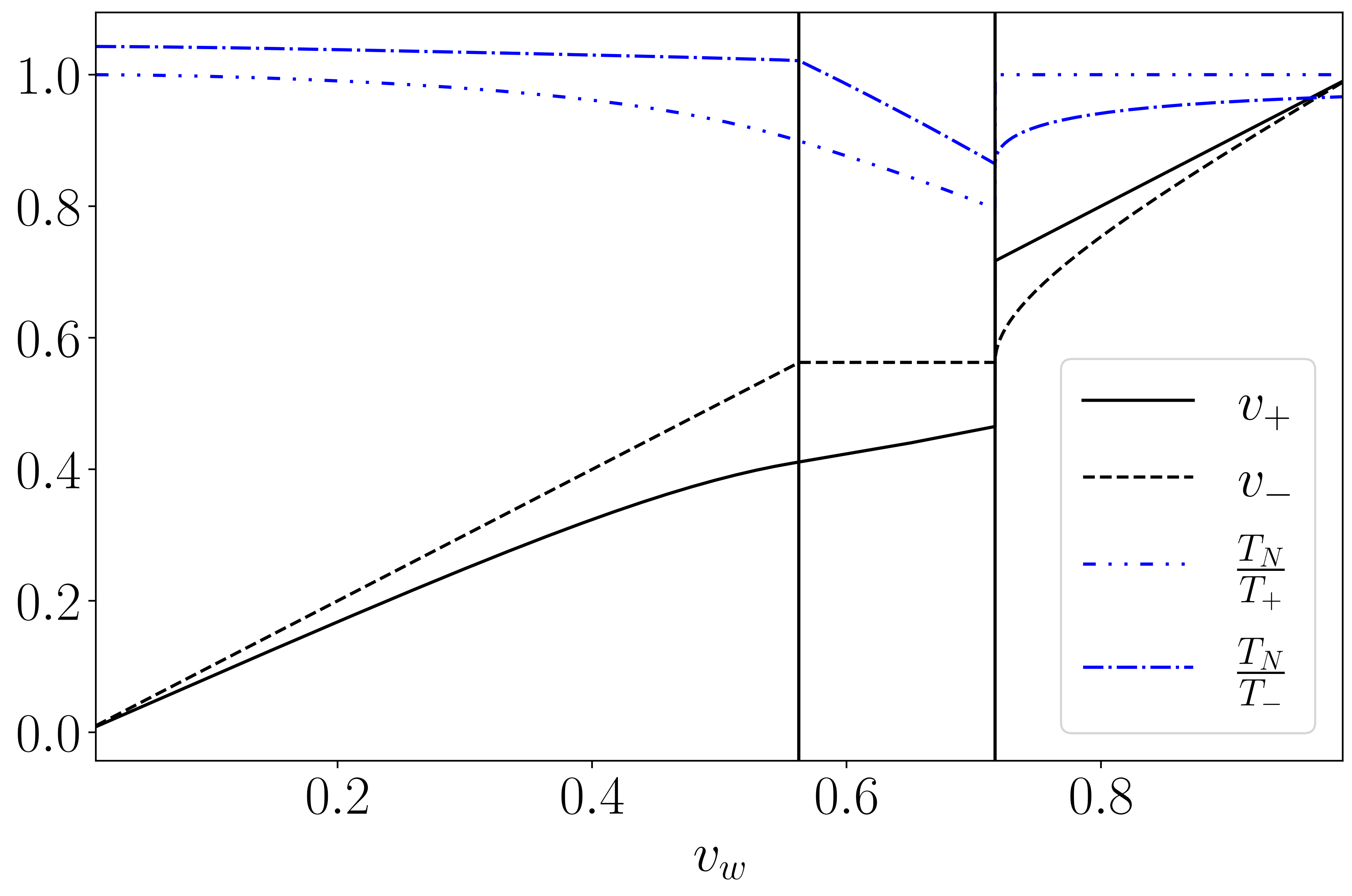

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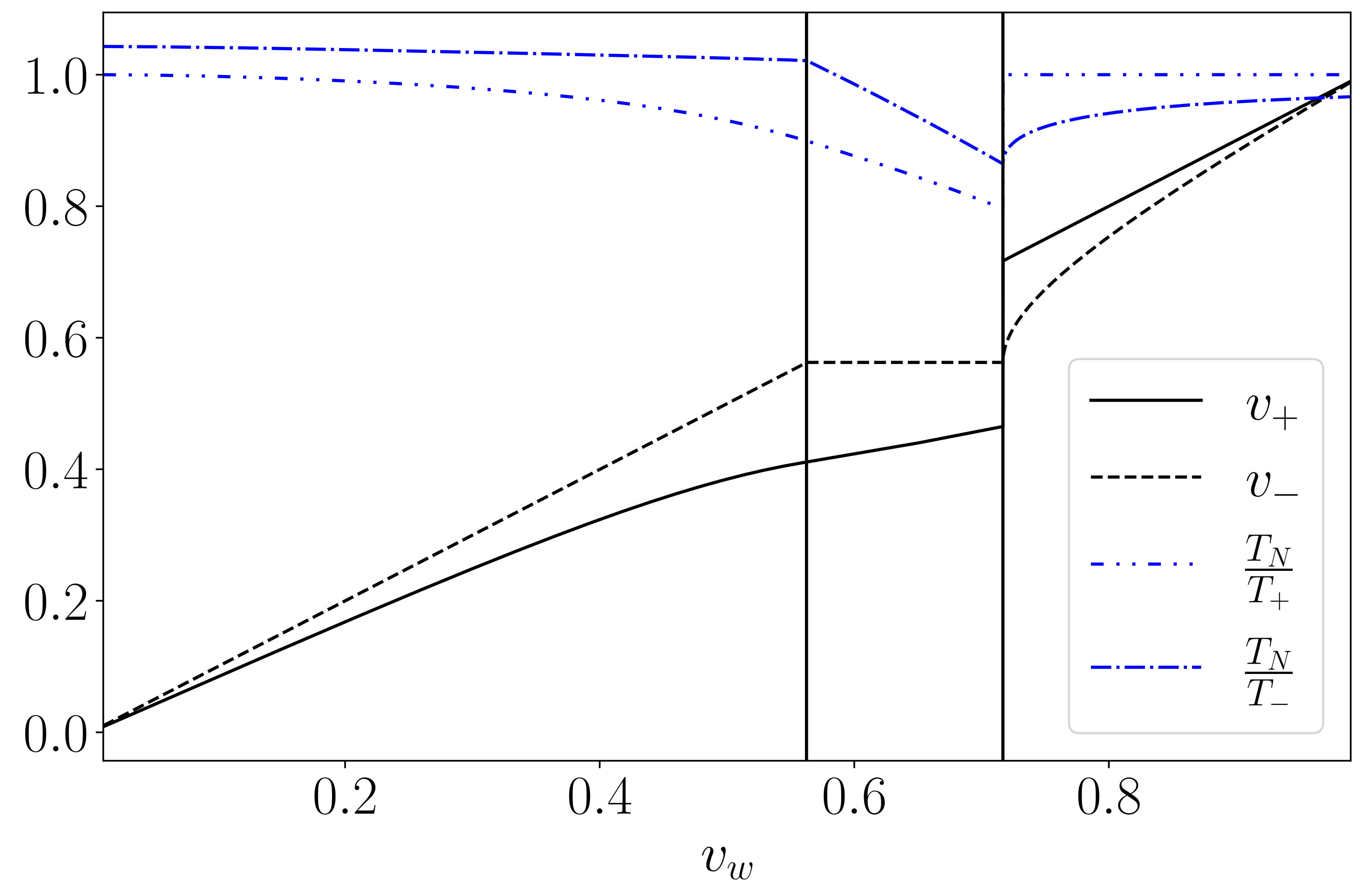
If the temperature and the velocity on the two sides of the wall are not the same, it would impact on the validity of the linearisation procedure.

[1004.4187] J.R. Espinosa et al.



The origin of the singularity

Energy-momentum conservation

The singularity arises because of an interplay between the energy-momentum conservation in the BE and the linearisation procedure

[2112.12548] C. Dorsch et al.

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We enforce energy-momentum conservation at full non linear level to find $v_{bg}(z)$ *and* $T_{bg}(z)$ *and implement perturbation theory on top of this.*

A new way for solving Boltzmann Equations

We define the background by imposing the conservation of its energy-momentum across the phase transition wall. This means solving

$$v_{\mathrm{bg}}^2 \gamma_{\mathrm{bg}}^2 \omega_{\mathrm{bg}} - \mathscr{F}_{\mathrm{bg}} + \frac{1}{2}(\partial_z \phi)^2 = \mathrm{k}_1,$$

$$v_{\mathrm{bg}} \gamma_{\mathrm{bg}}^2 \omega_{\mathrm{bg}} = \mathrm{k}_2 .$$

Scalar field contribution

Modified matching conditions for $\mathrm{v}_{\mathrm{bg}}(\mathrm{z})$ and $\mathrm{T}_{\mathrm{bg}}(\mathrm{z})$

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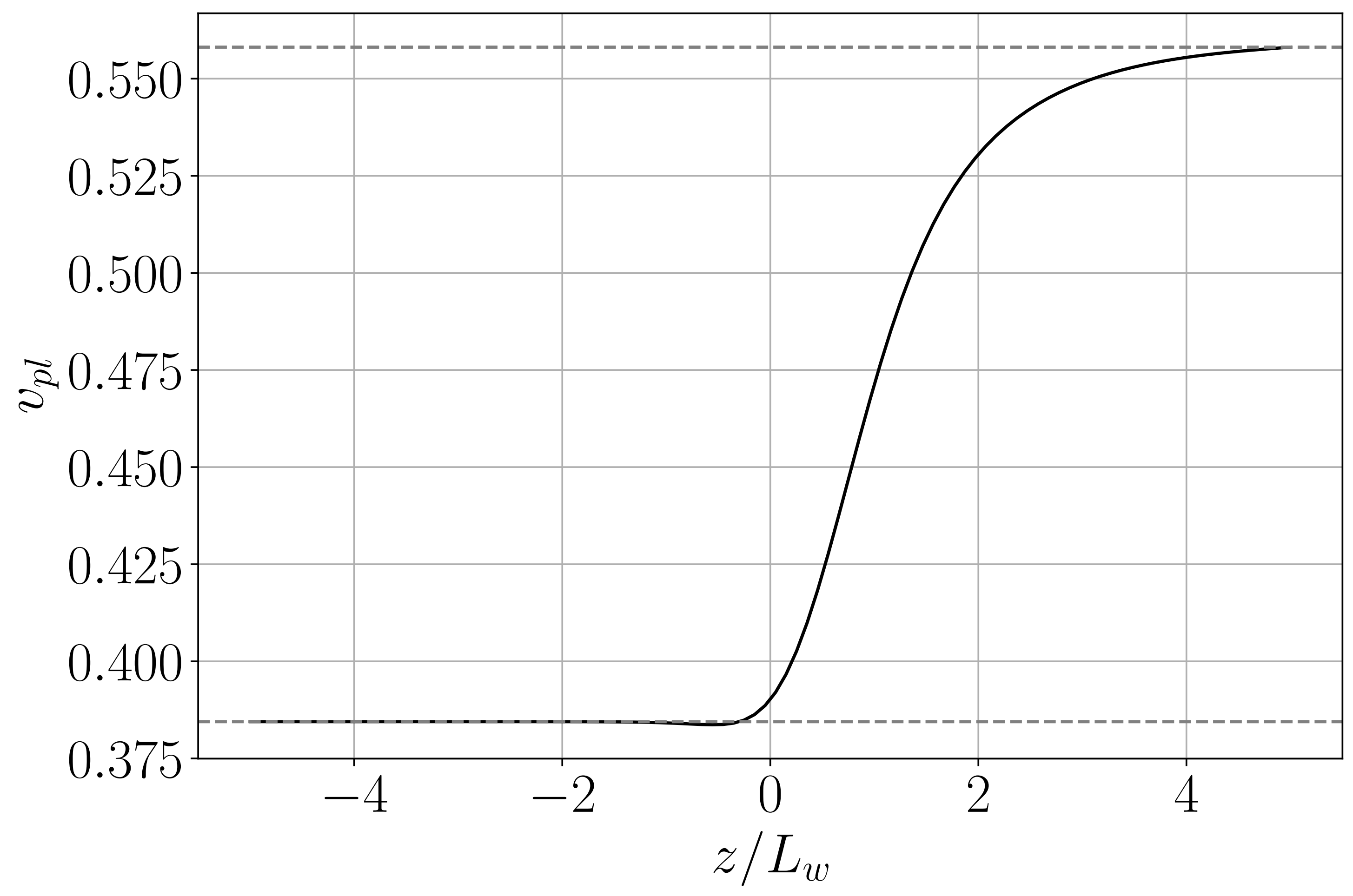

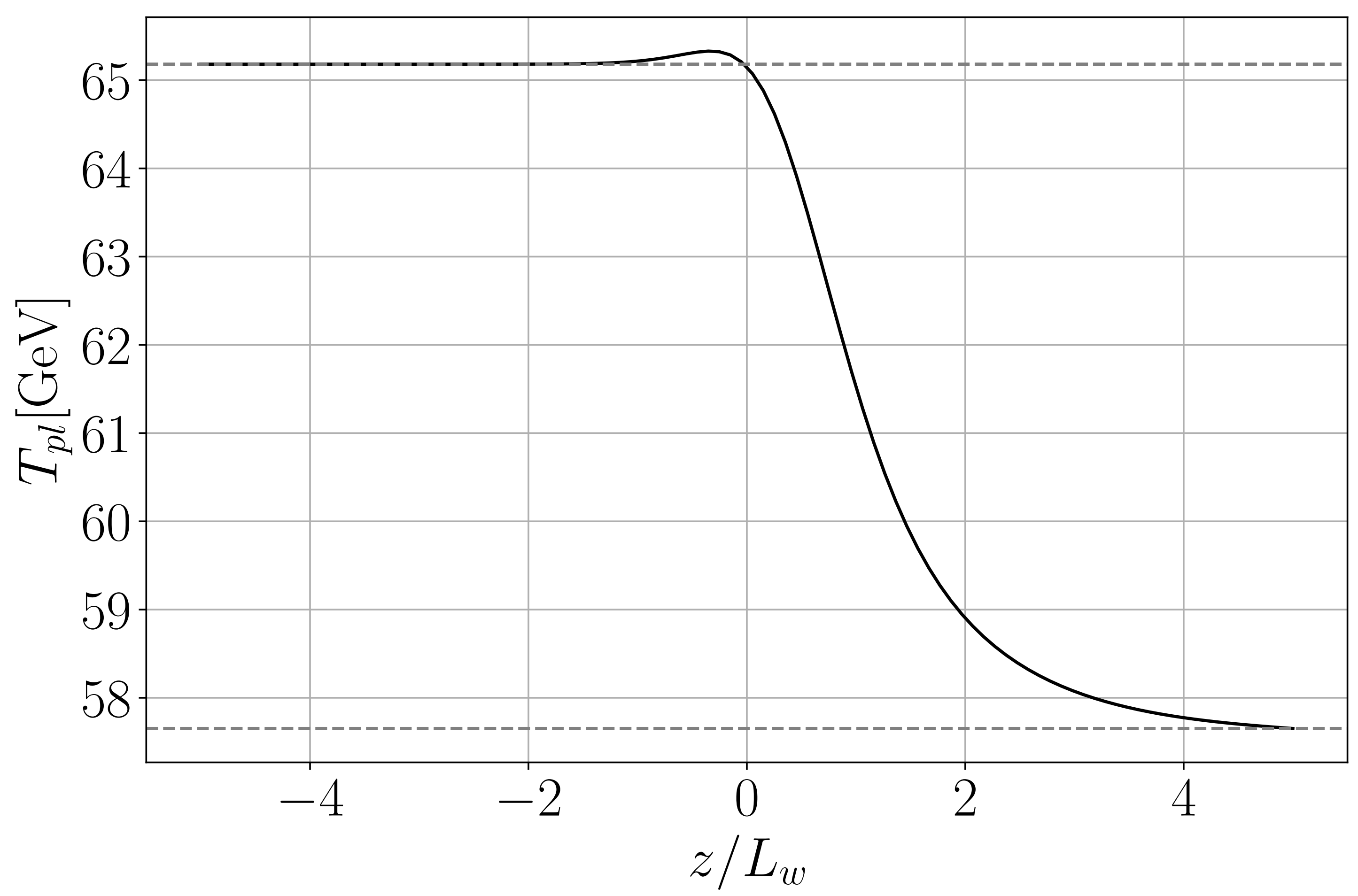

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Going back to the BEs: $$p^\mu \partial_\mu f_i \left(x^\mu, p^\mu\right) + \frac{1}{2}\partial_\mu m^2 \partial_{p^\mu} f_i \left(x^\mu, p^\mu\right) + \mathscr{C}_i = 0$$

Old approach	**Our approach**
$\mathrm{T_i(z) = \bar{T}_{bg} + \delta T_i(z)}$	$\mathrm{T_i(z) = T_{bg}(z) + \delta T_i(z)}$
$\mathrm{v_i(z) = \bar{v}_{bg} + \delta v_i(z)}$	$\mathrm{v_i(z) = v_{bg}(z) + \delta v_i(z)}$
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This conceptual difference implies the presence of a new term (one for every particle) in the BEs!

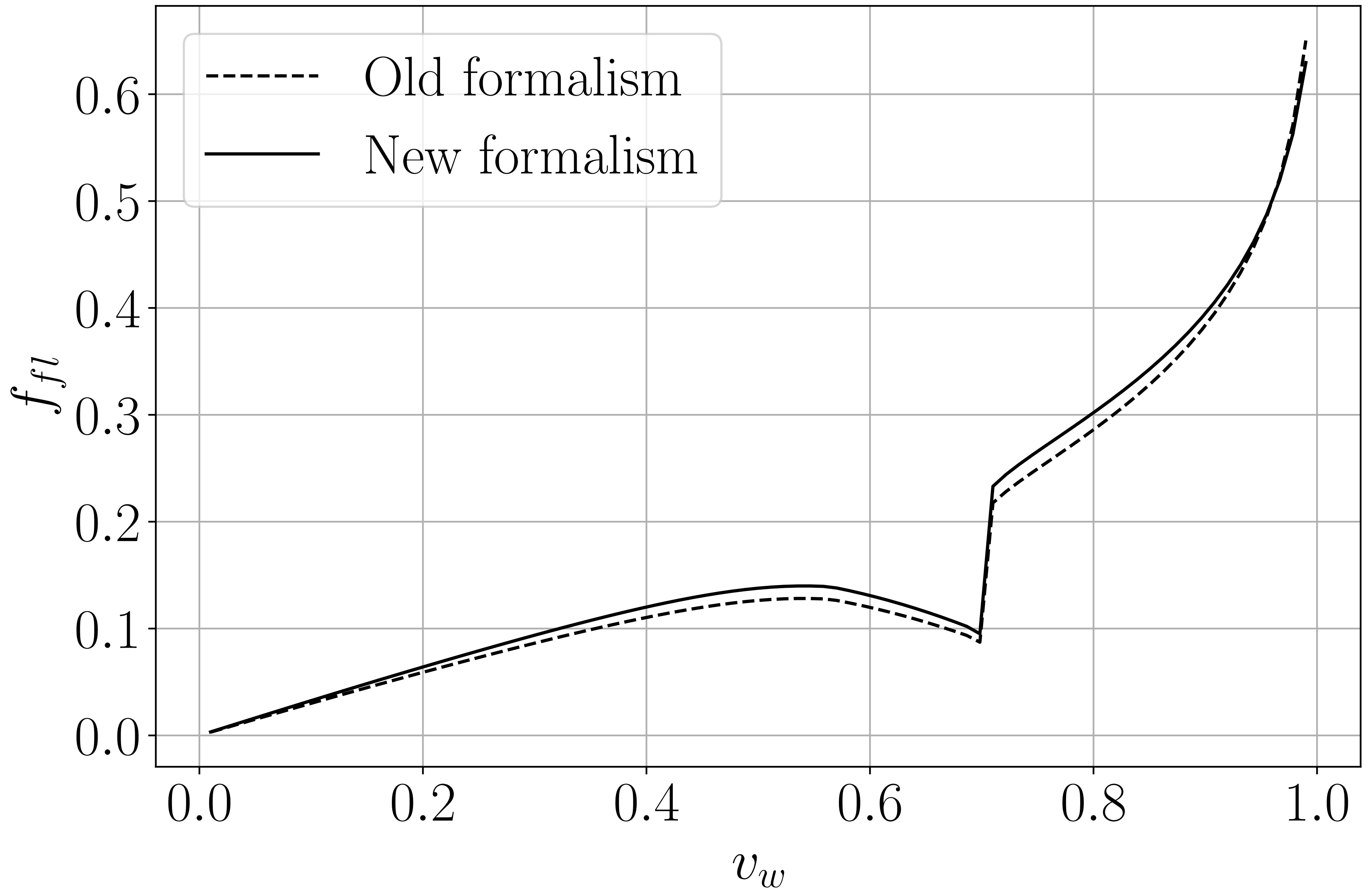
$$p^\mu \partial_\mu f_i^{\mathrm{bg}}\left(x, p\right) \supset (f_i^{\mathrm{bg}})' \frac{p^\mu p^\nu}{T}\left(u_\nu \frac{\partial_\mu T}{T} - \partial_\mu u_\nu\right)$$

This new "source" term is fundamental to ensure energy-momentum conservation at BEs level

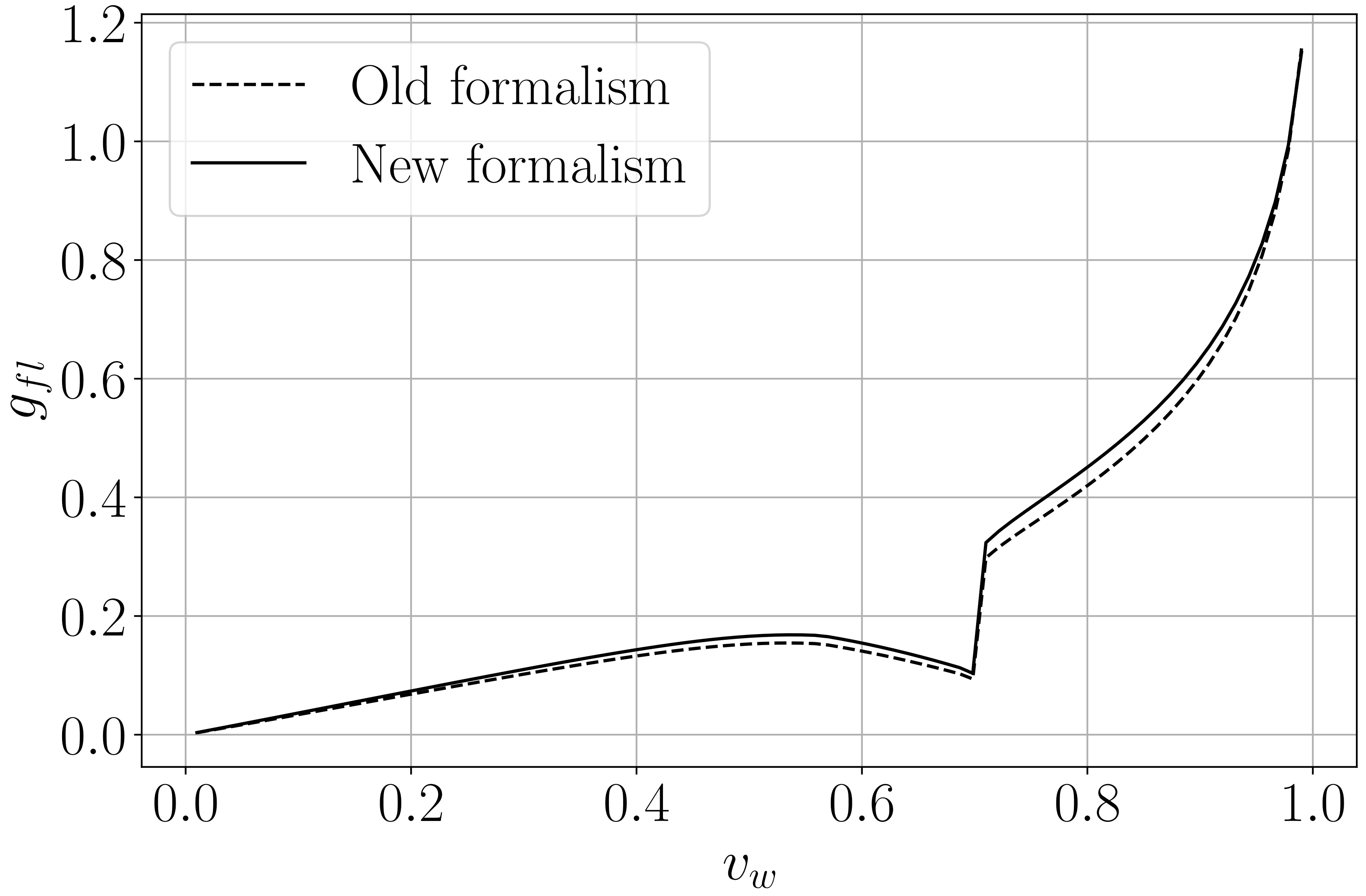
Results

Benchmark model: SM with low cutoff

$$\mathcal{F}(\phi,T) = V_0(\phi) - \frac{a}{3}T^4 + \frac{c}{2}\phi^2T^2, \quad V_0(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\tilde{\lambda}}{4}\phi^4 + \frac{1}{8\Lambda^2}\phi^6$$



For the heavy particles, the main source of friction comes from the $\partial m^2(z)$ term



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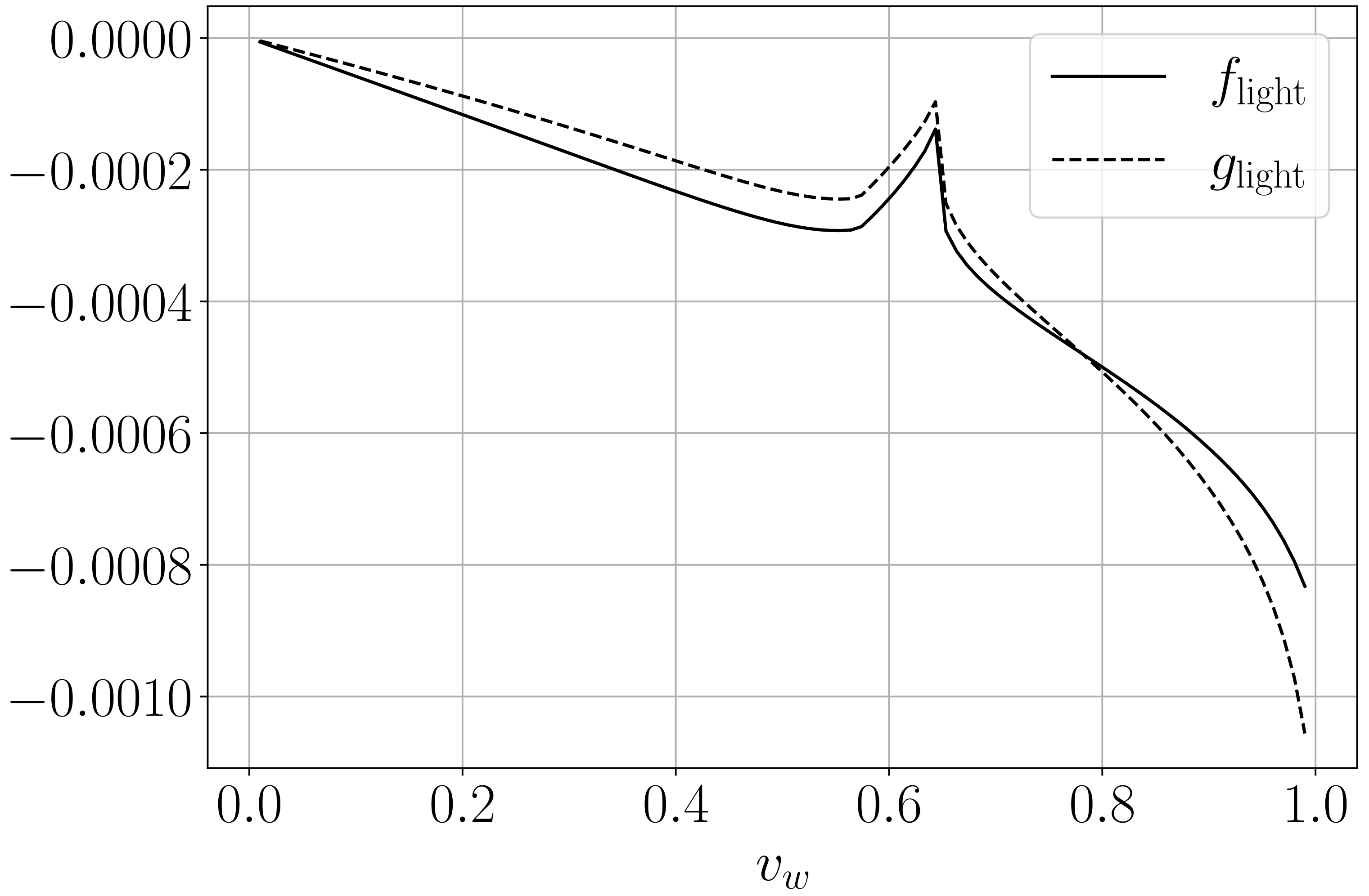
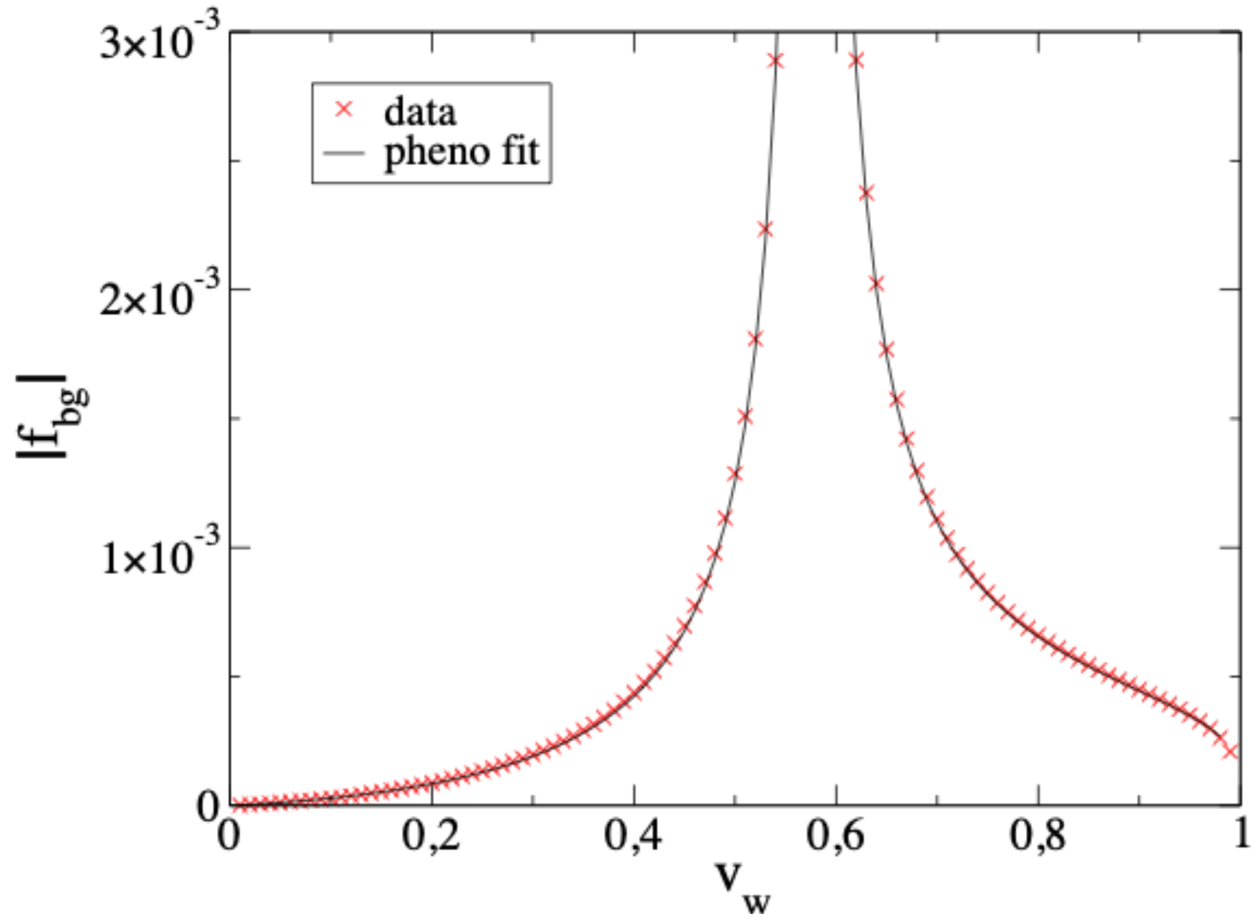
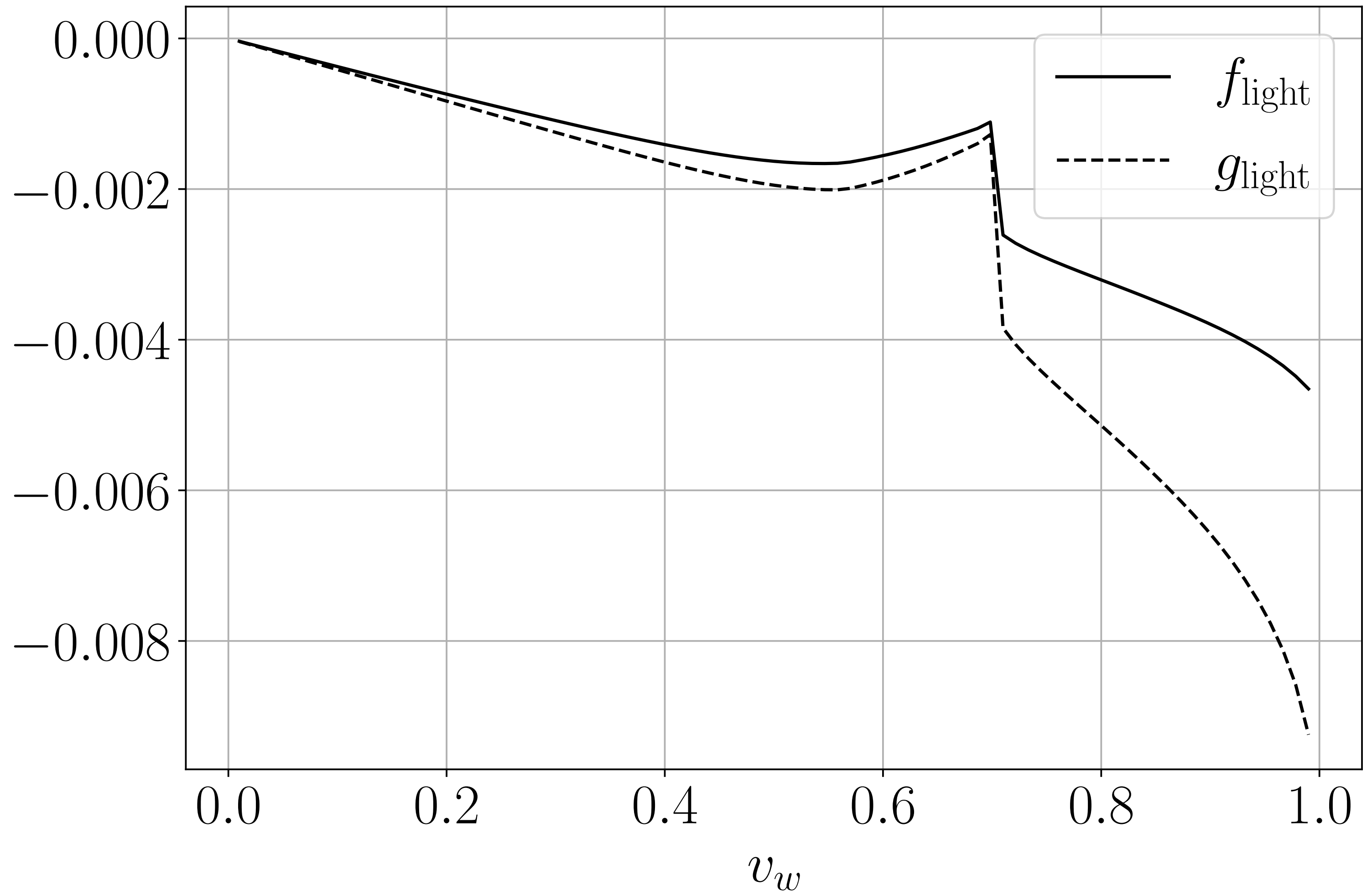


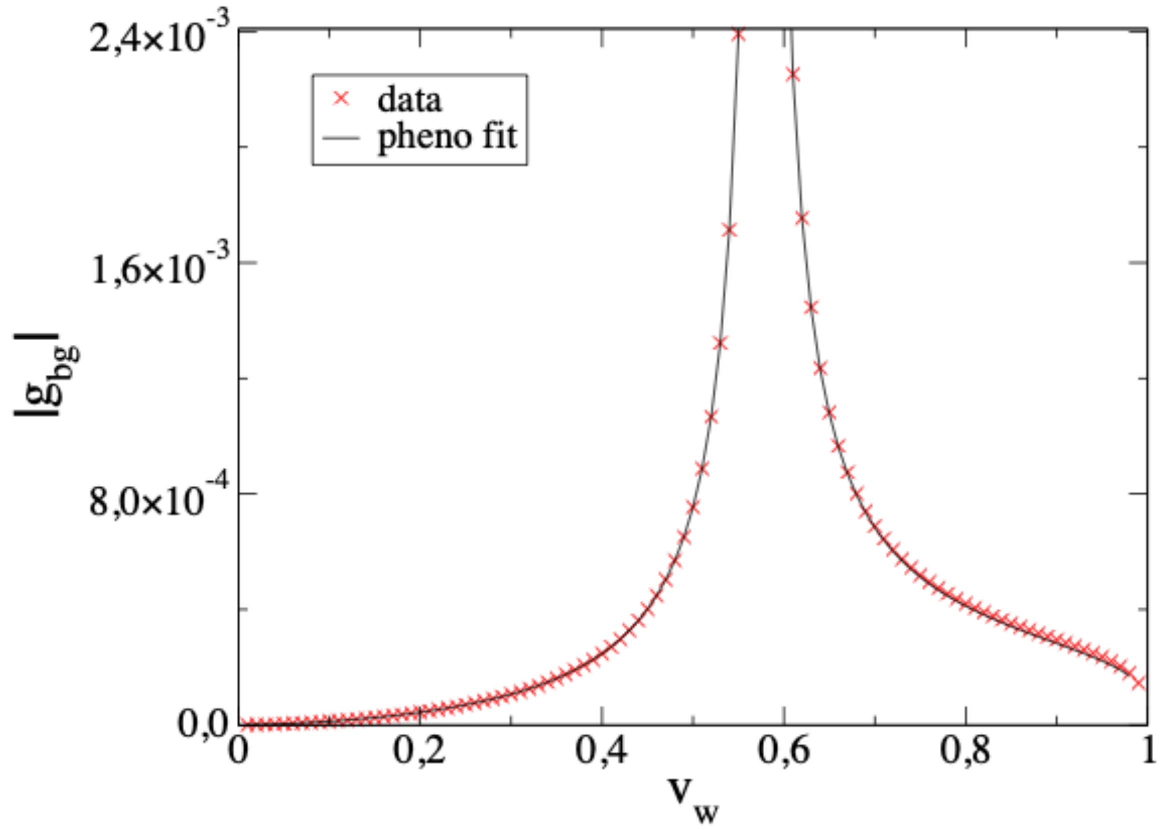
New

Old



For the "light" particles the new source term takes away the divergence!





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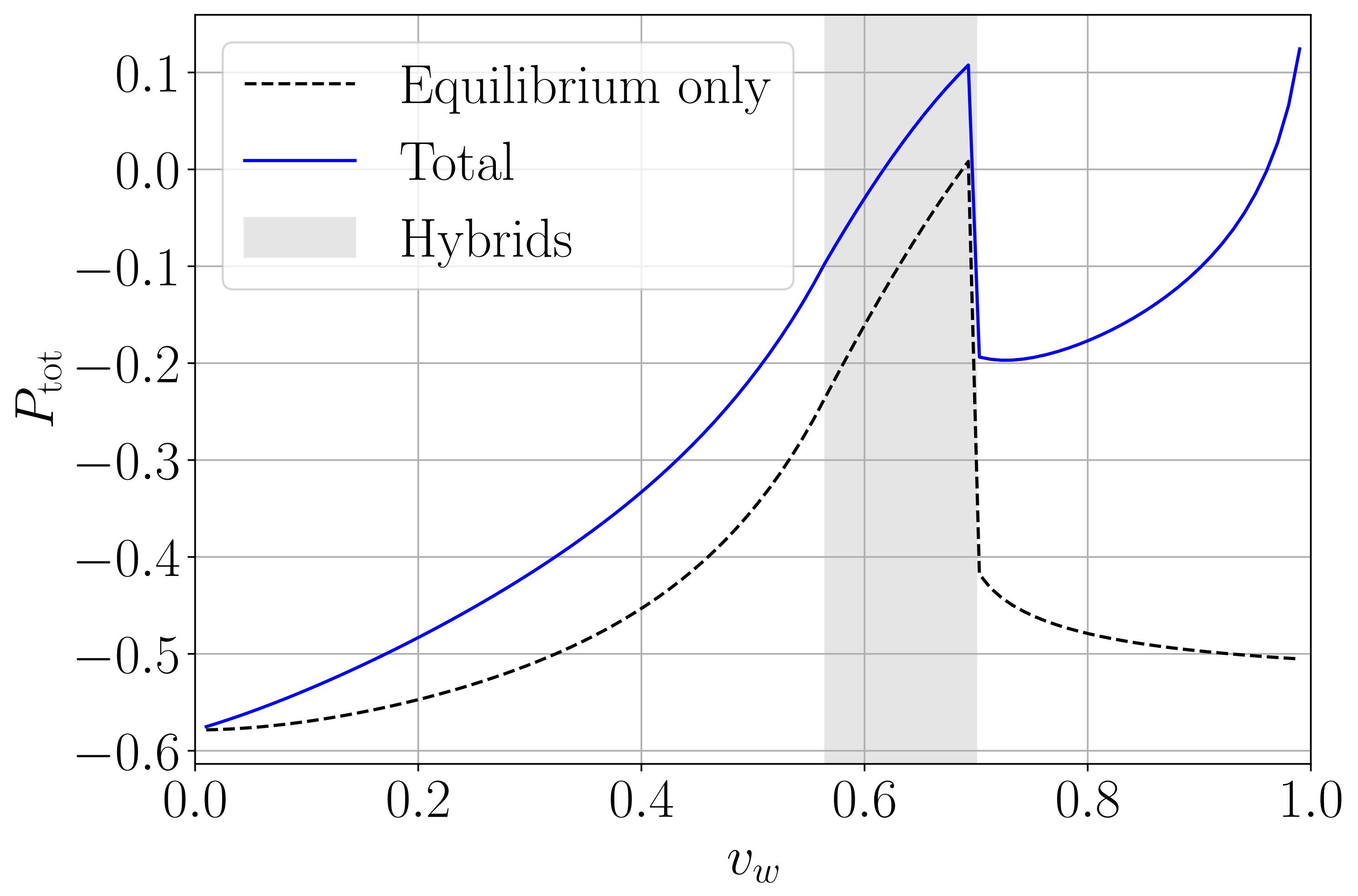
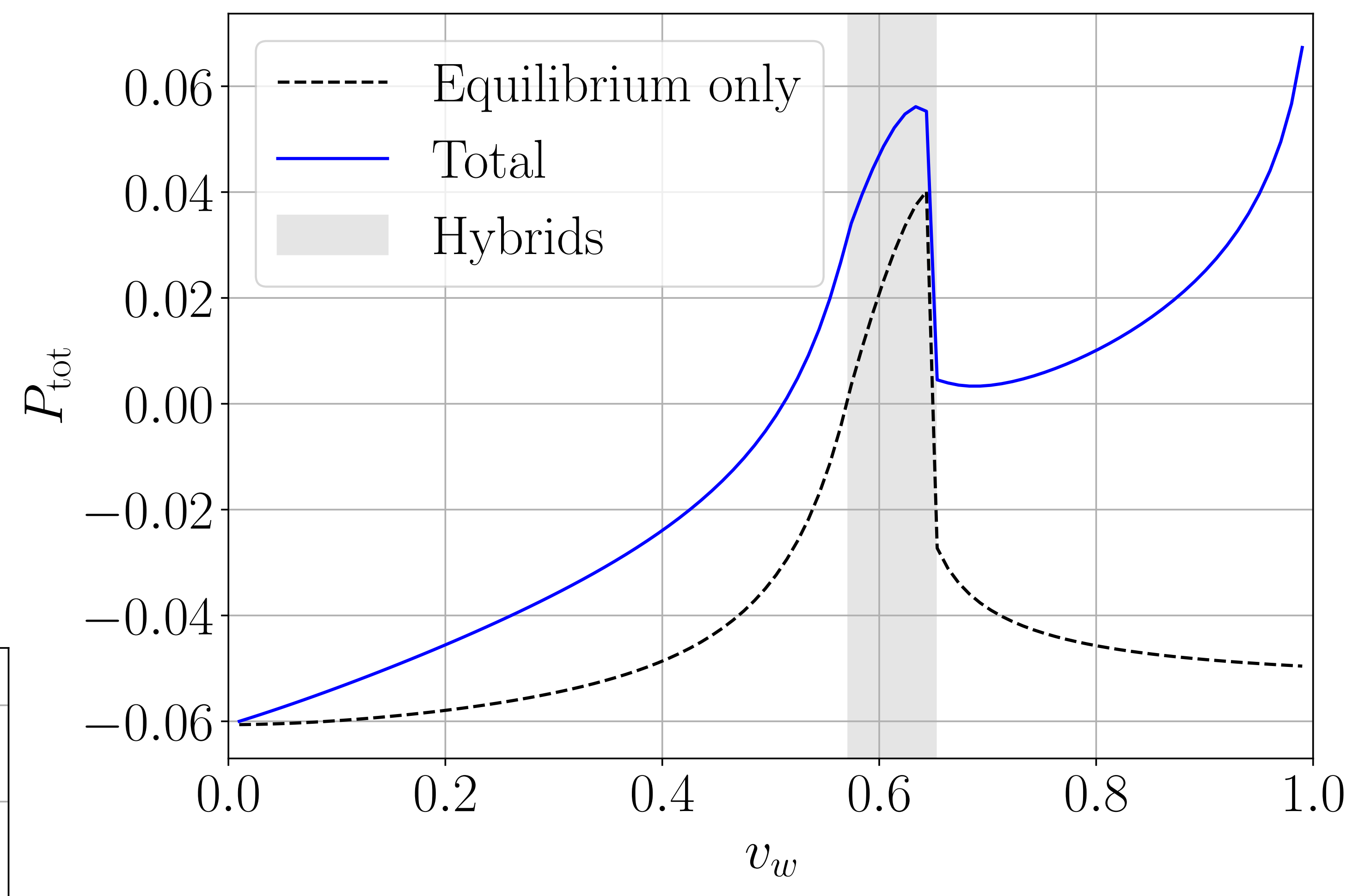
Pressure acting on the expanding wall for $\Lambda = 625$ GeV and $\Lambda = 690$ GeV





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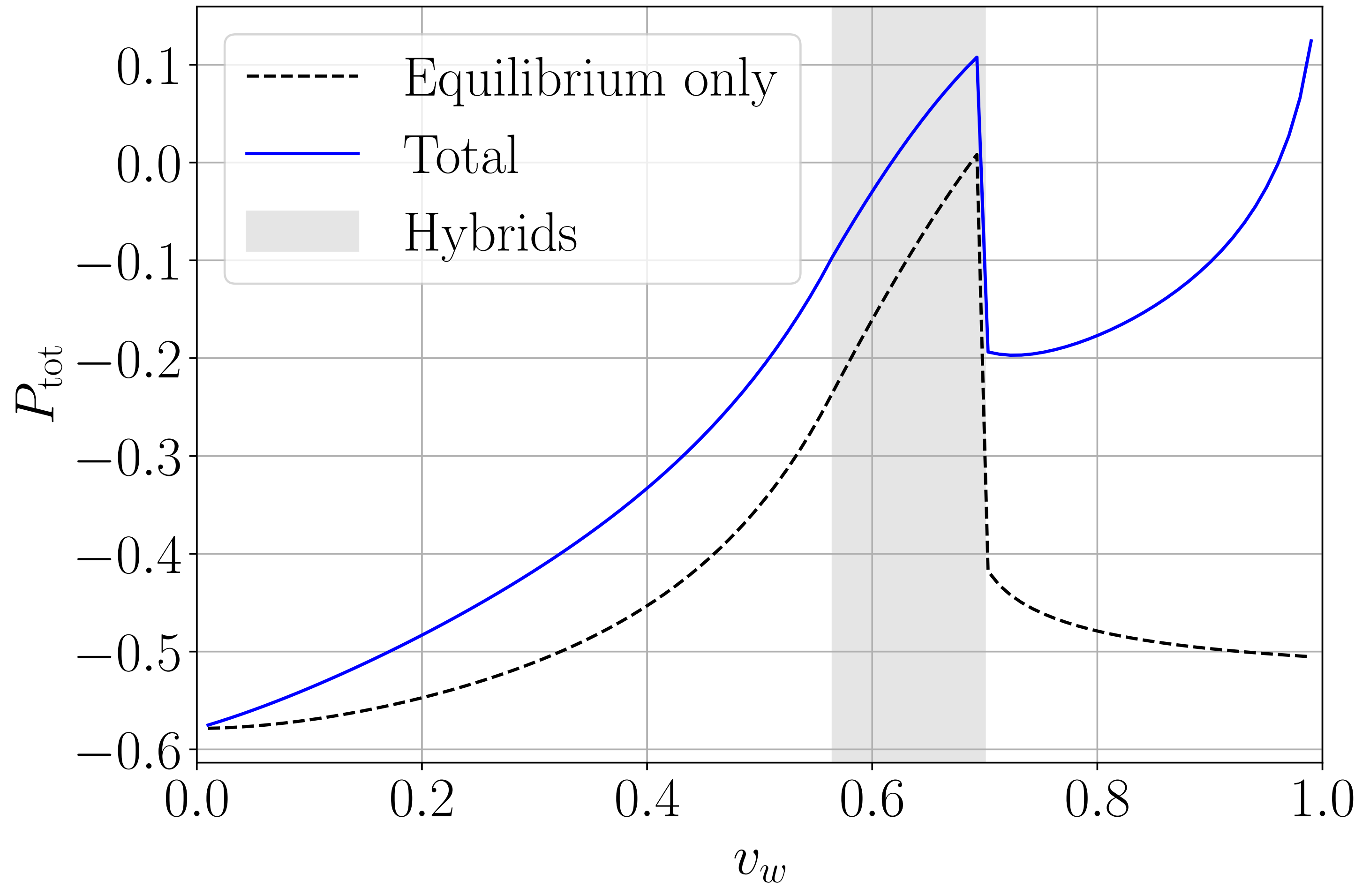
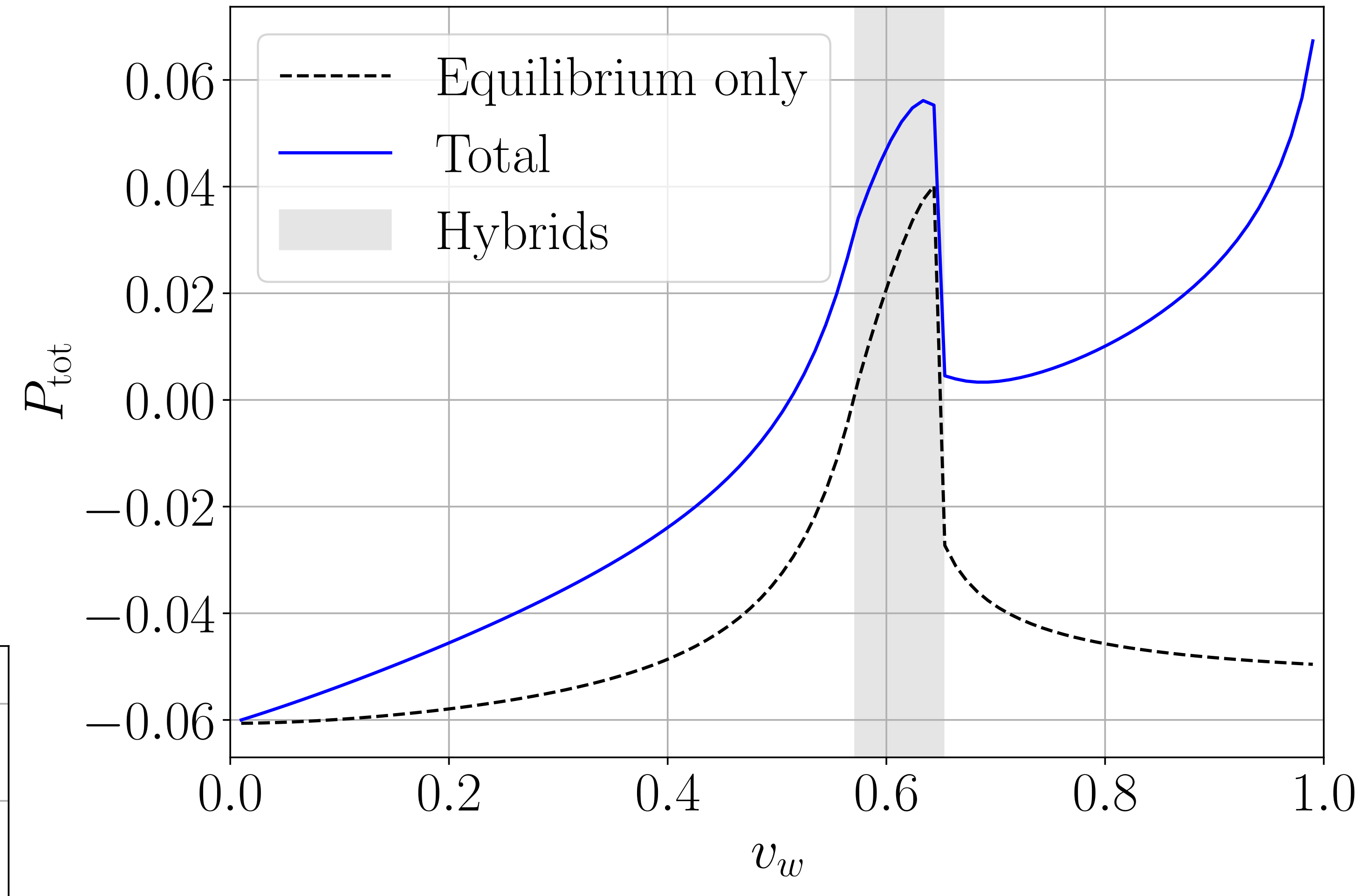
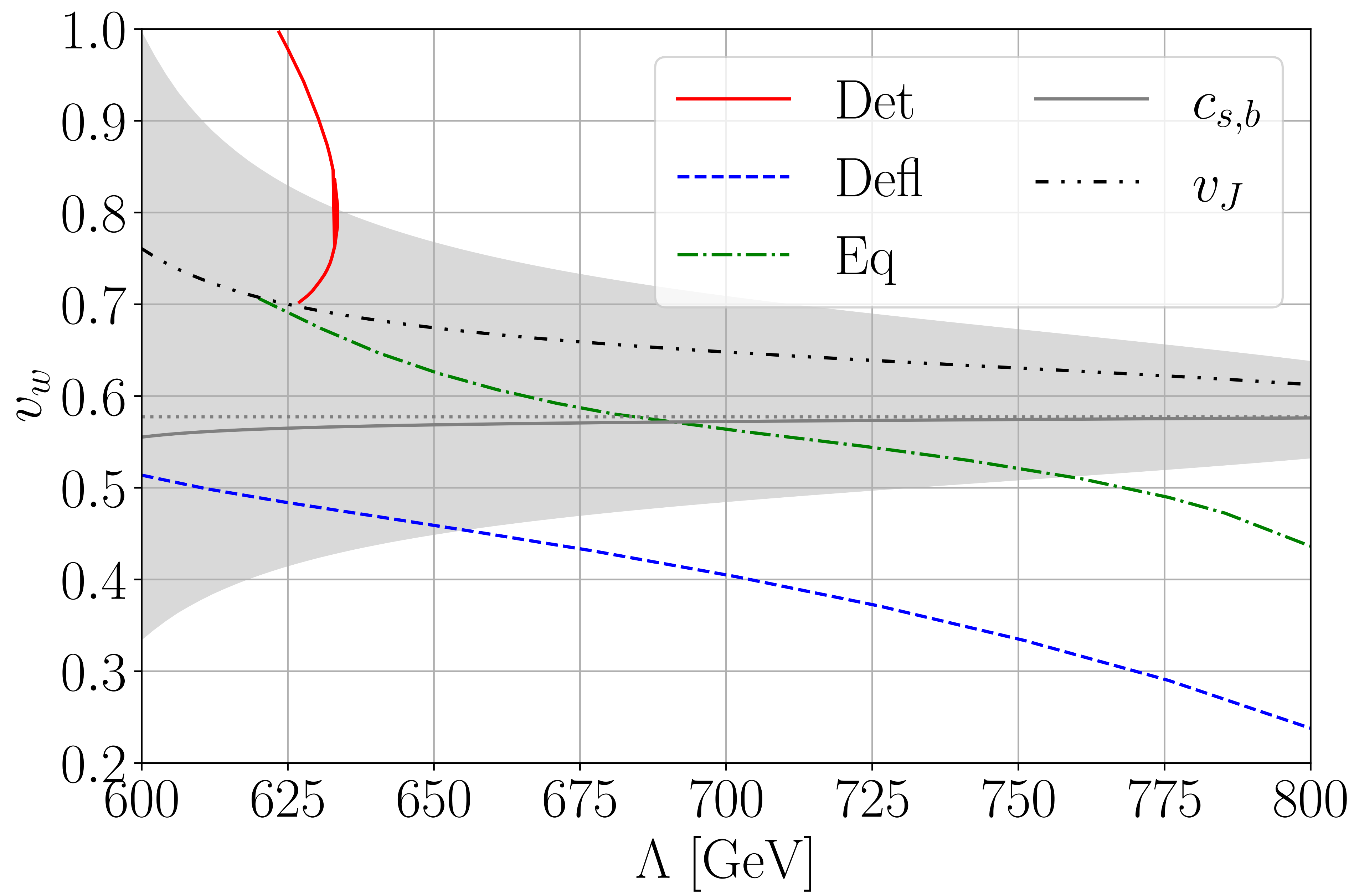




Solutions of $\xi_{\rm w}$ for different Λ



The grey region represents the limit where the old approach is not trustable anymore

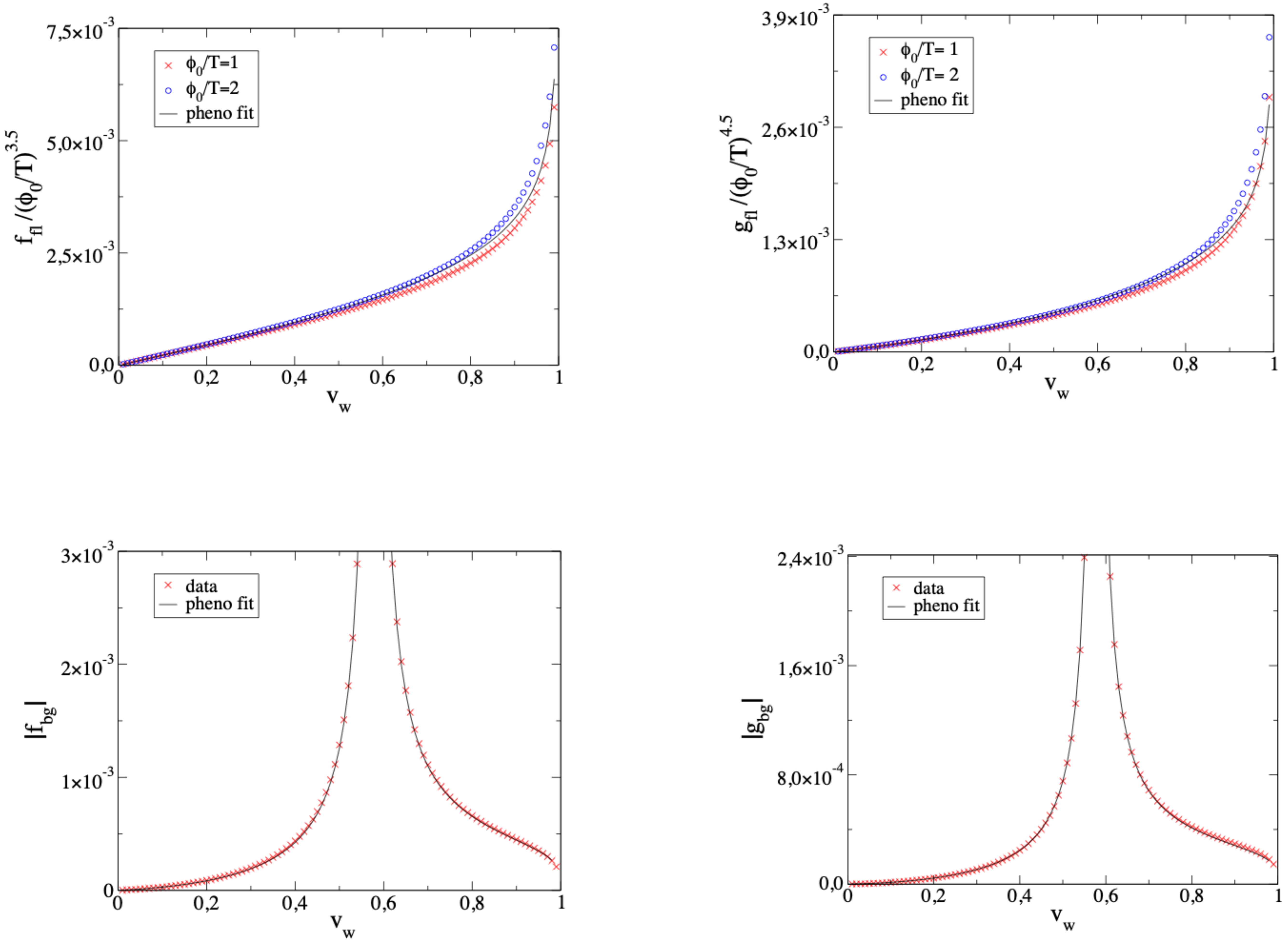
Thank you for your attention!

The origin of the singularity

$$v_+ = \frac{1}{1+\alpha_+}\left[\left(\frac{v_-}{2}+\frac{1}{6v_-}\right) \pm \sqrt{\left(\frac{v_-}{2}+\frac{1}{6v_-}\right)^2 + \alpha_+^2 + \frac{2}{3}\alpha_+ - \frac{1}{3}}\right] ,$$

which is regular if we send $v_- \to c_s$ but become singular if in a second moment we linearise with respect to α_+.

A “sonic boom” in the friction



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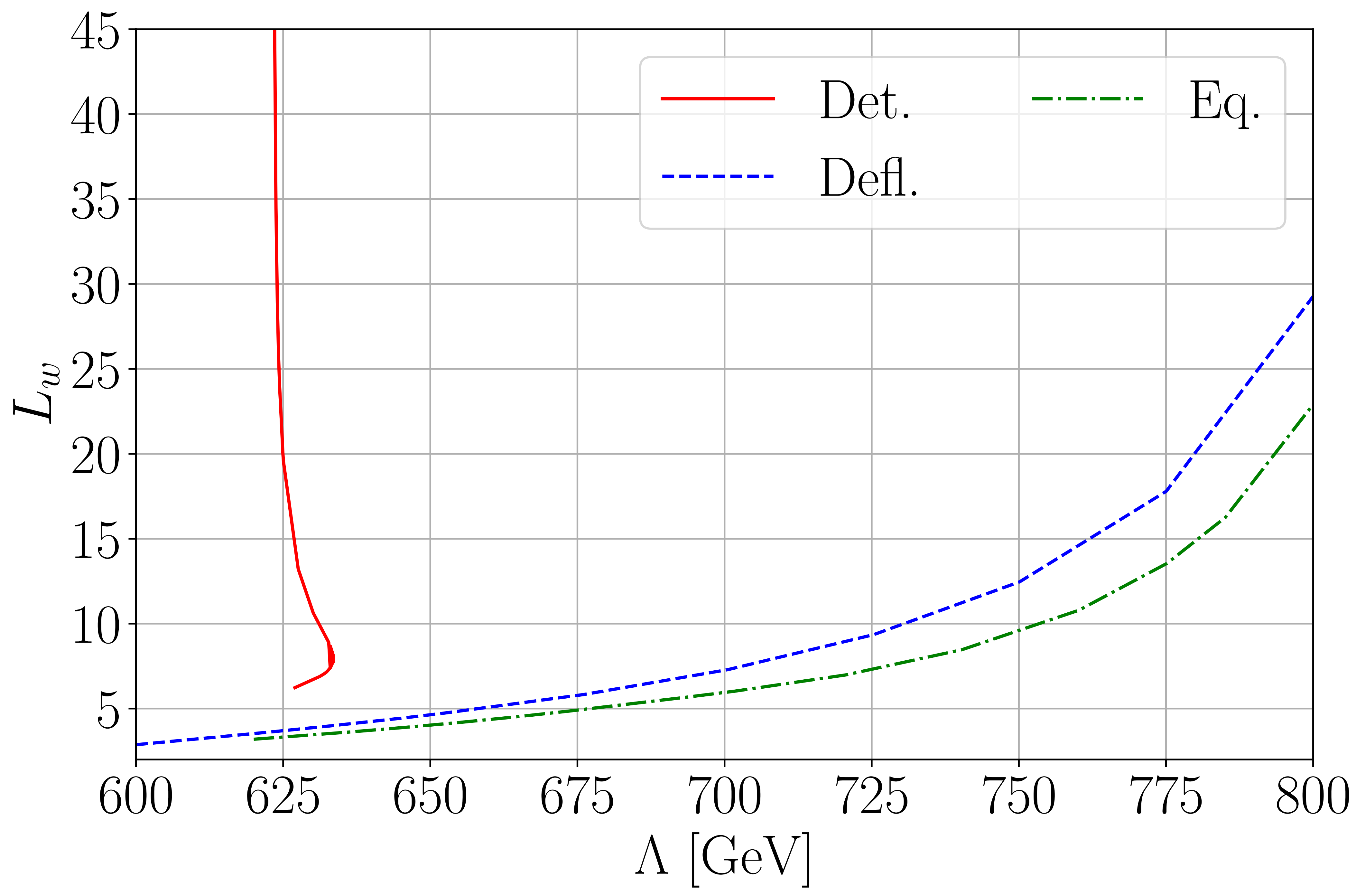
Is it physical?

The solutions for L_w



$$-\phi'' + \frac{\partial \mathcal{F}}{\partial \phi} + \sum_i \frac{dm_i^2}{d\phi} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_i} \delta f_i(p, x) = 0$$

1. $\int dz[\text{l.h.s of KG }] \times \phi' = 0$

$$f_{\text{fl}} \equiv \frac{N_t}{2T_+^2} \int dz\, \frac{dm_t^2}{dz} \left(c_{f1}\delta\mu_f + c_{f2}\delta\tau_f \right) + \frac{N_W}{2T_+^2} \int dz\, \frac{dm_W^2}{dz} \left(c_{b1}\delta\mu_b + c_{b2}\delta\tau_b \right)$$

$$f_{\text{light}} \equiv \frac{N_t}{2T_+^2} \int dz\, \frac{dm_t^2}{dz}\, c_{f2}\, \delta\tau_{\text{light}} + \frac{N_W}{2T_+^2} \int dz\, \frac{dm_W^2}{dz}\, c_{b2}\, \delta\tau_{\text{light}}$$

2. $\int dz[\text{l.h.s of KG}] \times \phi'(2\phi - \phi_0) = 0$