Challenging the lore on the strong CP problem

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The aim:

Challenge the conventional view of the strong CP problem by showing that **path integral** computations with a careful **infinite 4d volume** limit as well as calculations in **canonical quantization** imply that **QCD does not violate CP regardless** of the value of the θ **angle**

The plan:

1. Overview of the strong CP problem
2. The standard picture in the path integral
3. The standard picture in canonical quantization
4. Challenges to the lore
5. A new picture from the path integral
6. A new picture from canonical quantization
7. Conclusions

1. Overview of the strong CP problem

The QCD Lagrangian

$$S_{\text{QCD}} = \int \mathrm{d}^4x \left[-\frac{1}{4g^2} F^{a\,\mu\nu} F^a_{\mu\nu} + \frac{g^2\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} + \sum_{i=1}^{N_f} \overline{\psi}_i \left(i\gamma^\mu D_\mu - m_i e^{i\alpha_i\gamma_5} \right) \psi_i \right].$$

~~CP~~ ~~CP~~

θ-term $\supset \epsilon^{0123} F^a_{01} F^a_{23} \xrightarrow{CP} -\epsilon^{0123} F^a_{01} F^a_{23}$ **CP odd!** Behaves like a **phase**

Complex fermion masses

2 types of **CP-odd phases**: Naively **expect CP violation**

The θ term in the QCD Lagrangian

$$\frac{g^2}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \equiv \frac{g^2}{32\pi^2}F^{\mu\nu\,a}\widetilde{F}^a_{\mu\nu} = \partial_\mu K_\mu \quad K_\mu = \frac{1}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{1}{2}A_\nu\partial_\alpha A_\beta - \frac{\mathrm{i}}{3}A_\nu A_\alpha A_\beta\right].$$

θ-term is a total derivative and thus corresponds to a **boundary term**

→ it can never contribute in perturbation theory:

(Feynman rule $\propto \sum_i p_i = 0$)

▶ Possible effects of θ are **nonperturbative**

The neutron dipole moment

With **two types** of **CP-odd phases**, one **expects CP violation** in the strong interactions coming from **nonperturbative effects**.

This would lead to a **neutron electric dipole moment**, the **usual expectation** being

$$|d_n| \propto 10^{-16}(\theta + \alpha_u + \alpha_d + \alpha_s)\ e \cdot cm \equiv 10^{-16}\bar{\theta}\ e \cdot cm$$

However experiments constrain [neDM]

$$|d_n| < 1.8 \times 10^{-26} e \cdot cm \longrightarrow |\bar{\theta}| < 10^{-10}$$

while "naturally" one would expect $\bar{\theta} \sim \mathcal{O}(1)$ ⟶ **strong CP problem**

Proposed **solutions** involve new physics, e.g. **QCD axion**

The philosophy behind this talk

To ensure that we are looking for **new physics** in the best possible places, it is healthy to **cross-check / question the theoretical foundations** that motivate it

According to **our results**, there is **no strong CP problem**, and no need to have an axion to explain the neutron dipole moment. But axion-like-particles can still exist as generic remnants of symmetry breaking

2. The standard picture in the path integral

Nonperturbative effects from instantons

We saw that θ can only enter through **nonperturbative effects.**

These come from contributions to the **Euclidean path integral** around **nontrivial saddle points**

$$Z_E = \int \mathcal{D}\phi_i e^{-S_E}, \quad \phi_i = A_\mu, \Psi_k,$$

Usual perturbation theory

$$A_\mu = 0 + \delta A_\mu, \quad \delta A_\mu \ll 1$$

Instanton perturbation theory:

nonperturbative

$$A_\mu(x) = A_\mu^{\text{saddle}}(x) + \delta A_\mu(x), \quad \delta A_\mu \ll 1 \qquad e^{-S_{\text{saddle}}} \sim e^{-1/g^2}$$

Nonperturbative ‘t Hooft vertices in QCD

With these techniques [‘t Hooft] computed **fermion correlators** accounting for **fluctuations around instantons**

He deduced an **effective Lagrangian** whose tree-level correlators match nonperturbative ones

$$\mathcal{L}^{\rm QCD}_{\rm eff,'t\,Hooft} \sim e^{-i\theta}\prod_{i=1}^{N_f}\bar\psi_i P_R \psi_i + e^{i\theta}\prod_{i=1}^{N_f}\bar\psi_i P_L \psi_i$$

According to [‘t Hooft] : **phases misaligned with fermion masses: CP violation**

To link θ to observables, this can be **matched** to a **low-energy theory** that **includes** relevant d.o.f.s such as **the neutron**

3. The standard picture in the canonical formalism

Canonical quantization a la Jackiw

Following [Jackiw '80] stationary states $|\Psi^{(a)}\rangle$ are described by **wave functionals**

$$\Psi^{(a)}[A_i] = \langle A_i|\Psi^{(a)}\rangle$$

Eigenstate of field operator $\hat{A}_i$

The $|\Psi^{(a)}\rangle$ satisfy **Schrödinger's equation**

$$H\Psi^{(a)}[A] = E^{(a)}\Psi^{(a)}[A]$$

We quantize in the gauge $A_0 = 0$, which still allows gauge transformations $U(\mathbf{x})$

$$[A^{i,a}(\mathbf{x}), \Pi^{j,b}(\mathbf{x})] = \mathrm{i}\delta^{ij}\delta^{ab}\delta^3(\mathbf{x}-\mathbf{y}) \Rightarrow \mathbf{\Pi}^a = \frac{\delta}{\mathrm{i}\delta\mathbf{A}^a}\,.$$

$$\mathcal{H} = \frac{1}{2}\left[(\mathbf{E}^a)^2 + (\mathbf{B}^a)^2\right] = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta\mathbf{A}^a} - \frac{g^2}{8\pi^2}\theta\mathbf{B}^a\right)^2 + (\mathbf{B}^a)^2\right]\,.$$

The Chern-Simons Number Operator

With an **appropriate gauge choice**, one can show that the θ−dependent part of S is

$$S_\theta = \theta(W[\mathbf{A}_f] - W[\mathbf{A}_i]) \qquad W[\mathbf{A}] = \frac{1}{4\pi^2}\varepsilon_{ijk}\int_S \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}A_i\partial_j A_k - \frac{\mathrm{i}}{3}A_iA_jA_k\right]$$

Considering **gauge transformations** $\mathbf{A} \to \mathbf{A}_U$, it turns out that if one **imposes** [Jackiw]

$$U(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 1$$

then transformations fall into **equivalence classes** $U^{(n)}, n \in \mathbb{N}$ such that

$$W[\mathbf{A}_{U^{(n)}}] = W[\mathbf{A}] + n$$

As **gauge transformations do not change the physics,** one demands that they act as a **rephasing**

$$U^{(n)}|\Psi^{(a)}\rangle = e^{in\tilde{\theta}}|\Psi^{(a)}\rangle$$

Canonical quantization a la Jackiw

One can perform a **change of basis** which **gets rid of θ in the Hamiltonian**

$$\Psi[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\Psi'[\mathbf{A}]$$

$$\mathcal{H}' = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta \mathbf{A}^a}\right)^2 + (\mathbf{B}^a)^2\right].$$

In the new basis the **rephasing** under gauge transformations is **not cancelled**:

$$U^{(n)}|\Psi'^{(a)}\rangle = e^{in(\tilde{\theta}-\theta)}|\Psi'^{(a)}\rangle$$

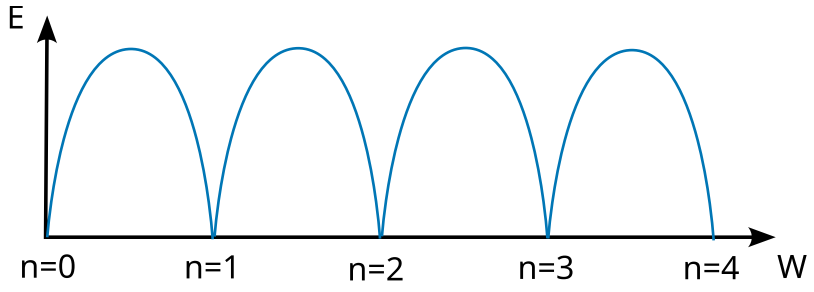
One **cannot get rid** of the **combination of phases** $\tilde{\theta} - \theta$ ▶ Can't exclude CP violation

The θ-vacua

The **classical vacua** are given by **pure gauge configurations** with **integer CS number**

$$A_\mu^{0,(m)} = \frac{i}{g} U^{(m)}(x) \partial_\mu U^{mn)}(x),$$

$$W[A_\mu^{0,(m)}] = n$$



We consider **states** $|m\rangle$ which are the quantum analogue of the classical vacua:

$$\hat{W}|m\rangle = m|m\rangle \qquad U^{(n)}|m\rangle = |m+n\rangle$$

Then one can get **states** satisfying **Jackiw's gauge invariance**

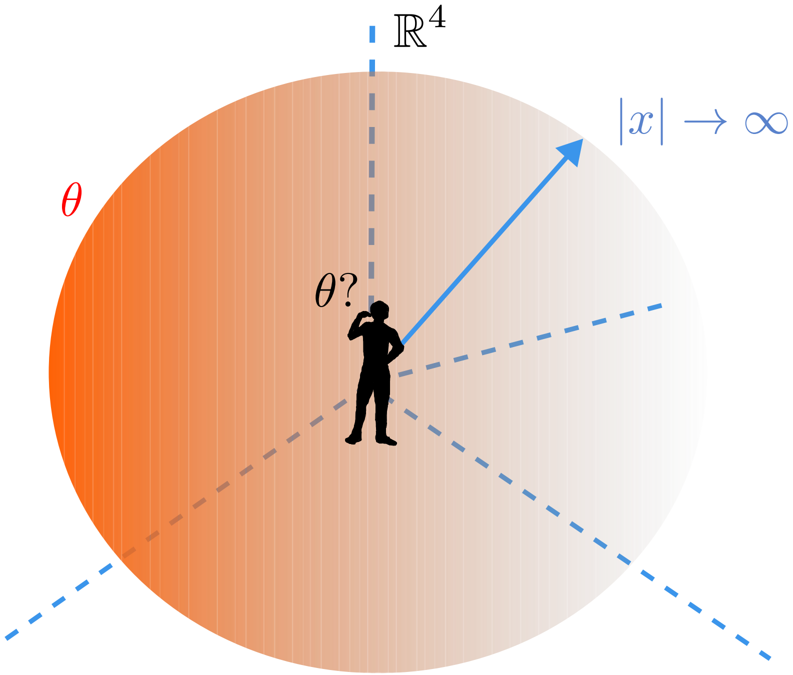
$$|\tilde{\theta}\rangle = \sum_m e^{i\tilde{\theta} m}|m\rangle = \sum_m e^{i\tilde{\theta}\hat{W}}|m\rangle \qquad U^{(n)}|\tilde{\theta}\rangle = e^{in\tilde{\theta}}|\tilde{\theta}\rangle$$

θ-vacua

4. The cracks in the picture

Challenges to the lore

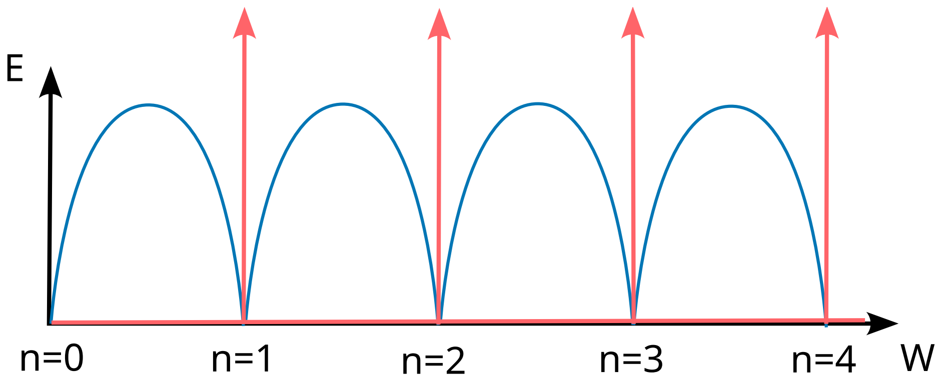
With S_θ being a **boundary term**, how come one gets physical effects when sending the boundary to infinity? Isn't this **against locality**?



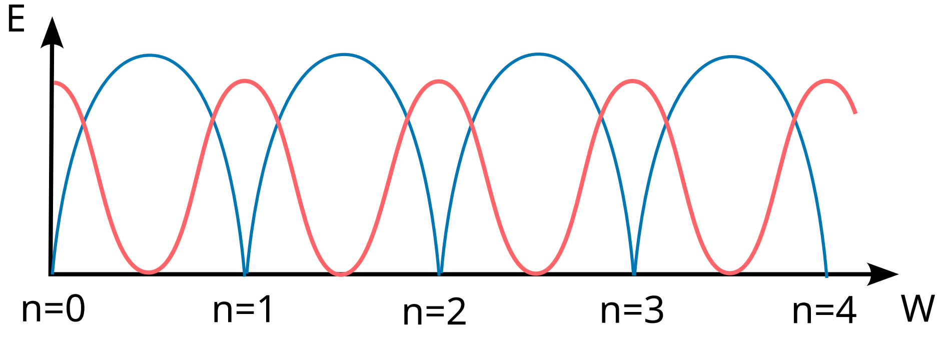
Challenges to the lore

The ***θ*-vacua** only have **support** on **classical field configs** with **integer W**

But in QM, wave functions do not vanish outside classical minima!



Theta vacua



QM expectation

Challenges to the lore

The θ**-vacua are not normalizable**, **contradicting postulates of QM** [Okubo & Marshak]

$$\langle\hat{\theta}|\hat{\theta}'\rangle = \sum_{mn} e^{-i\hat{\theta}m+i\hat{\theta}'n}\langle m|n\rangle = \sum_{n} e^{-in(\hat{\theta}-\hat{\theta}')} = \delta(\hat{\theta}-\hat{\theta}')$$

While it is widely believed that the θ-vacua are invariant (up to a phase) under all gauge transformations, this is not true!

Gauge invariance only applies to spatial gauge transformations with

$$U(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 1$$

If one demands normalizability or treats all gauge transformations as a redundancy, the θ-vacua are unacceptable

Our work

We have **three types of calculations** supporting **CP conservation in QCD**:

Using **path integral methods**:

- We have **recomputed Green's functions in the dilute instanton gas**, in Euclidean and Minkowski spacetime, and found **no CPV**

  This talk

- We also have a UV **computation** of fermion correlators **which does not rely on instantons**, yielding the same conclusion

Using **canonical quantization**, we have rederived how θ drops out of observables and P is conserved

This talk

5. A new picture from the path integral

Towards correlators: vacuum path integral

Ordinary path integrals are not **vacuum transition amplitudes**:

$$\int_{\phi_i,\phi_f,T} \mathcal{D}\phi\, e^{\mathrm{i}S_T} = \langle \phi_f | e^{-\mathrm{i}HT} | \phi_i \rangle = \sum_n e^{-\mathrm{i}E_n T} \langle \phi_f | n \rangle \langle n | \phi_i \rangle$$

To get the **vacuum transition amplitude** it is necessary to take the **infinite** ***T*** **limit,**

$$Z = \lim_{T\to\infty e^{-i0+}} \int_T \mathcal{D}\phi\, e^{\mathrm{i}S_T} \sim \lim_{T\to\infty e^{-i0+}} \langle 0 | e^{-\mathrm{i}HT} | 0 \rangle$$

B.c.s. arbitrary!

To recover the vacuum amplitude for **finite** ***T***, one **needs to know the wave functional of the vacuum**

$$\langle 0 | e^{-iHT} | 0 \rangle = \int [\mathcal{D}\phi_f]_{T/2} [\mathcal{D}\phi_i]_{-T/2} \langle 0 | \phi_f \rangle \langle \phi_i | 0 \rangle \int_{\phi_i,\phi_f,T} \mathcal{D}\phi\, e^{\mathrm{i}S}$$

B.c.s fixed by wave functional, need additional reweighting

Wrap-up: the importance of boundary conditions

- To ensure projection into vacuum, we first use the Euclidean **path integral for infinite** ***VT*****,** without the need to enforce particular b.c.s

- Later we will use **canonical quantization** to determine the θ dependence of the **wave functional**

Euclidean saddles and topological charge

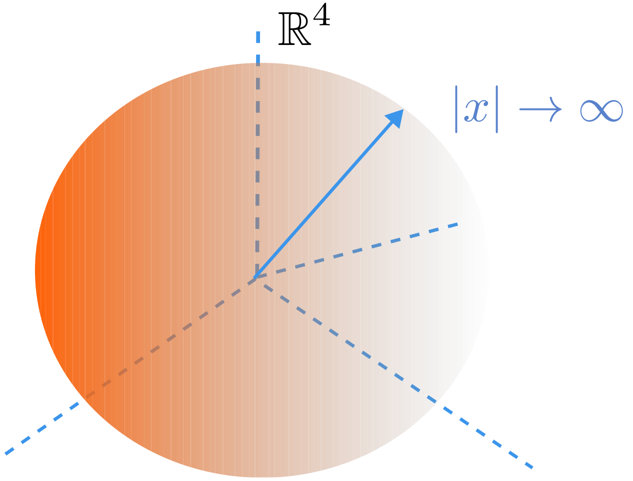
Only saddles with **finite action** contribute

$S_E = \int d^4x \mathcal{L}_E$ finite ➡ $\mathcal{L}_E \to 0, |x| \to \infty$ ➡ **fields at ∞ gauge equivalent to 0!**

This leads to a **mapping** between the **sphere at infinity** $|x| \to \infty$ and the **gauge group**

$$A_\mu \to \frac{i}{g} U(x)^\dagger \partial_\mu U(x)$$

$$U(x) \in \mathrm{SU}(3)$$

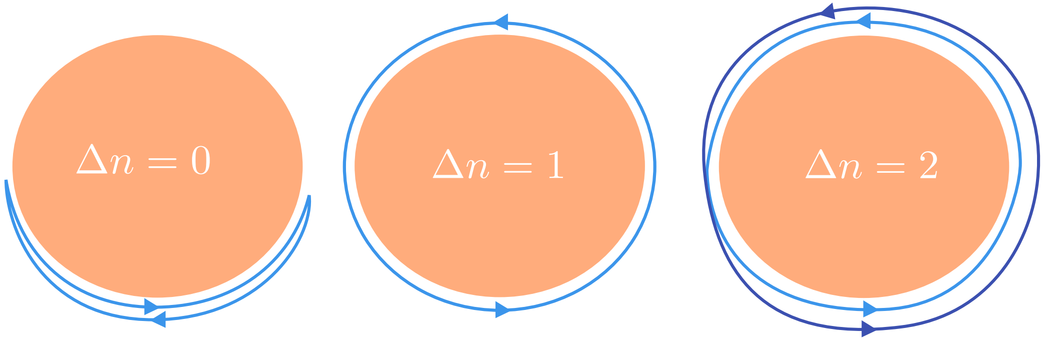


Why integer topological charge?

These **mappings** are **characterized by integers** **(topological charge Δn)** counting how many times the sphere is wrapped around the group. Furthermore, it turns out that

$$S_\theta = i\theta\Delta n$$

1D analogy: Wrapping a rubber band around a cylinder



> **Remember: in QCD, we only get topological sectors in infinite 4D volume, from requiring finite action!**

Strategy to compute correlators

Local fermionic Green functions are obtained by:

Calculating **fermion propagator** for a given saddle

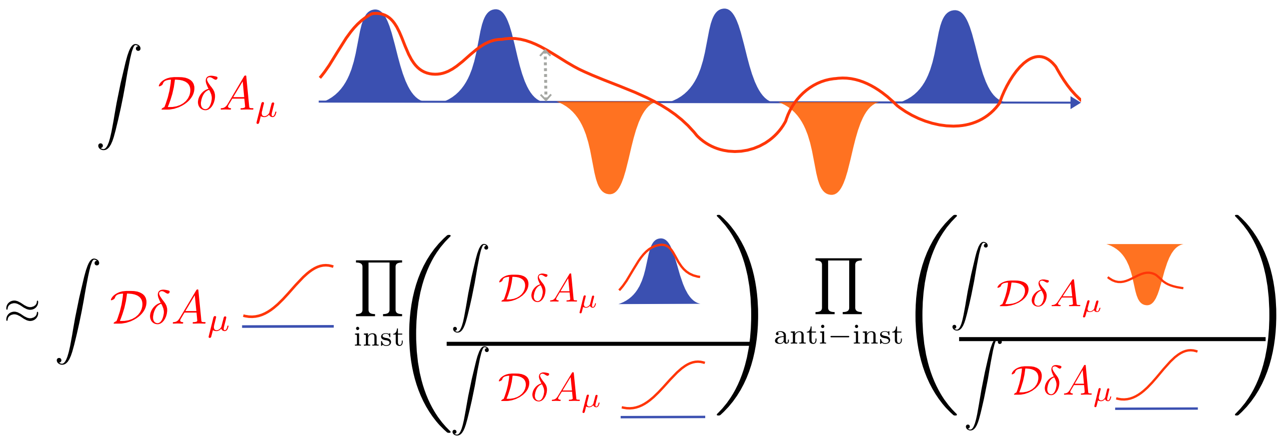
Determining **fluctuation determinants** for a given saddle

Summing over all saddle points in **all** Δn sectors

$$Z\langle\psi(x)\bar{\psi}(y)\rangle \approx \sum_{\Delta n} \sum_{\substack{\text{saddle d.o.f} \\ \Delta n \text{ fixed}}} e^{i\Delta n\theta} e^{-S_{E,\text{saddle}}} \Pi_f (\text{Fermionic det})|_{\text{saddle}} (\text{Gauge det})^{-1/2}|_{\text{saddle}} (\text{Propagator(x,y)})$$

The key points of our calculation

Integration of fluctuations **factorizes** in a **standard way** [Callan, Coleman]



We use **standard approximations** in the literature for the **fermion propagators** in instanton backgrounds [Diakonov]

Crucially, we make use of the fact that the **classification into topological sectors** with integer Δn **only applies in infinite volume**

Results ($N_f=1$)

$$\langle\psi(x)\bar{\psi}(x')\rangle = \lim_{\substack{N\to\infty \\ N\in N}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\langle\langle\psi(x)\bar{\psi}(x')\rangle\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\tilde{Z}_{\Delta n}} = S^E_{0\text{inst}}(x,x') + \kappa\bar{h}(x,x')m^{-1}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}.$$

Free propagator

$\propto$ Projector into fermion zero modes for $\Delta n=1$

Topological classification only **enforced in infinite volume**, which **fixes ordering**

$$S^E_{0\text{inst}}(x,x') = (-\not{D} + me^{-\mathrm{i}\alpha\gamma_5})\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-x')}}{p^2+m^2}$$

Alignment between perturbative and non-perturbative phases: **No CP violation**

Results with the usual order of limits

The alternative order of limits gives standard results:

$$\lim_{VT\to\infty}\lim_{\substack{N\to\infty\\ N\in N}}\frac{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\langle\langle\psi(x)\bar{\psi}(x')\rangle\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\tilde{Z}_{\Delta n}} = S^{E}_{0\mathrm{inst}}(x,x') - \kappa\bar{h}(x,x')m^{-1}\mathrm{e}^{\mathrm{i}\theta\gamma_5}$$

Misaligned phases: CPV

The **order of limits** is the **only point of departure** from **standard results**

6. A new picture in canonical quantization

Goals

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}A_f]_{T/2}[\mathcal{D}A_i]_{-T/2}\langle 0|A_f\rangle\langle A_i|0\rangle \int_{A_i,A_f,T} \left(\prod \mathcal{D}A\right) e^{\mathrm{i}S}$$

- Understand the **θ-dependence of wave functionals**
- Fix **θ-dependence of correlators without infinite volume limit**
- **Show cancellation of θ-dependence** to confirm $VT \to \infty$ results

For simplicity we focus on the case of a **pure gauge theory**

Going beyond Jackiw's picture

As we saw, according to Jackiw one can to to a **basis of states** $\psi'[\mathbf{A}]$ in which

$$\mathcal{H}' = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta\mathbf{A}^a}\right)^2 + (\mathbf{B}^a)^2\right].$$

$$U^{(n)}|\Psi'^{(a)}\rangle = e^{in(\tilde{\theta}-\theta)}|\Psi'^{(a)}\rangle$$

i.e., **CP-odd phases** only enter in the rephasing of states under gauge transf.

Let's reconsider the action of gauge transformations from the start and challenge the 2nd equation above. We want **physics to be invariant under all gauge transformations**

$$U(\mathbf{x})|\Psi'^{(a)}\rangle = e^{i\alpha(U(\mathbf{x}))}|\Psi'^{(a)}\rangle$$

Going beyond Jackiw's picture

Compatibility with the **group structure** leads to

$$U_2(\mathbf{x})U_1(\mathbf{x})|\Psi'^{(a)}\rangle = e^{i\alpha(U_2(\mathbf{x}))+\alpha(U_1(\mathbf{x}))}|\Psi'^{(a)}\rangle \equiv e^{i\alpha(U_2(\mathbf{x})U_1(\mathbf{x}))}|\Psi'^{(a)}\rangle$$

i.e. the $\alpha(U)$ furnish a **one-dimensional representation of the gauge group**.

QCD is based on **SU(3)**, which is a **connected simple group**. For such groups, it is known that the **only possible representation is the trivial one**!

$$\alpha(U) = 0$$

Going beyond Jackiw's picture

Hence, **demanding** that **all gauge transformations** $U(\mathbf{x})$ lead to **rephasings**, one is led to basis of states in which

$$\mathcal{H}' = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta \mathbf{A}^a}\right)^2 + (\mathbf{B}^a)^2\right] .$$

$$U(\mathbf{x})|\Psi'^{(a)}\rangle = |\Psi'^{(a)}\rangle$$

In this basis there are **no CP-odd phases at all! No CP violation**

Only difference w.r.t. **Jackiw's treatment** is **lifting the restriction** to $U(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 1$

Is there a good reason for restricting $U(\mathrm{x})$?

[Jackiw, 1980]

We shall make a very important hypothesis concerning the physically admissible finite transformations. While some plausible arguments can be given in support of this hypothesis (see below) in the end we must recognize it as an assumption, without which the subsequent development cannot be made. We shall assume that the allowed gauge transformation matrices U tend to a definite limit as r passes to infinity.

$$\lim_{r \to \infty} U(\mathbf{r}) = U_\infty \, . \qquad (36b)$$

Is there a good reason for restricting $U(\mathbf{x})$?

As we used **ordinary quantization without constraints related to the $U(\mathbf{x})$**, we have to consider all of these transformations

Any $U(\mathbf{x})$ applied on a quark state gives another quark with the same mass, spin, etc. But we see a **finite number of quarks!**

- We should **allow all gauge transformations** and **treat them as redundancies**

(First quantize, then constraint)

- With the trivial rephasing under $U(\mathbf{x})$, one can define **normalizable states** under **inner product** on **gauge fixed surfaces** that respects **hermiticity** of H

Back to the basis with θ in the Hamiltonian

$$\Psi[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\Psi'[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\tilde{\Psi}_{\mathrm{g.i}}[\mathbf{A}]$$

Evolve with CP-even H'

No ~~CP~~ phase in b.c.'s

$$\mathrm{e}^{-\mathrm{i}\theta W[\mathbf{A}_f]}\tilde{\Psi}_{0,\mathrm{g.i.}}[\mathbf{A}_f] \qquad \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}_i]}\Psi_{0,\mathrm{g.i.}}[\mathbf{A}_i] \qquad \mathrm{e}^{\mathrm{i}\theta(W[\mathbf{A}_f]-W[\mathbf{A}_i])}$$

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\mathbf{A}_f]_{T/2}[\mathcal{D}\mathbf{A}_i]_{-T/2}\Psi_0[\mathbf{A}_f]^*\Psi_0[\mathbf{A}_i]\int_{\mathbf{A}_i,\mathbf{A}_f,T}\mathcal{D}\phi\, e^{\mathrm{i}S+\mathrm{i}S_\theta}$$

$$= \int [\mathcal{D}\mathbf{A}_f]_{T/2}[\mathcal{D}\mathbf{A}_i]_{-T/2}\tilde{\Psi}_{0,\mathrm{g.i.}}[\mathbf{A}_f]^*\tilde{\Psi}_{0,\mathrm{g.i.}}[\mathbf{A}_i]\int_{\mathbf{A}_i,\mathbf{A}_f,T}\mathcal{D}\phi\, e^{\mathrm{i}S}$$

θ **disappears** from the **partition function** → **CP conservation**

FAQs

—for which we have answers—

Chiral Lagrangian and the neutron dipole moment

[Crewther et al]'s calculation of $n_{\rm EDM}$

The η' mass

[Albandea et al]'s calculation of the topological susceptibility in quantum rotor

[Shifman et al]'s general conclusions about CP violation in QCD

7. Conclusions

A consistent picture

Infinite *T* method (no wave functionals needed)

$$Z = \lim_{T\to\infty e^{-i0+}} \int_T \mathcal{D}\phi\, e^{\mathrm{i}S_T} \sim \lim_{T\to\infty e^{-i0+}} \langle 0|e^{-\mathrm{i}HT}|0\rangle$$

- θ-dependence disappears when taking $VT\to\infty$ before summing over Δn, as required by consistency with integer Δn

Wave functional method for finite *T*

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_i|0\rangle \int_{\phi_i,\phi_f,T} \mathcal{D}\phi\, e^{\mathrm{i}S}$$

- θ-dependence cancels between wave functionals and S_θ.
- Behaviour of states under $U(\mathbf{x})$ compatible with finite norm

Where do we depart from standard results?

- **We only enforce topological quantization when it is mathematically necessary**, i.e. for

$$VT \to \infty$$

- **We allow all gauge transformations**, lifting the restriction to

$$U(\mathbf{x}) \underset{|\mathbf{x}| \to \infty}{\longrightarrow} 1$$

and demand that all $U(\mathbf{x})$ just give rephasings

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Thank you!

Additional material

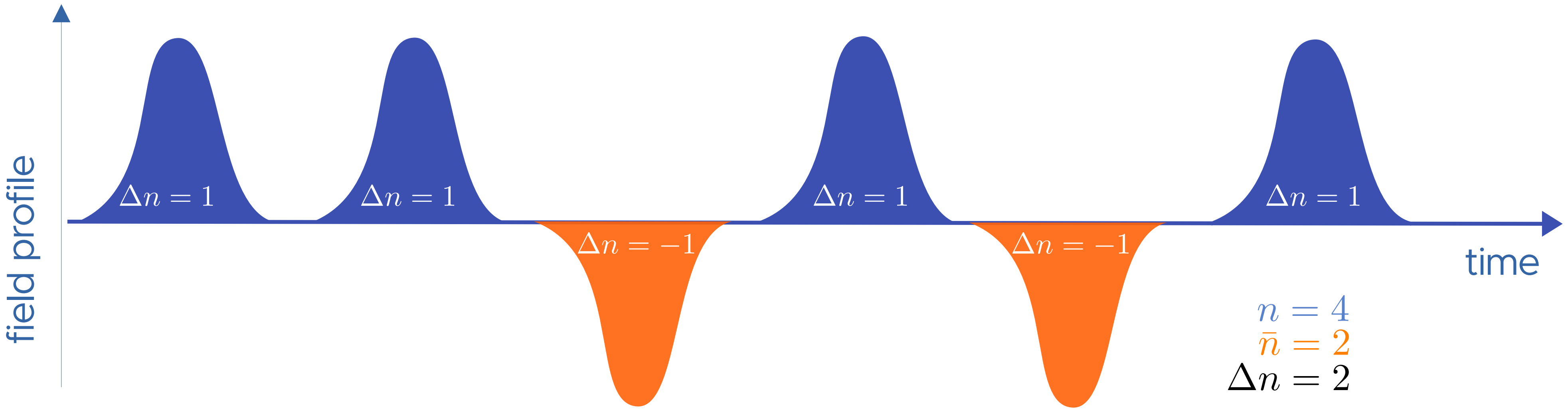
$\bar{n}$

The dilute instanton gas

A **saddle point** of topological charge Δn can be approximated as a **superposition** of:

- n localized saddles with Δn=1 **(instantons)** and
- $\bar{n}$ localized saddles with Δn=−1 **(anti-instantons)**

$$\Delta n = n - \bar{n}$$



The IR perspective: Chiral Lagrangian

At **low energies**, **QCD confines**. The quarks get expectation values $\langle \bar{q} q \rangle$ which **break** the approximate **global flavour symmetries**

$$U(3)_L \times U(3)_R \to U(3)_V$$

The **relevant dynamical fields** are:

Goldstones of the broken symmetries (e.g. pions) ⟷ broken gens acting on vacuum

$$U = \langle U \rangle e^{\mathrm{i} \frac{\Pi^a \sigma^a}{\sqrt{2} f_\pi}} \sim \bar{\psi} P_R \psi$$

Neutron proton doublet

$$N = \begin{pmatrix} p \\ n \end{pmatrix}$$

The IR perspective: Chiral Lagrangian

Most general **Lagrangian compatible with the symmetries**

$$\mathcal{L}_{\pi,\mathrm{p,n}} \supset \frac{1}{4} f_\pi^2 \mathrm{Tr} D_\mu U D^\mu U^\dagger + (a f_\pi^3 \mathrm{Tr} M U + |b| e^{-i\xi} f_\pi^4 \mathrm{det} U + \mathrm{h.c.})$$

$$+ i\bar{N}\not{D}N - (m_N \bar{N}\tilde{U}P_L N + ic\,\bar{N}\gamma^\mu \tilde{U}^\dagger D_\mu \tilde{U} P_L N + d\,\bar{N}\tilde{M}^\dagger P_L N + e\,\bar{N}\tilde{U}\tilde{M}\tilde{U}P_L N + \mathrm{h.c.})$$

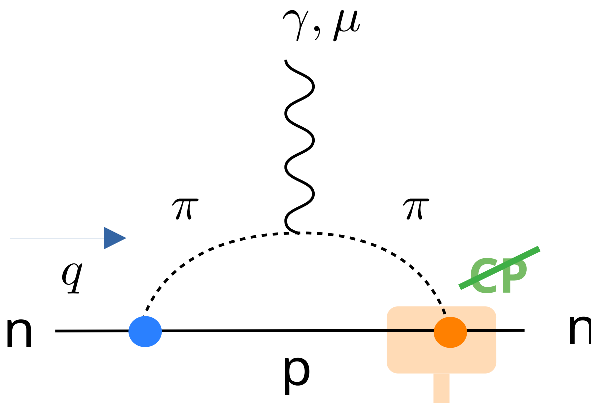
Given $U \sim \bar{\psi} P_R \psi$**, comparing with** $\mathcal{L}_{\mathrm{QCD}}$ → **M =quark mass matrix**

$$M = \begin{pmatrix} m_u e^{i\alpha_u} & & \\ & m_d e^{i\alpha_d} & \\ & & m_s e^{i\alpha_s} \end{pmatrix}$$

CP-odd phases

($\tilde{U}$: projection into u,d flavours)

Neutron dipole moment



$$\mathcal{L}_{\text{eff}} \supset (\xi + \alpha_u + \alpha_d + \alpha_s) f(q^2) \bar{N} (\vec{S} \cdot \vec{E}) N$$

$d_n \propto (\xi + \alpha_u + \alpha_d + \alpha_s)$ neutron dipole moment

Matching the UV and the IR a la 't Hooft

UV: 't Hooft vertices

$$\mathcal{L}^{\rm QCD}_{\rm eff, 't\,Hooft} \sim e^{-i\theta}\prod_{i=1}^{N_f}\bar{\psi}_i P_R \psi_i + e^{i\theta}\prod_{i=1}^{N_f}\bar{\psi}_i P_L \psi_i$$

IR: Chiral Lagrangian

$$\mathcal{L}_{\rm pion} \supset |b| e^{-i\xi} f_\pi^4 \det U + \text{h.c.}, \qquad U \sim \bar{\psi} P_R \psi$$

Matching leads to

$$\xi = \theta$$

Neutron dipole moment: $|d_n| \propto (\xi + \alpha_u + \alpha_d + \alpha_s) = \theta + \sum_i \alpha_i \equiv \bar{\theta}$

Experimental bounds: $\bar{\theta} < 10^{-10}$

Matching the UV and the IR our way

UV: 't Hooft vertices

$$\mathcal{L}^{\rm QCD}_{\rm eff,'t\,Hooft} \sim e^{\mathrm{i}\sum_j \alpha_j}\prod_{i=1}^{N_f}\bar\psi_i P_R \psi_i + e^{-\mathrm{i}\sum_j \alpha_j}\prod_{i=1}^{N_f}\bar\psi_i P_L \psi_i$$

IR: Chiral Lagrangian

$$\mathcal{L}_{\rm pion} \supset |b| e^{-i\xi} f_\pi^4 {\rm det} U + {\rm h.c.}, \qquad U \sim \bar\psi P_R \psi$$

Matching leads to

$$\xi = -\sum_i \alpha_i$$

Neutron dipole moment:

$$|d_n| \propto (\xi + \alpha_u + \alpha_d + \alpha_s) = 0$$

Baluni’s CP-violating effective Lagrangian

Baluni’s CP-violating Lagrangian (used by [Crewther et al]) is based on searching for field redefinitions that minimize the QCD mass term

$$\mathcal{L}_M(U_{R,L}) = \bar{\psi} U_R^\dagger M U_L \psi_L + \text{h.c.}, \quad U_{R,L} \in SU_{R,L}(3)$$

$$\langle 0|\delta\mathcal{L}|0\rangle = \min_{U_{R,L}} \langle 0|\mathcal{L}_M(U_{R,L})|0\rangle$$

However, there is an **extra assumption**: that the **phase of the fermion condensate is aligned with**

$$\langle \bar{\psi}_R \psi_L \rangle = \Delta e^{\mathrm{i} c\theta} \mathbb{I}$$

This assumption **does not hold** for the chiral Lagrangian with $\xi = -\alpha$, but is valid for $\xi = \theta$

Crewther et al's calculations

Using [Baluni]'s CP-violating Lagrangian and current algebra [Crewther et al] get

$$\langle 0|\delta\mathcal{L}|\eta'\pi^0\pi^0\rangle = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2}{f_\pi}\bar{\theta}\,.$$

From our general Chiral Lagrangian we get

$$\mathcal{L}_{\rm M}^{\rm EFT} \supset \frac{B_0 \sin(\xi+\alpha_u+\alpha_d)}{f_\pi\sqrt{\frac{1}{m_u^2}+\frac{1}{m_d^2}+\frac{2\cos(\xi+\alpha_u+\alpha_d)}{m_u m_d}}}\left[(\pi^0)^2+2\pi^+\pi^-\right]\eta'$$

$$\langle 0|\delta\mathcal{L}|\eta'\pi^0\pi^0\rangle = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2}{f_\pi}(\xi+\bar{\alpha}),\,.$$

Match for $\xi = \theta$

So once more, traditional results are built on the **(hidden) assumption** $\xi = \theta$

The η' mass

Chiral Lagrangian with alignment in the phases of mass terms and anomalous terms still predicts a **nonzero value of the η' mass**

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \mathrm{Tr} MU + |b| e^{i\arg\det M} f_\pi^4 \det U + \mathrm{h.c.}$$

$$m_{\eta'}^2 = 8|b| f_\pi^2$$

Can be seen to be **proportional** to the **topological susceptibility** over **finite volumes** of the **pure gauge theory**, in line with [Witten, Di Vecchia & Veneziano]

Classic arguments linking topological susceptibility to CP violation ([Shifman et al]) rely on analytic expansions which **don't apply** with our limiting procedure

***Z* from infinite-volume partition function becomes non-analytic in θ.**
This possibility has been mentioned by [Witten]

Partition function and analiticity

Usual partition function is analytic in θ

$$Z_{\text{usual}} = \lim_{VT\to\infty} \lim_{\substack{N\to\infty \\ N\in N}} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = \mathrm{e}^{2\mathrm{i}\kappa_{N_f} VT \cos(\bar{\alpha}+\theta+N_f\pi)}$$

θ-dependence of observables (giving CP violation) usually relies on expansion. e.g.

$$\frac{\langle \Delta n \rangle}{\Omega} = \mathrm{i}\,(\theta-\theta_0) \left.\frac{\langle \Delta n^2 \rangle}{\Omega}\right|_{\theta_0} + \mathcal{O}(\theta-\theta_0)^2$$

topological susceptibility

[Shifman et al]

In our limiting procedure the former is not valid, as Z becomes nonanalytic in θ

$$Z = \lim_{\substack{N\to\infty \\ N\in N}} \lim_{VT\to\infty} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = I_0(2\mathrm{i}\kappa_{N_f} VT) \lim_{\substack{N\to\infty \\ N\in N}} \sum_{|\Delta n|\leq N} \mathrm{e}^{\mathrm{i}\Delta n(\bar{\alpha}+\theta+N_f\pi)}$$

θ is part of normalization constant that drops out from observables, there is no CP violation

[Witten, Nucl. Phys. B 156 (1979)]

the physics is of order e^{-N}, contrary to the basic assumptions of this paper, or else the physics is non-analytic as a function of θ, In the latter case, which is quite plausible, the singularities would probably be at $\theta = \pm\pi$, as Coleman found for the massive Schwinger model [10]. It is also quite plausible that θ is not really an angular variable.)

To write a formal expression for $d^2E/d\theta^2$, let us think of the path integral formulation of the theory:

$$Z = \int dA_\mu \exp i \int \mathrm{Tr}\left[-\tfrac{1}{4}F_{\mu\nu} + \frac{g^2\theta}{16\pi^2 N} F_{\mu\nu}\tilde{F}_{\mu\nu}\right]. \qquad (5)$$

The quantum rotor

[Albandea et al] have claimed that our order of limits gives wrong results in a toy model of QCD given by the 1D quantum rotor:

$$\mathcal{L} = \frac{\mu}{2}\,\dot{q}(t)^2 + \frac{1}{2\pi}\,\theta\dot{q}(t)$$

Total deriv. Analogue to θ term in QCD

Their claim is that our limiting procedure gives a zero result for the Euclidean topological susceptibility, defined as

$$\chi = \frac{1}{\Delta\tau}\int d\tau \int d\tau' \frac{d}{d\tau}\,\frac{d}{d\tau'}\langle T q(\tau) q(\tau')\rangle\Big|_{\theta=0}$$

The quantum rotor

Their calculation goes as follows. One starts assuming “topological sectors” with

$$\int d\tau \dot{q}(\tau) = q(\tau_f) - q(\tau_i) = 2\pi\Delta n \Rightarrow S_\theta = i\Delta n$$

(Note that in contrast to QCD at infinite volume, this boundary conditions are not necessary)

$$\chi = \sum_{\Delta n} \frac{1}{4\pi^2 \Delta\tau} \int d\tau \int d\tau' \frac{d}{d\tau} \frac{d}{d\tau'} \langle T q(\tau) q(\tau') \rangle_{\Delta n},$$

$$= \frac{1}{4\pi^2 \Delta\tau \sum_{\Delta m} Z_{\Delta m}} \sum_{\Delta n} \int d\tau \int d\tau' \int_{\Delta n} \mathcal{D}q \, \dot{q}(\tau) \dot{q}(\tau') e^{-S_E}$$

The quantum rotor

Exchanging the order of time integrals and the path integration:

$$\chi = \frac{1}{\Delta\tau \sum_{\Delta m} Z_{\Delta m}} \sum_{\Delta n} \Delta n^2 Z_{\Delta n}$$

Taking the infinite volume limit $\Delta\tau \to \infty$, before summing over all Δ n gives zero, in conflict with the lattice results

It turns out that this calculation is wrong because for $\Delta\tau \to \infty$

$$\int d\tau \int d\tau' \int_{\Delta n} \mathcal{D}q\, \dot{q}(\tau)\dot{q}(\tau')e^{-S_i} \neq \int_{\Delta n} \mathcal{D}q \int d\tau \int d\tau'\, \dot{q}(\tau)\dot{q}(\tau')e^{-S_i}$$

as can be seen from explicit computation of the correlator

The quantum rotor

From explicit computation of the correlator:

$$\chi = \frac{1}{4\pi^2 \Delta\tau} \int d\tau \int d\tau' \left(\frac{1}{\mu} \delta(\tau - \tau') - \frac{1}{\Delta\tau\mu} + \frac{4\pi^2}{\Delta\tau^2} \frac{\sum_{\Delta n} \Delta n^2 Z_{\Delta n}}{\sum_{\Delta n} Z_{\Delta n}} \right)$$

Taking $\Delta\tau \to \infty$ first, the last 2 terms in the parenthesis can be ignored with respect to the first, and one is left with

$$\chi = \frac{1}{4\pi^2 \mu}$$

which agrees with the result in the other order of limits and the lattice. Hence χ is independent of the order of limits and it cannot be concluded that our preferred ordering of limits is wrong

Technical remarks on canonical quantization

We assumed that in the $\psi'[\mathbf{A}]$ basis one can work with states

$$U(\mathbf{x})|\Psi'^{(a)}\rangle = e^{i\alpha(U(\mathbf{x}))}|\Psi'^{(a)}\rangle$$

One still has to prove that this is possible for eigenstates of the Hamiltonian!

We have done it by looking at the **functional Schrödinger equation** in the $\psi'[\mathbf{A}]$ basis

Is there a symmetry related to parity?

Even in the presence of θ, the Hamiltonian has a **discrete symmetry**, which can be seen to enforce

$$\Psi[\mathbf{A}^P] = \pm e^{-2i\theta W[\mathbf{A}]}\Psi[\mathbf{A}]$$

parity = rephasing

$$iS \underset{P}{\longrightarrow} iS - 2i\theta(W[\mathbf{A}_f] - W[\mathbf{A}_i])$$

$+2i\theta W[\mathbf{A}_f]$ $\quad -2i\theta W[\mathbf{A}_i]$ $\quad -2i\theta W[\mathbf{A}_f] + 2i\theta W[\mathbf{A}_i]$

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\Psi_0[\phi_f]^*\Psi_0[\phi_i]\int_{\phi_i,\phi_f,T}\mathcal{D}\phi\, e^{iS}$$

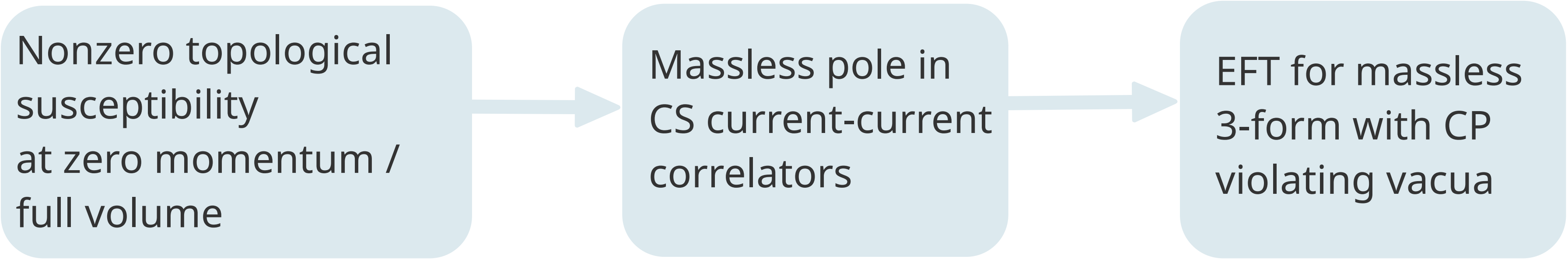
The **partition function** is **parity invariant!**

Dvali's footnote

[2] The 3-form language of [14] clarifies the claim of [24] that by changing the order of limits in ordinary instanton calculation, one ends up with $\vartheta = 0$. In this approach one performs calculation in the finite volume and then takes it to infinity. In 3-form language the meaning of this is rather transparent. The finite volume is equivalent of introducing an infrared cutoff in form of a shift of the massless pole in (28) away from zero. This effectively gives a small mass to the 3-form. For any non-zero value of the cutoff, the unique vacuum is $E_0 = 0$ which is equivalent to $\vartheta = 0$. Other states $E \neq 0$ (corresponding to $\vartheta \neq 0$) have finite lifetimes which tend to infinity when cutoff is taken to zero. In this way the $\vartheta \neq 0$ vacua are of course present but one is constrained to $\vartheta = 0$ by the prescription of the calculation. Thus, changing the order of limits by no means eliminates the ϑ-vacua. As usual, when taking the limit properly, one must keep track of states that become stable in that limit. These are the states with $\vartheta \neq 0$ ($E \neq 0$), which become the valid vacua in the infinite volume limit. The effect is in certain sense equivalent to introducing an auxiliary axion and then decoupling it.

Dvali's 3-form formalism

[Dvali] has the following line of reasoning from which he concludes that QCD violates CP



[Dvali] argues that in a calculation at finite volume which is then sent to infinity,
CP violation can't be captured because the infrared regulation gives a mass to the 3 form.

Dvali’s criticism

We make the following observations:

- [`t Hooft]’s original calculations (at finite volume, taken to ∞ in the end) lead to CP violation for arbitrary θ, in conflict with Dvali’s argument

- If finite volume is problematic, more reason to take the infinite volume limit as soon as possible, as we do, leading to no CP violation for arbitrary θ

- Dvali’s formalism has no explicit/direct link to UV θ parameter. Our UV computations (and the Chiral Lagrangian that matches them) would only select the CP conserving vacua in Dvali’s EFT.