How common are Grand Unified Theories?

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in collaboration with

Maximilian Ruhdorfer (Stanford)

DESY Theory Workshop 2025

based on [2408.11089]

24.09.2025

The Puzzle

Standard Model of particle physics: $SU(3) \times SU(2) \times U(1)$

$$3\times \quad \begin{pmatrix} \nu \\ e_L \end{pmatrix} \; e_R \; \begin{pmatrix} u_{L,1} u_{L,2} u_{L,3} \\ d_{L,1} d_{L,2} d_{L,3} \end{pmatrix} \; \begin{pmatrix} d_{R,1} d_{R,2} d_{R,3} \end{pmatrix} \; \begin{pmatrix} u_{R,1} u_{R,2} u_{R,3} \end{pmatrix}$$

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Georgi-Glashow: this unifies into SU(5) : $3\times\ (\bar{\mathbf{5}},\ \mathbf{10})$

$$3\times \quad \begin{pmatrix} d^c_{R,1} \\ d^c_{R,1} \\ d^c_{R,1} \\ e_L \\ -\nu \end{pmatrix} \qquad \begin{pmatrix} 0 & u^c_{R,3} & -u^c_{R,2} & u_{L,1} & d_{L,1} \\ -u^c_{R,3} & 0 & u^c_{R,1} & u_{L,2} & d_{L,2} \\ u^c_{R,2} & -u^c_{R,1} & 0 & u_{L,3} & d_{L,3} \\ -u_{L,1} & -u_{L,2} & -u_{L,3} & 0 & e^c_R \\ -d_{L,1} & -d_{L,2} & -d_{L,3} & -e^c_R & 0 \end{pmatrix}$$

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What are the odds!?

Motivation

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interesting answer

- if common: no need to be surprised by grand unifiability, can stay GUT agnostic
- if rare: purely group-theoretical bottom-up indication for Grand Unification

Unifiability

observation

1. fermion sector of the SM *remarkably complete*
   - anomaly free generation-by-generation
   - no evidence for BSM fermios charged under SM
   - LHC: new fermions chiral under SM all but ruled out by Higgs measurements
2. SM *fermions unify neatly* into representation of SU(5)
   - unification of all gauge forces (simple group)
   - no additional fermions in GUT rep $\longrightarrow$ *closure*

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How common is neat unifiability of fermions among SM-like theories?

$\longrightarrow$ *we do not consider gauge coupling unification*

SM-like theories

characteristic features

- 3 gauge forces $\sim$ reductive rank-3 semisimple $\times U(1)$ gauge algebra
- $D = 15$ fermions per generation
- three separately anomaly free generations
- integer hypercharges $|Q| \leq 6$
- fermion representation is chiral

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SM-like theories (single generation)

- anomaly-free, chiral representations of SM-like gauge group
  - SM gauge algebra
  - any semi-simple gauge algebra with rank≤ 3 and a $U(1)$ factor i.e. $SU(2) \times U(1)$ (rank-1), $\{SO(5), SU(2) \times SU(2), SP(4), G_2\} \times U(1)$ (rank-2) and $\{SU(2)^3, SU(4), SU(3) \times SU(2), SP(6), SO(5) \times SU(2), SO(7)\} \times U(1)$ (rank-3)
- restrict fermion dimension $D \leq D_{\text{max}}$, charges $|Q| \leq Q_{\text{max}}$

  $\Rightarrow$ *result depends on these assumptions; can be discussed*

Base set of anomaly-free (=consistent) representations

1. find all non-anomalous reps of semisimple part ($SU(3) \times SU(2)$)
2. assign $U(1)$ charges, keep those that satisfy anomaly cancellation
3. remove equivalent representations
   - rescaling of $U(1)$ charge, conjugate reps

$\Rightarrow$ solved using Mathematica packages SuperFlocci or GroupMath.

Some subtleties in efficient $U(1)$ charge assignment.

many thanks to Joseph Tooby-Smith!

examples:

$$(\mathbf{1},\mathbf{2})_0 \oplus (\mathbf{3},\mathbf{1})_{-1} \oplus (\bar{\mathbf{3}},\mathbf{1})_1$$ smallest, $D = 8$

$$(\mathbf{3},\mathbf{2})_0 \oplus (\bar{\mathbf{3}},\mathbf{1})_{-1} \oplus (\bar{\mathbf{3}},\mathbf{1})_1$$ smallest chiral, $D = 12$

$$(\mathbf{1},\mathbf{1})_{-6} \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\bar{\mathbf{3}},\mathbf{2})_{-1} \oplus (\mathbf{3},\mathbf{1})_{-2} \oplus (\mathbf{3},\mathbf{1})_4$$ you know this one, $D = 15$

SuperFlocci – checking unifiability bottom-up

https://github.com/jstoobysmith/Superfloccinaucinihilipilification [2306.16439]

Semisimple unifications of any gauge theory

Andrew Gomes,[*] Maximilian Ruhdorfer,[†] and Joseph Tooby-Smith[‡]
Laboratory for Elementary Particle Physics, Cornell University, Ithaca, New York 14853, USA

We present a *Mathematica* package that takes any reductive gauge algebra and fully-reducible fermion representation, and outputs all semisimple gauge extensions under the condition that they have no additional fermions, and are free of local anomalies. These include all simple completions, also known as grand unified theories. We additionally provide a list of all semisimple completions for 5835 fermionic extensions of the one-generation Standard Model.

$\Longrightarrow$ *plug in every theory in base set and check for simple gauge extension*

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------------------------------------------------------------
SuperFlocci Output
Program by: Andrew Gomes, Maximillian Ruhdorfer, and Joseph Tooby-Smith. 2022
------------------------------------------------------------
Input algebra: su(2)⊕su(3)⊕u(1)
Input representation: (2,3,0)⊕(1,3̄,-1)⊕(1,3̄,1)
Date of generation: Sun 9 Jun 2024 11:58:36
------------------------------------------------------------
------------------------------------------------------------
Maximal algebras:
1) su(2)⊕su(2)⊕su(3) (1,2,3)⊕(2,1,3̄)
------------------------------------------------------------
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```

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Input algebra: su(2)⊕su(3)⊕u(1)
Input representation: (1,1,-6)⊕(2,1,3)⊕(2,3̄,-1)⊕(1,3,-2)⊕(1,3,4)
Date of generation: Sun 9 Jun 2024 12:00:08
------------------------------------------------------------
------------------------------------------------------------
Maximal algebras:
1) su(5) (5̄)⊕(10)
------------------------------------------------------------
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GroupMath – top-down decoposing all candidate GUTs

https://renatofonseca.net/groupmath [R.Fonseca'2011.01764]

- candidate GUTs with non-singlet fermion rep with $D \leq D_{\text{max}}$

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{{SU2, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}}, {SU3, {1, 3, $\overline{3}$, 6, $\overline{6}$, 8, 10, $\overline{10}$, 15, $\overline{15}$, 15', $\overline{15'}$}},
{SU4, {1, 4, $\overline{4}$, 6, 10, $\overline{10}$, 15, 20, $\overline{20}$, 20', 20'', $\overline{20''}$}}, {SU5, {1, 5, $\overline{5}$, 10, $\overline{10}$, 15, $\overline{15}$}}, {SU6, {1, 6, $\overline{6}$, 15, $\overline{15}$, 20}}, {SU7, {1, 7, $\overline{7}$}},
{SU8, {1, 8, $\overline{8}$}}, {SU9, {1, 9, $\overline{9}$}}, {SU10, {1, 10, $\overline{10}$}}, {SU11, {1, 11, $\overline{11}$}}, {SU12, {1, 12, $\overline{12}$}}, {SU13, {1, 13, $\overline{13}$}}, {SU14, {1, 14, $\overline{14}$}},
{SU15, {1, 15, $\overline{15}$}}, {SU16, {1, 16, $\overline{16}$}}, {SU17, {1, 17, $\overline{17}$}}, {SU18, {1, 18, $\overline{18}$}}, {SU19, {1, 19, $\overline{19}$}}, {SU20, {1, 20, $\overline{20}$}},
{SP4, {1, 4, 5, 10, 14, 16, 20}}, {SP6, {1, 6, 14, 14'}}, {SP8, {1, 8}}, {SP10, {1, 10}}, {SP12, {1, 12}}, {SP14, {1, 14}}, {SP16, {1, 16}},
{SP18, {1, 18}}, {SP20, {1, 20}}, {SO5, {1, 4, 5, 10, 14, 16, 20}}, {SO6, {1, 4, $\overline{4}$, 6, 10, $\overline{10}$, 15, 20, $\overline{20}$, 20', 20'', $\overline{20''}$}},
{SO7, {1, 7, 8}}, {SO8, {1, 8_v, 8_s, 8_c}}, {SO9, {1, 9, 16}}, {SO10, {1, 10, 16, $\overline{16}$}}, {SO11, {1, 11}}, {SO12, {1, 12}}, {SO13, {1, 13}},
{SO14, {1, 14}}, {SO15, {1, 15}}, {SO16, {1, 16}}, {SO17, {1, 17}}, {SO18, {1, 18}}, {SO19, {1, 19}}, {SO20, {1, 20}}, {G2, {1, 7, 14}}}

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- candidate GUTs with non-singlet fermion rep with $D \leq D_{\max}$

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{SO14, {1, 14}}, {SO15, {1, 15}}, {SO16, {1, 16}}, {SO17, {1, 17}}, {SO18, {1, 18}}, {SO19, {1, 19}}, {SO20, {1, 20}}, {G2, {1, 7, 14}}}

- GroupMath computes all distinct decompositions, fast!

In[∘]:= DecomposeRep[SO10, -16, {SU3, SU2, U1}]

There are 2 non-equivalent ways of embedding {SU3, SU2, U1} in SO10.

Under each of them, the representation

-16 decomposes as follows (x1 is a free real number):

Embedding	Decomposition					
#1	$3 \otimes 2 \otimes -1$	$\overline{3} \otimes 1 \otimes (1 + x1)$	$\overline{3} \otimes 1 \otimes (1 - x1)$	$1 \otimes 2 \otimes 3$	$1 \otimes 1 \otimes (-3 + x1)$	$1 \otimes 1 \otimes (-3 - x1)$
#2	$3 \otimes 2 \otimes -1$	$\overline{3} \otimes 2 \otimes 1$	$1 \otimes 2 \otimes 3$	$1 \otimes 2 \otimes -3$		

- assign charges and count

Result: SM gauge algebra

most restrictive base set

$$\left.\frac{\#\text{ unifiable reps}}{\#\text{ SM-like reps}}\right|_{D_{\max}=15}^{\text{SM algebra, completely chiral}} = \frac{1}{2}\,,$$

→ SM is special

→ clearly need to broaden view

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$$\left.\frac{\#\text{ unifiable reps}}{\#\text{ SM-like reps}}\right|^{\text{SM algebra, completely chiral}}_{D_{\text{max}}=15} = \frac{1}{2}\,,$$

→ SM is special

→ clearly need to broaden view

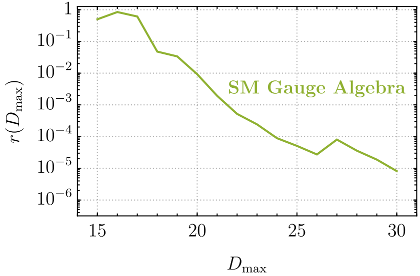
allow for larger neighborhood

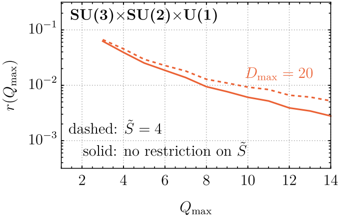
$$\left.\frac{\#\text{ unifiable reps}}{\#\text{ SM-like reps}}\right|^{\text{SM algebra, completely chiral}}_{D_{\text{max}}=20,\ |Q|\leq 10} = \frac{11}{1186}\,.$$

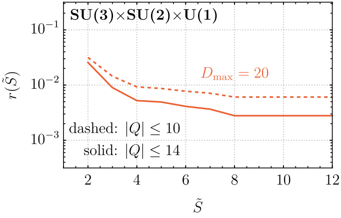
→ *but: arbitrary cuts!*

Dependence on definitions

most SM-like gives most conservative result



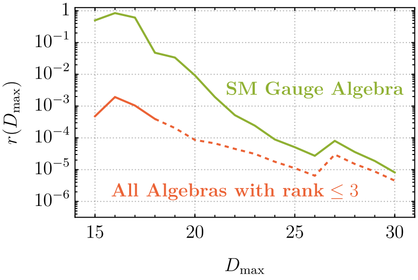




Is the SM gauge algebra special?

considering all reductive semisimple $\times\, U(1)$ algebras (rank ≤ 3)

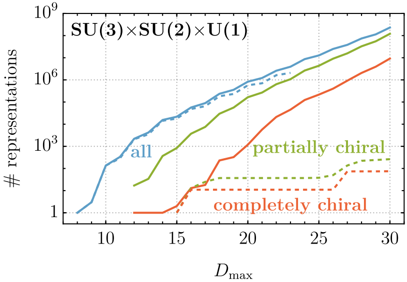
$$\left.\frac{\#\text{ unifiable reps}}{\#\text{ SM-like reps}}\right|_{D_{\max}=15,\ |Q|\leq 6}^{\text{all algebras, completely chiral}} = \frac{1}{365}$$



The role of chirality

allowing for partially chiral representations

$$\left.\frac{\#\text{ unifiable reps}}{\#\text{ SM-like reps}}\right|_{D_{\max}=15,\,|Q|\leq 6}^{\text{SM algebra, partially chiral}} = \frac{1}{111}\,,$$



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It unifies. Surprised?

$$3\times \quad \begin{pmatrix} d^c_{R,1} \\ d^c_{R,1} \\ d^c_{R,1} \\ e_L \\ -\nu \end{pmatrix} \qquad \begin{pmatrix} 0 & u^c_{R,3} & -u^c_{R,2} & u_{L,1} & d_{L,1} \\ -u^c_{R,3} & 0 & u^c_{R,1} & u_{L,2} & d_{L,2} \\ u^c_{R,2} & -u^c_{R,1} & 0 & u_{L,3} & d_{L,3} \\ -u_{L,1} & -u_{L,2} & -u_{L,3} & 0 & e^c_R \\ -d_{L,1} & -d_{L,2} & -d_{L,3} & -e^c_R & 0 \end{pmatrix}$$

"Evidence"?!

Conclusions

- The SM fermions unify neatly into a representation of a simple group
- *Is that surprising?*
  - $\mathcal{O}(1)$ of the handful of anomaly free theories in the immediate neighborhood of the SM unify
  - once we generalize in any way, unifiability becomes rare $\mathcal{O}(10^{-2})$

    larger fermion dimension, different gauge algebra, partially chiral theories

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    - once we generalize in any way, unifiability becomes rare $\mathcal{O}(10^{-2})$

      larger fermion dimension, different gauge algebra, partially chiral theories
- Bottom-up indication for Grand Unification, relying only on group theory
    - without measure in theory space $\rightarrow$ no "evidence"
- Result comparable to a fine-tuning measure

Many thanks to collaborators Max Ruhdorfer and Joseph Tooby-Smith.

Representations