Positivity properties of Wilson loop with Lagrangian insertion in $\mathcal{N} = 4$ SYM

SHUN-QING ZHANG (MPI FOR PHYSICS, GARCHING)

+ with **Dmitry Chicherin, Johannes Henn, Jaroslav Trnka (2410.11456)**
+ with **Dmitry Chicherin, Johannes Henn, Elia Mazzucchelli, Jaroslav Trnka, Qinglin Yang (2503.05443)**
+ with **Dmitry Chicherin, Johannes Henn, Yu Wu, Zhihao Wu, Yongqun Xu, Yang Zhang (To appear)**

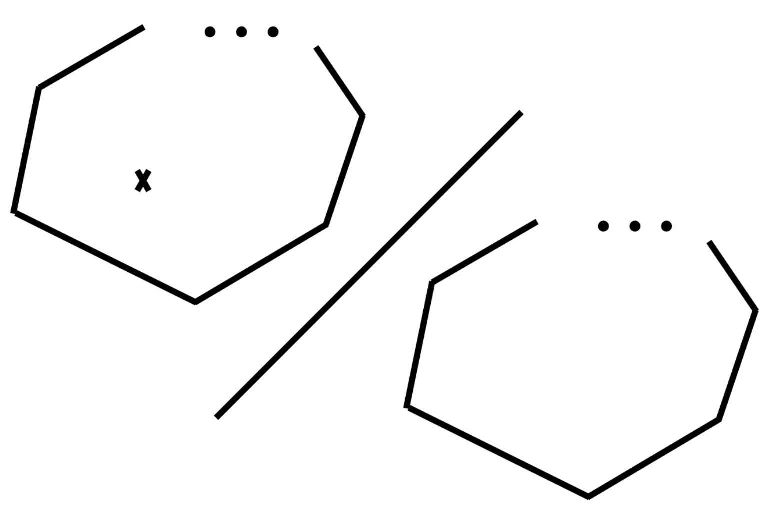
DESY Theory Workshop 2025

MAX-PLANCK-INSTITUT
FÜR PHYSIK

European Research Council
Established by the European Commission

Wilson loops with Lagrangian insertion :

• n-cusp Wilson loop with an insertion point x_0
[Alday, Tseytlin '11; Alday, Buchbinder, Tseytlin '11]



$$F_n = \pi^2 \frac{\langle W[x_1, \cdots, x_n] \mathscr{L}(x_0) \rangle}{\langle W[x_1, \cdots, x_n] \rangle}$$

• **Finite** quantities, similar to QCD hard functions

• Integrand: *logarithm* of scattering amplitudes by **Wilson-loop/Amplitude** duality. [Drummond, Korchemsky, Henn, Sokatchev; Brandhuber, Heslop, Travaglini]

This talk

• Novel geometric expansion: [Arkani-Hamed, Henn, Trnka '21]
- Shed light on all-loop structure
- Testing ground for **higher-point multi-loop** Feynman integrals

• Surprising properties: conformal invariant, **positivity**, duality to all-plus Yang-Mills amplitudes [Chicherin, Henn '22]

Perturbative data: $F_n^{(L)}$

★ : Symbol result

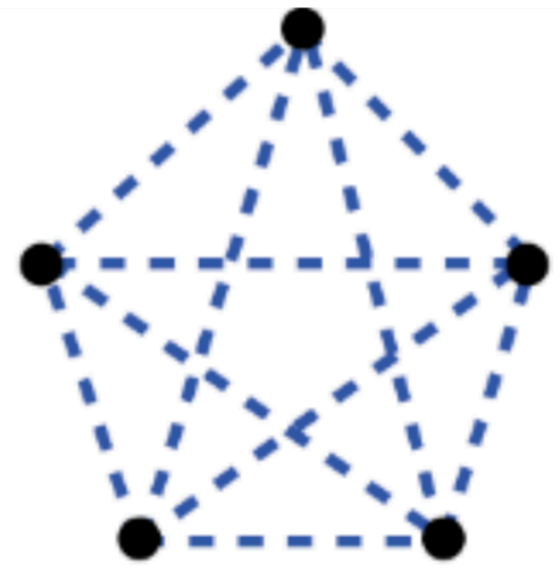
- $n = 4$
  - One- and two-loop [Alday, Heslop, Sikorowski '12] [Alday, Henn, Sikorowski '13]
  - Three-loop [Henn, Korchemsky, Mistlberger '19]
  - Strong coupling [Alday, Buchbinder, Tseytlin '11]
  - Two-loop "**Negative geometric**" expansion [Arkani-Hamed, Henn, Trnka '21]
  - Two-loop with Two Lagrangian insertions [Abreu, Chicherin, Sotnikov, Zoia '24]
- $n = 5$
  - One- and two-loop [Chicherin, Henn '22]
  - Two-loop "Negative geometric" expansion [Chicherin, Henn, Trnka, SQZ '24]
  - Three-loop (ladder) geometry [Chicherin, Henn, Mazzucchelli, Trnka, Yang, SQZ '25] ★

★

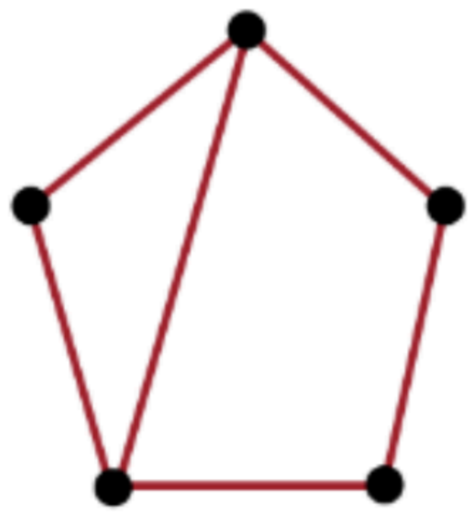
★

Negative geometry expansion

[Arkani-Hamed, Henn, Trnka '21]

$\Omega_L =$ 

single Amplituhedron geometry for the amplitude

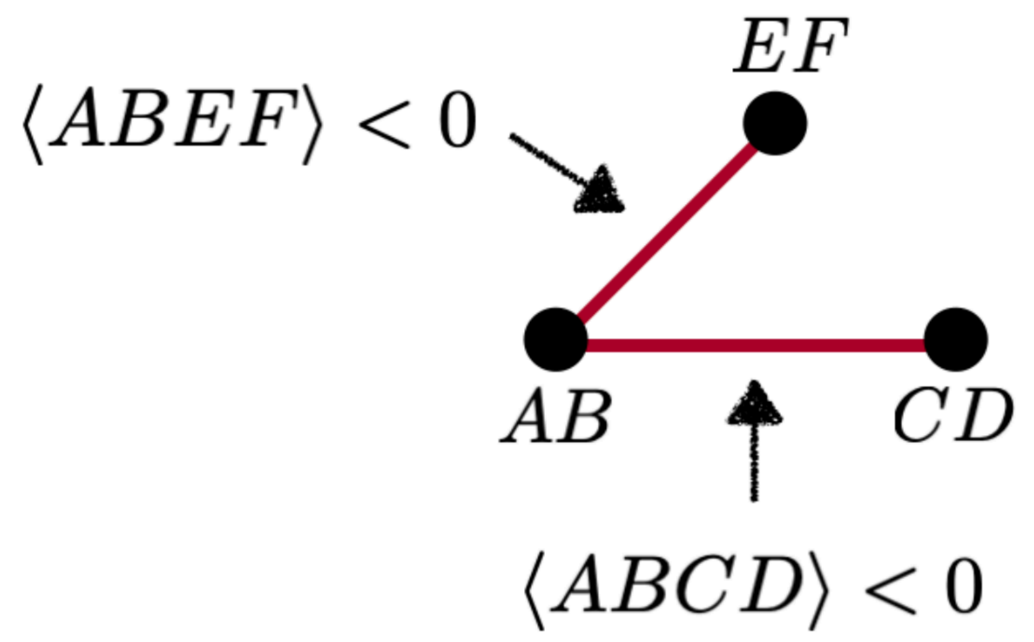
$\widetilde{\Omega}_L = \sum_G (-1)^{E(G)}$ 

sum over geometries for the logarithm of the amplitude

Talk by Trnka Amplitudes 2025

· Each diagram represents a positive geometry with certain set of inequalities → canonical form

(Vertex = loop momentum, edges = "quadratic" constraints)



· L-loop corrections $F_n^{(L)}$

$$F_n^{(1)} = F_n^{(\otimes\!-\!\bullet)} \, . \qquad F_n^{(2)} = -F_n^{(\otimes\!-\!\bullet\!-\!\bullet)} - \frac{1}{2} F_n^{(\bullet\!-\!\otimes\!-\!\bullet)} + \frac{1}{2} F_n^{(\otimes\triangle)} \, .$$

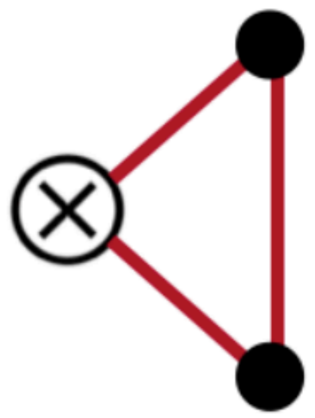
Four-point results [Arkani-Hamed, Henn, Trnka '21]

$L=1$:  $= \log^2 z + \pi^2\,.$

$L=2$

 $= -\frac{1}{2}\left[\pi^2 + \log^2 z\right]^2$

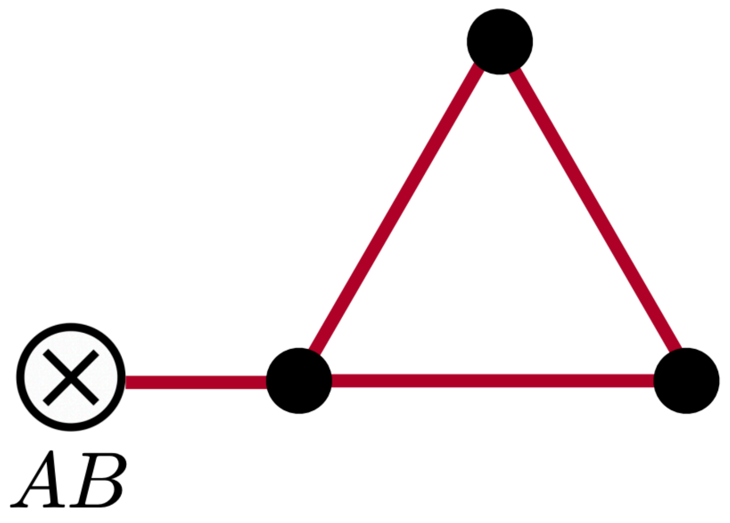
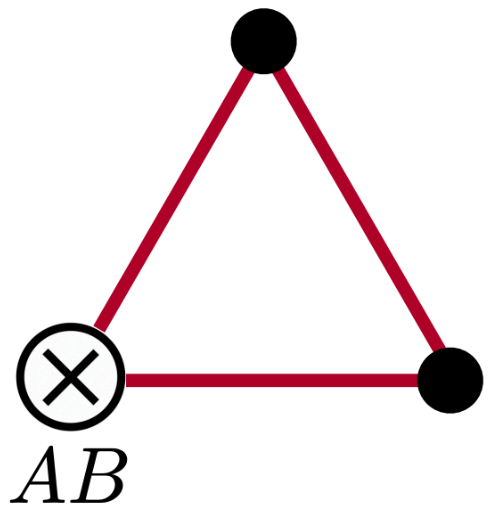
 $= -\frac{1}{12}\left[\pi^2 + \log^2 z\right] \times \left[5\pi^2 + \log^2 z\right]$

 $= 8H_{0,0,0,0} + 8H_{-1,0,0,0} - 16H_{-1,-1,0,0} + 8H_{-2,0,0} - 8\zeta_3\left(2H_{-1} - H_0\right) + 4\pi^2\left(H_{-1,0} - 2H_{-1,-1} + H_{-2}\right) + \frac{13\pi^4}{45}.$

Properties of integrated geometries:

- **IR-finite, Uniform Weight-2L**
- ***Tree* graphs: simpler**
- **Uniform sign for z > 0.**

- **d'Alembertian DE:**
- **All-loop resummation for Γ_{cusp}**
- [Brown, Oktem, Paranjape, Trnka '23]

$\Box_{x_0}$  $=$ 

Higher points

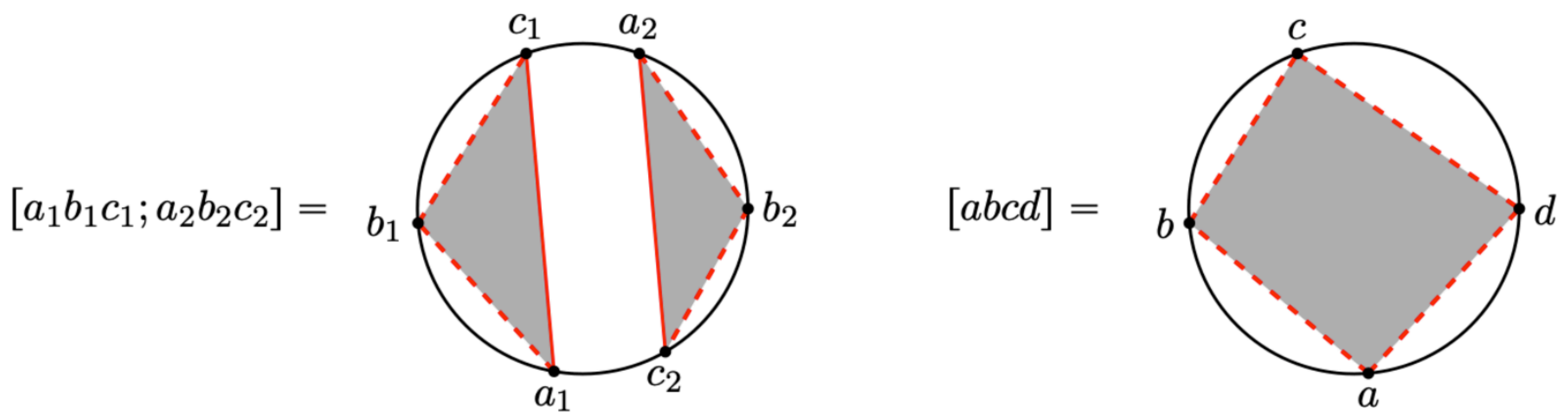
General structure:

$$F_n^{(L)} = \sum_j r_{n,j}\, g_{n,j}^{(L)}$$

$r_{n,j}$: rational prefactor= **leading singularities** (LS)
All-loop conjecture: LS given by **Kermit**
[Brown, Henn, Mazzucchelli,Trnka,'25]
Conformal invariant!

$g_{n,j}^{(L)}$: pure polylogarithmic functions of **weight 2L**
e.g. **Pentagon functions**

$$[a_1b_1c_1;a_2b_2c_2] = \frac{\langle AB(a_1b_1c_1)\cap(a_2b_2c_2)\rangle^2}{\langle ABa_1b_1\rangle\langle ABb_1c_1\rangle\langle ABa_1c_1\rangle\langle ABa_2b_2\rangle\langle ABb_2c_2\rangle\langle ABa_2c_2\rangle}$$

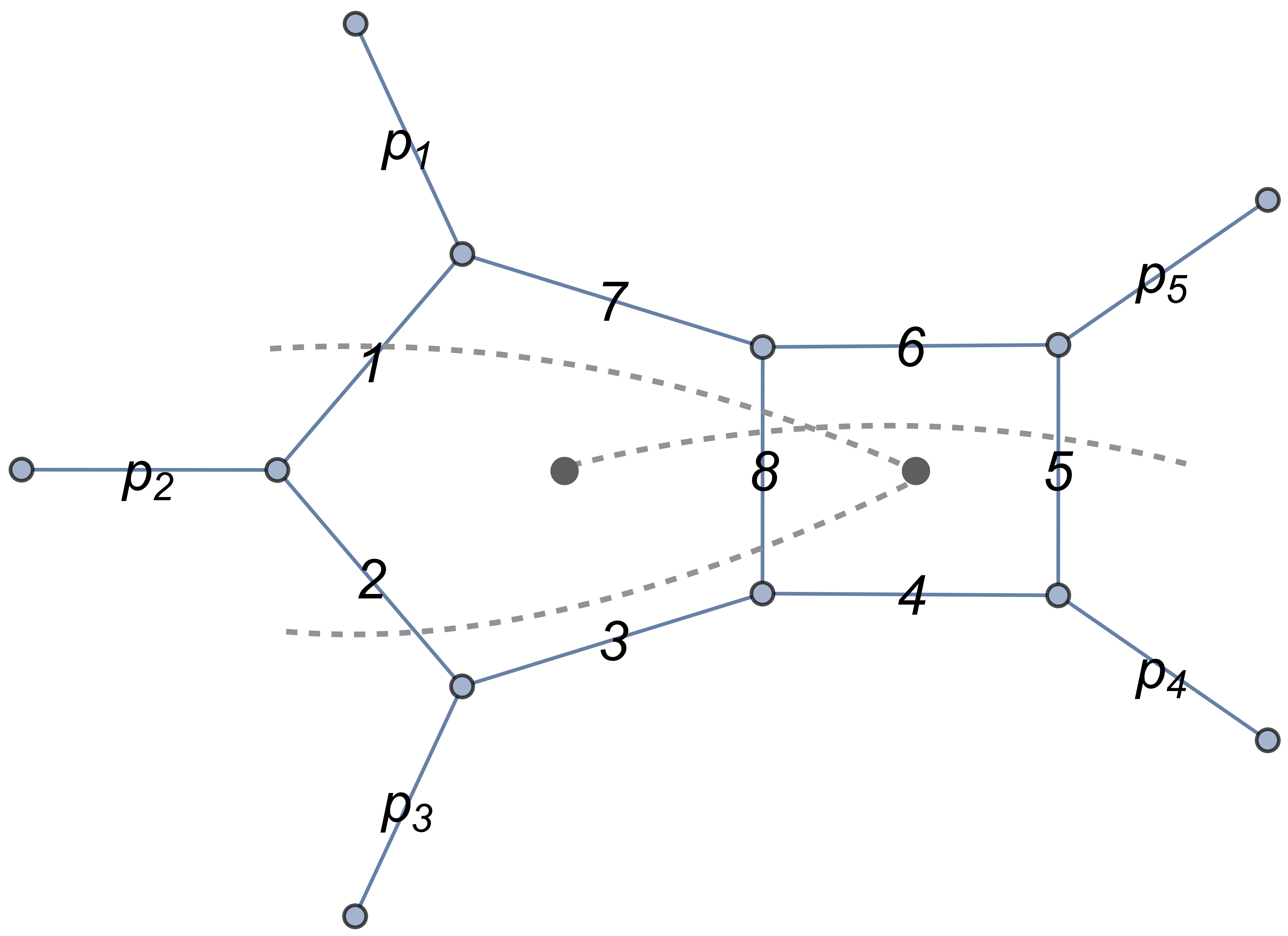


	n	5	6	7	8	9	10	11	12
$L=1$	$\frac{n(n-3)}{2}$	5	9	14	20	27	35	44	54
$L\geqslant 2$	$\frac{(n-1)(n-2)^2(n-3)}{12}$	6	20	50	105	196	336	540	825

Counting of LS

Canonical Differential Equations [Henn, '13]

• n=5, L=2



- **Topology for WL: Penta-Box + 1L×1L** [Chicherin, Henn '22]
- **Topology for Negative Geometries:**
  8 + 3 ISP → 11 propagators: "Non-planar" in dual-momentum sapce [Chicherin, Henn, Trnka, SQZ '24]

Only planar alphabet $\mathbb{A}_{n=5}^{\text{2-loop}}$ needed

- Choose $\mathcal{O}(10^5)$ random kinematic points from Eucl+.
- Evaluate **pentagon functions** using the **C++** code [Gehrmann, Henn, Lo Presti '18]

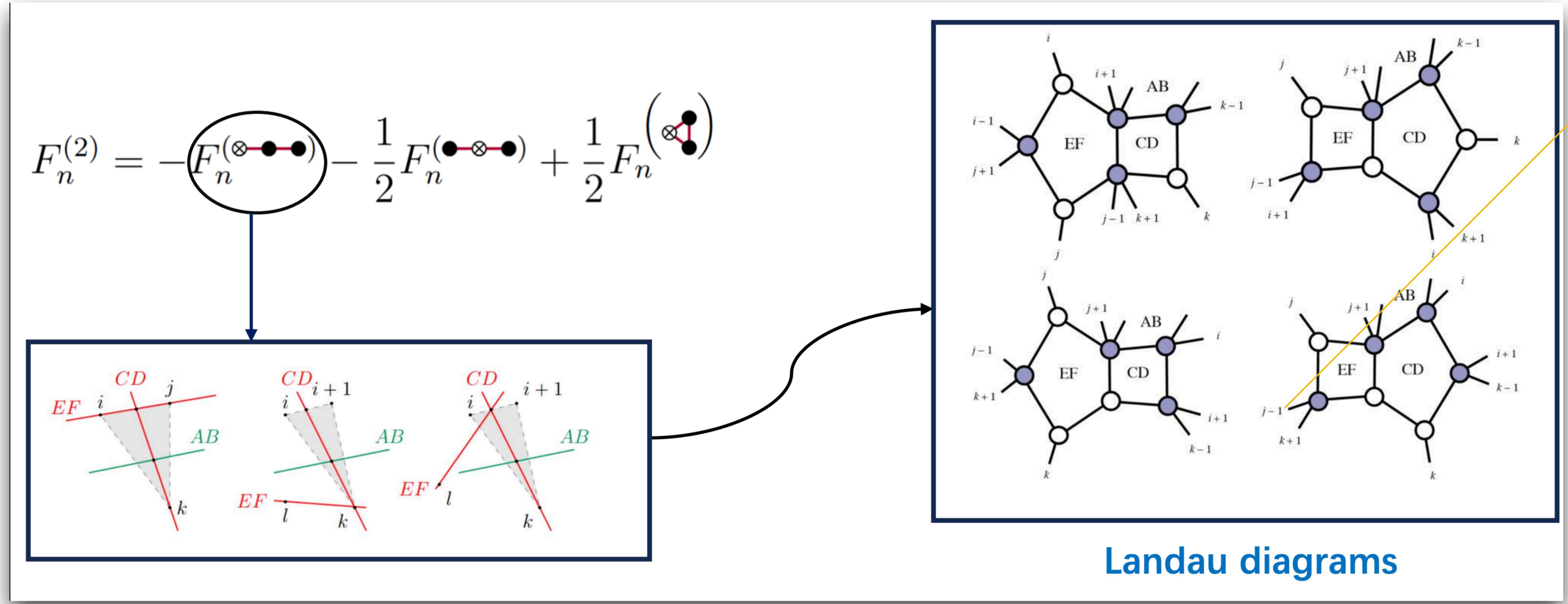
$$\text{Eucl}^+ \;:\qquad \epsilon_5 > 0\,, \qquad s_{i\,i+1} > 0\,, \qquad i = 1, \ldots, 5\,.$$

bootstrap

$$F_5^{(\otimes\!-\!\bullet\!-\!\bullet)}\,,\; F_5^{(\bullet\!-\!\otimes\!-\!\bullet)}\,,\; F_5^{(\text{triangle})}\Big|_{\text{Eucl}^+} > 0\,, \quad \text{and} \quad F_5^{(2)}\Big|_{\text{Eucl}^+} < 0\,.$$

Geometric Landau bootstrap for negative geometries

[Chicherin, Henn, Mazzucchelli, Trnka, Yang, SQZ '25]



[Talk by Qinglin Yang Amplitudes 2025]

Bootstrap conditions:

- Collinear, soft limits
- No spurious poles, e.g. $s_{1,3}$
- Last entries from d'Alembertian

New pentagon alphabets

Fix:

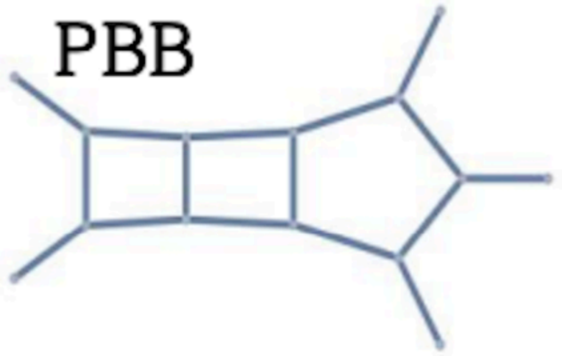
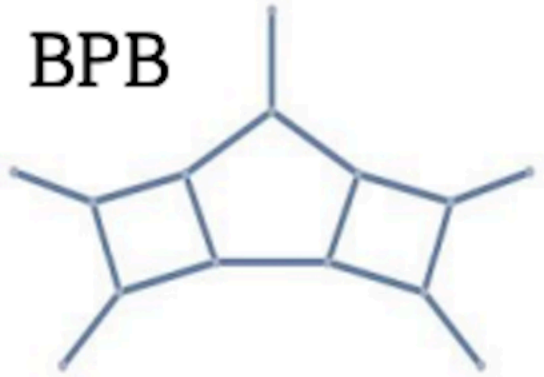
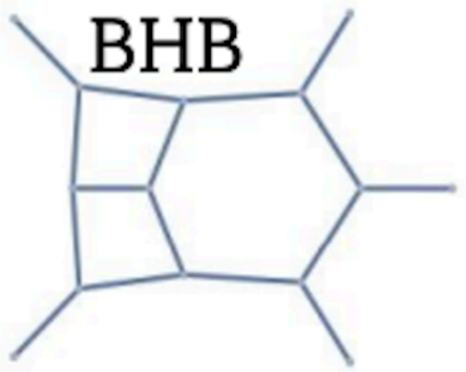
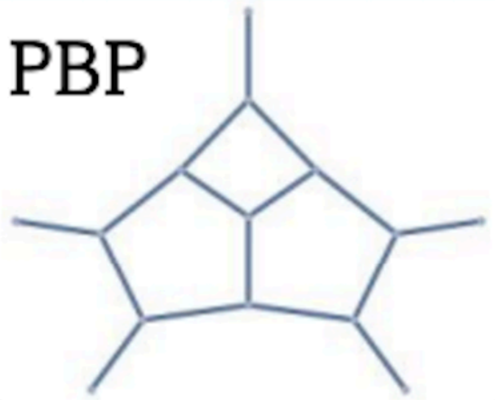
- five-point three loop ladder
- six-pint two loop ladder

All planar Three-Loop Five-Point Feynman integrals

[Chicherin, Henn, Wu, Wu, Xu, SQZ, Zhang]

PRD Editor's Suggestion

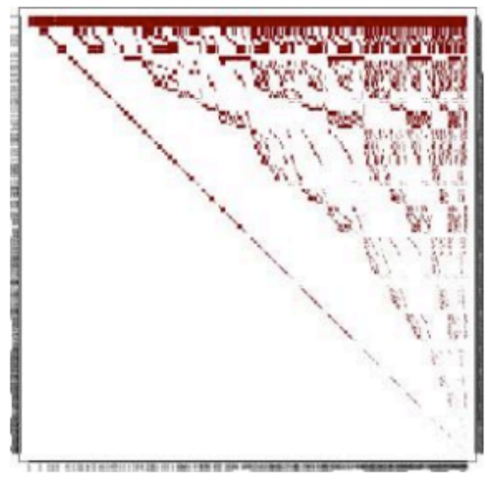
[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]

Family	PBB 	BPB 	BHB 	PBP 
Master Integrals	316	367	**431**	**734**
Five Point Integrals	115	156	164	323

Gong show by Yongqun Xu

Amplitudes 2025

$$\mathbb{A}_{\rm pl}^{\text{3-loop}} = \mathbb{A}_{\rm pl}^{\text{2-loop}} \cup \left\{\widetilde{W}_i\right\}_{i=1}^{5} \cup \left\{\widetilde{W}_i\right\}_{i=16}^{20} \cup \left\{\widetilde{W}_i\right\}_{i=41}^{55} \cup \left\{\widetilde{W}_i\right\}_{i=76}^{80}$$



734×734

Conjectured in [Chicherin, Henn, Trnka, Zhang '24]

Proved to be completed here.

$$\left\{\widetilde{W}_i\right\}_{i=76}^{80} = \frac{q_i - \sqrt{\Delta_4^{(1)}}\sqrt{\Delta_5}}{q_i + \sqrt{\Delta_4^{(1)}}\sqrt{\Delta_5}}, \quad \left\{\widetilde{W}_i\right\}_{i=41}^{55} = \frac{q_i - \sqrt{\Delta_4^{(1)}}}{q_i + \sqrt{\Delta_4^{(1)}}}$$

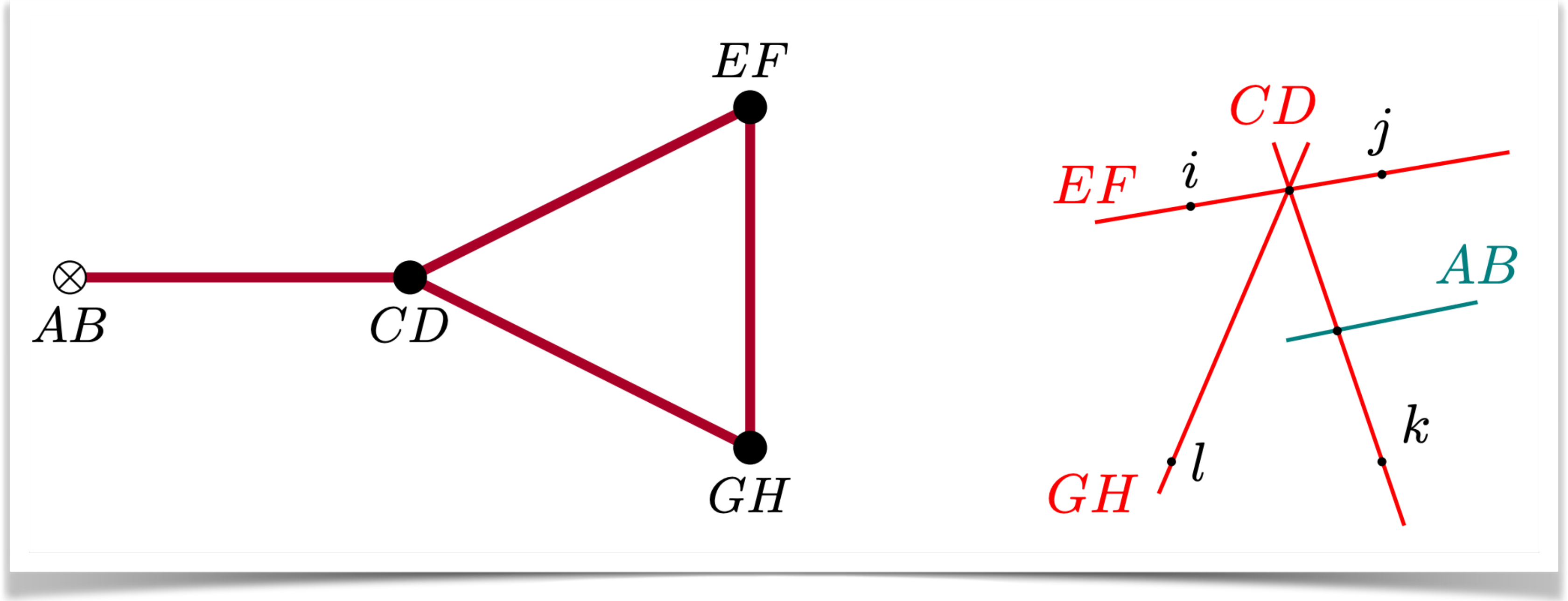
New square roots

$\Delta_4^{(i)}, \quad i = 1,\cdots,5$

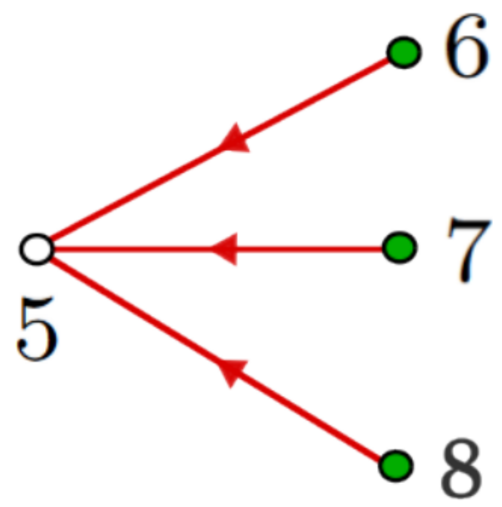
Outlook

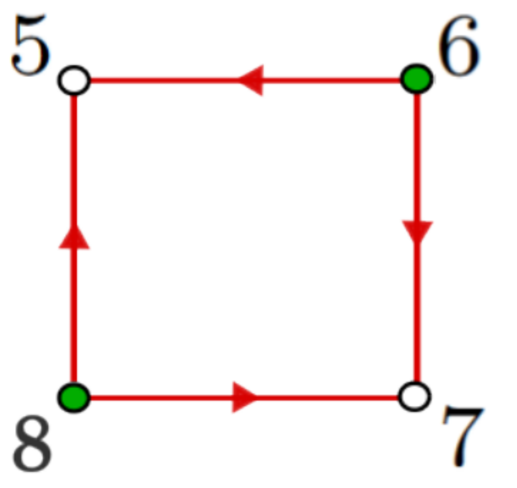
- **Next step : Five-point 3L WL**
  - **Positivity**? Three-loop pentagon functions?
  - Duality to **pure Yang-Mills** and **QCD** amplitudes [Chicherin, Henn'22]
  - **Cluster Algebra**? **[Talk by Rigers Aliaj]**

- **Negative geometries** with higher cycles
  Elliptic integrals?
  [Talks by Sara Maggio and Yoann Sohnle]



- **ABJM.** [Henn, Lagares, SQZ '23 '24]
  - **ABJM amplituhedron, 4pt integrand** to **five loops** [He, Huang, Kuo '23]
  - We compute $F^{(3)}_{\text{ABJM, 4pt}}$ and **4-loop cusp**: Agree with **Integrability** [Gromov, Vieira '08]
  - **Higher points, e.g. 6pts ? LS ? d'lembertian DE?**





$$\Gamma^{(\text{box})}_{\text{cusp}} = 0$$

simpler structure in ABJM?

Thank you !