

Integrable auxiliary field deformations of dimensionally reduced gravity

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DESY Theory Workshop: Synergies towards the future Standard Model

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Based on [MC, A. Kleinschmidt, D.Osten, arXiv:2409.04523]
[MC, D.Osten, arXiv:2502.01750]

- 1 Integrability: what do we mean?
- 2 Auxiliary field deformations: KK reduction of $D = 4$ GR to $D = 2$
- 3 Conclusions

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In $D = 2$, $x^\mu = (t, x)$.

- 1 *Lax integrability*: the Euler-Lagrange equations are equivalent to the flatness of the *Lax connection* $\mathfrak{L}(t, x; z) = \mathfrak{L}_\mu(t, x; z)dx^\mu$,

$$\partial_t \mathfrak{L}_x(t, x; z) - \partial_x \mathfrak{L}_t(t, x; z) + [\mathfrak{L}_t(t, x; z), \mathfrak{L}_x(t, x; z)] = 0.$$

- 2 *Hamiltonian integrability*: one of the possible ways to show the involution of conserved charges is when **[J.M. Maillet, '86]**

$$\begin{aligned} \{ \mathfrak{L}_{x, \underline{1}}(x, z), \mathfrak{L}_{x, \underline{2}}(x', z') \} &= [r_{\underline{12}}(z, z'), \mathfrak{L}_{x, \underline{1}}(x, z)] - [r_{\underline{21}}(z', z), \mathfrak{L}_{x, \underline{2}}(x', z')] \\ &\quad - (r_{\underline{12}}(z, z') + r_{\underline{21}}(z', z)) \partial_x \delta(x - x'), \end{aligned}$$

where

$$\begin{aligned} X_{\underline{1}} &\equiv X \otimes \mathbb{1} = X^A(T_A \otimes \mathbb{1}), & X_{\underline{2}} &\equiv \mathbb{1} \otimes X = X^A(\mathbb{1} \otimes T_A), \\ r_{\underline{12}} &= r^{AB} T_A \otimes T_B, & r_{\underline{21}} &= r^{BA} T_A \otimes T_B \end{aligned}$$

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Dimensionally reduced GR

KK reduction of $D = 4$ GR to $D = 2$

$$D = 4$$

Gravity with two space-like isometries

$$D = 3$$

$$D = 2$$

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KK reduction of $D = 4$ GR to $D = 2$

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$D = 3$

$\mathbb{R}^+ \times \text{SL}(1)$ symmetry + dualisation of KK vector
 $\mathbb{R}^+ \times \text{SL}(2)_{(\text{E})}$ symmetry

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T^2

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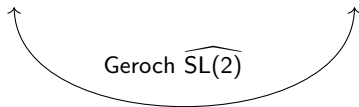
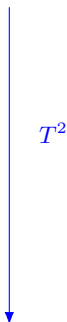
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KK reduction of $D = 4$ GR to $D = 2$

The Lagrangian in $D = 2$ after dimensional reduction is

$$\mathcal{L}_{D=2} = \partial_\mu \rho \partial^\mu \sigma - \frac{1}{2} \rho \text{Tr}(P_\mu P^\mu),$$

with

$$V^{-1} \partial_\mu V = P_\mu + Q_\mu, \quad V \in \text{SL}(2), \quad Q_\mu \in \mathfrak{so}_2, \quad P_\mu \in \mathfrak{sl}_2 \oplus \mathfrak{so}_2.$$

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Equations of motion:

$$D_\mu(\rho P^\mu) = 0,$$

$$\square \sigma + \frac{1}{2} \text{Tr}(P_\mu P^\mu) = 0,$$

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supplemented by Virasoro constraints

$$V_{\mu\nu} = \partial_\mu \rho \partial_\nu \sigma - \frac{1}{2} \partial_\nu \partial_\mu \rho - \frac{1}{2} \rho \text{Tr}(P_\mu P_\nu) = 0.$$

In [B. Julia, H. Nicolai, '96] was proven the existence of an even larger duality-symmetry,

$$\frac{\mathcal{W} \ltimes \widehat{\mathrm{SL}(2)}}{K(\mathcal{W}) \ltimes K(\widehat{\mathrm{SL}(2)})}.$$

Dimensionally reduced GR

Integrability of the undeformed model

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For integrability, define

$$\hat{\mathcal{Y}} = \left(\hat{\mathcal{V}}(z) \cdot \Gamma, e^{\hat{\sigma}} K \right) \in \frac{\mathcal{W} \ltimes \widehat{\mathrm{SL}(2)}}{K(\mathcal{W}) \ltimes K(\widehat{\mathrm{SL}(2)})},$$

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with

$$\hat{\mathcal{V}}(z) \in \widehat{\mathrm{SL}(2)}, \quad \mathbf{\Gamma} \in \mathcal{W}.$$

The conformal factor is rescaled:

$$\hat{\sigma} = \sigma - \frac{1}{2} \ln(\partial_+ \rho \partial_- \rho),$$

so that transforms as a proper scalar under conformal transformations.

Generalised linear system + twisted self duality constraints:

$$\hat{\mathcal{Y}}^{-1} \partial_\mu \hat{\mathcal{Y}} = \left(\mathbf{\Gamma}^{-1} \hat{\mathcal{V}}^{-1} \partial_\mu (\hat{\mathcal{V}} \mathbf{\Gamma}) , \left[\partial_\mu \hat{\sigma} - \Omega'(\hat{\mathcal{V}}, \hat{\mathcal{V}}^{-1} \partial_\mu \hat{\mathcal{V}}) \right] \mathbf{K} \right) \cong (\mathfrak{L}_\mu(z), 0),$$

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where

$$\mathfrak{L}_{\mu}(z) \equiv Q_{\mu} + \frac{1+z^2}{1-z^2} P_{\mu} + \frac{2z}{1-z^2} \epsilon_{\mu\nu} P^{\nu} + \left(\frac{1+z^2}{1-z^2} \frac{\partial_{\mu} \rho}{\rho} + \frac{2z}{1-z^2} \epsilon_{\mu\nu} \frac{\partial^{\nu} \rho}{\rho} \right) z \frac{\partial}{\partial z}.$$

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Then

$$d\mathfrak{L} + \frac{1}{2} [\mathfrak{L}, \mathfrak{L}] = 0 \quad \Longleftrightarrow \quad \text{Euler-Larange field equations.}$$

Deformed reduced GR

The deformation in $D = 2$

Deform the Lagrangian through introduction of auxiliary fields [MC, D. Osten, '25]. An approach pioneered for the PCM by [C.Ferko, L. Smith, '24] and inspired by [E. Ivanov, B. Zupnik, '02].

$$\begin{aligned}\tilde{\mathcal{L}} = & -\partial_\mu \rho \partial^\mu \sigma - 2(\chi_{1\mu} \partial^\mu \rho + \chi_{2\mu} \partial^\mu \sigma + \chi_1^\mu \chi_{2\mu}) \\ & + \rho \left(\text{Tr} \left[\frac{1}{2} (P_\mu P^\mu) + 2P_\mu v^\mu + v_\mu v^\mu \right] \right) + E(\nu),\end{aligned}$$

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with

$$\nu = (\eta^{\alpha\beta} \eta^{\gamma\sigma} + \epsilon^{\alpha\beta} \epsilon^{\gamma\sigma}) \left(\chi_{1\alpha} \chi_{2\gamma} - \frac{\rho}{2} \text{Tr}(v_\alpha v_\gamma) \right) \left(\chi_{1\beta} \chi_{2\sigma} - \frac{\rho}{2} \text{Tr}(v_\beta v_\sigma) \right).$$

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We define:

$$\mathcal{P}_\mu = -(P_\mu + 2v_\mu), \quad \mathcal{R}_\mu = -(\partial_\mu \rho + 2\chi_{2\mu}), \quad \mathcal{S}_\mu = -(\partial_\mu \sigma + 2\chi_{1\mu}).$$

Deformed reduced GR

The deformation in $D = 2$

Equations of motion: dynamical for scalars , dynamical for conformal factor and algebraic ones ,

$$\begin{cases} \partial_\mu \mathcal{R}^\mu = 0, \\ D_\mu(\rho \mathcal{P}^\mu) = 0, \\ \partial_\mu \mathcal{S}^\mu = \text{Tr} \left[\frac{1}{2} (P_\mu P^\mu) + 2P_\mu v^\mu + v_\mu v^\mu \right] + K^{\mu\nu}(\chi_{1,2}, v, \rho) \text{Tr}(v_\mu v_\nu), \\ \begin{cases} P_\mu = -v_\mu + K_\mu{}^\nu(\chi_{1,2}, v, \rho) v_\nu, \\ \partial_\mu \sigma = -\chi_{1\mu} + K_\mu{}^\nu(\chi_{1,2}, v, \rho) \chi_{1\nu}, \\ \partial_\mu \rho = -\chi_{2\mu} + K_\mu{}^\nu(\chi_{1,2}, v, \rho) \chi_{2\nu}, \end{cases} \end{cases}$$

where

$$K_\mu{}^\nu(\chi_{1,2}, v, \rho) = E'(\nu)(\delta_\mu^\alpha \eta^{\gamma\nu} - \eta_{\mu\rho} \epsilon^{\rho\alpha} \epsilon^{\gamma\nu}) \left(\chi_{1\alpha} \chi_{2\gamma} - \frac{\rho}{2} \text{Tr}(v_\alpha v_\gamma) \right)$$

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The deformation in $D = 2$: integrability

With the same parametrisations, the same twisted self-duality conditions, impose the generalised linear system:

$$\hat{\mathcal{Y}}^{-1} \partial_\mu \hat{\mathcal{Y}} = \left(\mathbf{\Gamma}^{-1} \hat{\mathcal{V}}^{-1} \partial_\mu (\hat{\mathcal{V}} \mathbf{\Gamma}), \left[\tilde{\mathcal{S}}_\mu - \Omega'(\hat{\mathcal{V}}, \hat{\mathcal{V}}^{-1} \partial_\mu \hat{\mathcal{V}}) \right] \mathbf{\kappa} \right) \cong (\tilde{\mathcal{L}}_\mu(z), 0),$$

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The flatness of this connection is equivalent to the **EOMs** [MC, D. Osten, '25].

Virasoro constraints stemming from the vanishing of the central term are

$$\tilde{S}_\pm = -\frac{1}{2} \rho \operatorname{Tr} \left(\frac{v_\pm v_\pm}{\chi_{2\pm}} + \frac{K_\pm^\mp v_\mp v_\mp}{\chi_{2\mp}} \right),$$

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On Hamiltonian integrability: we recover the same r -matrix of [D. Bernard, N. Regnault, '01].

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